On the Definiteness of the Weighted Laplacian and its Connection to Effective Resistance

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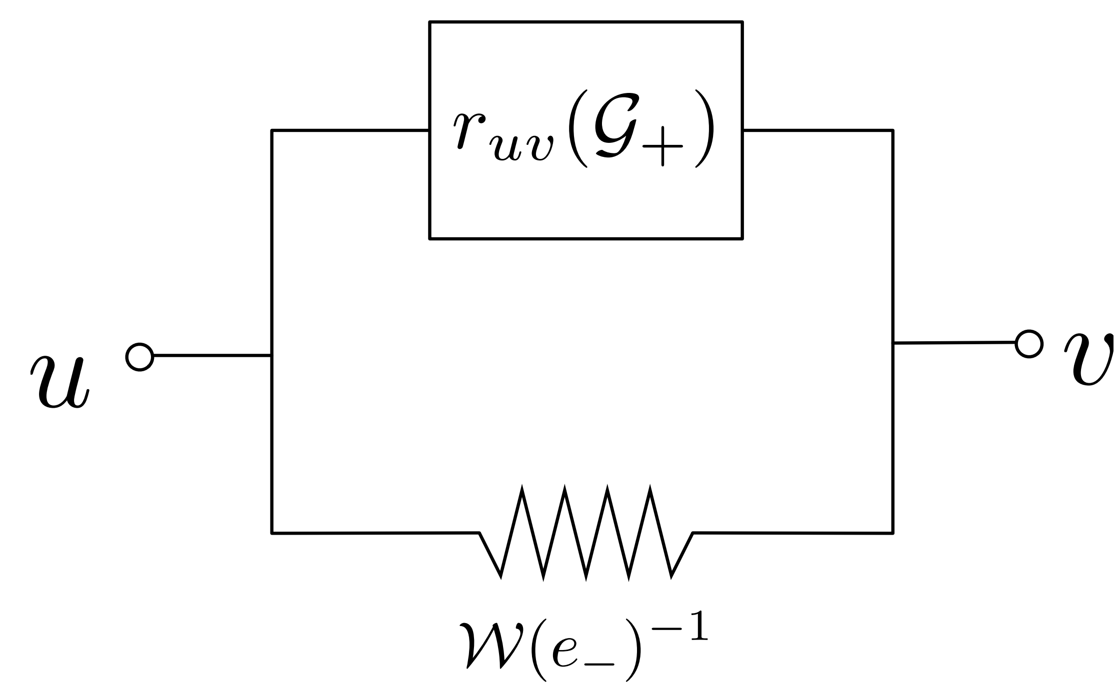
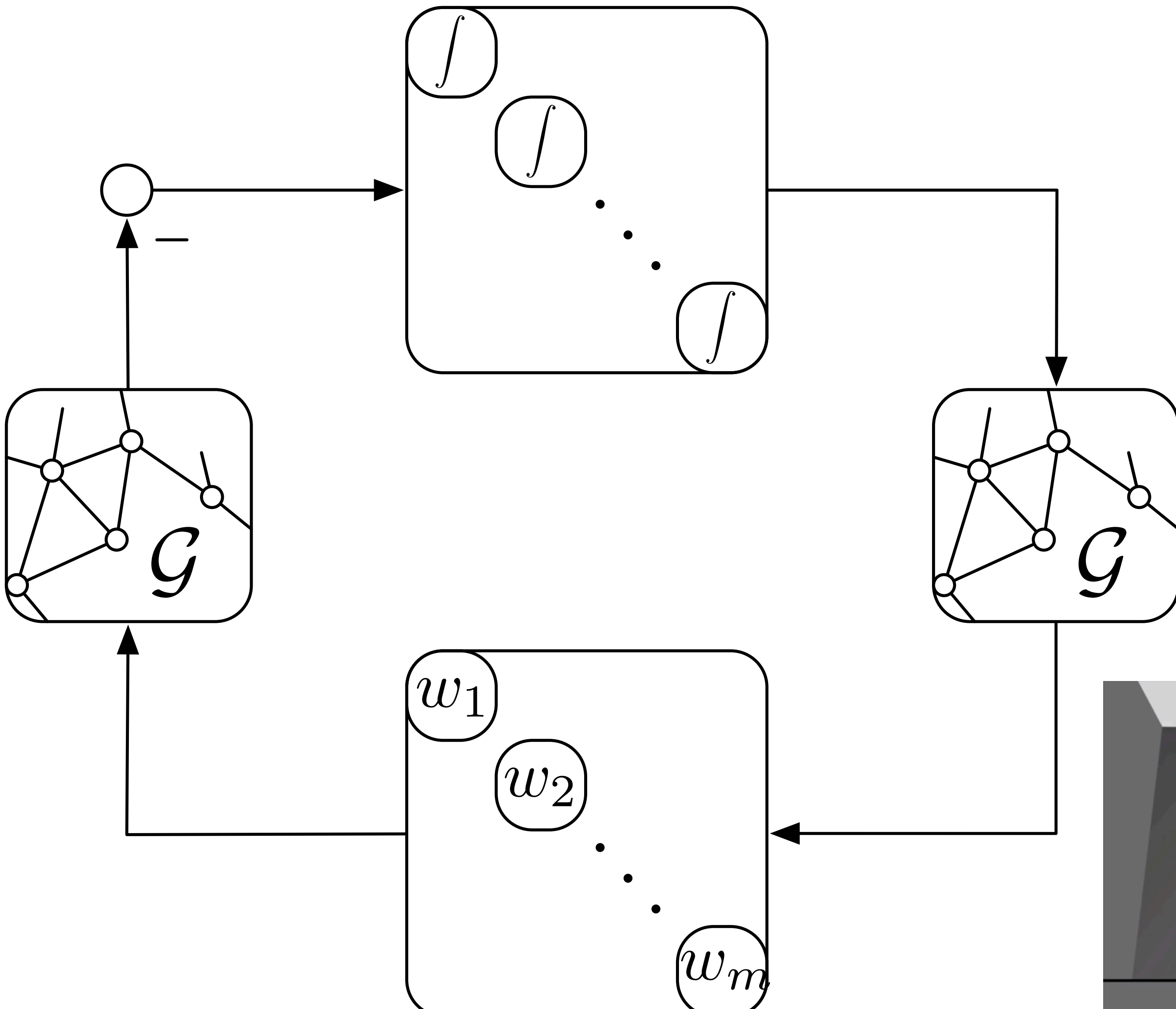


Diffusively Coupled Networks

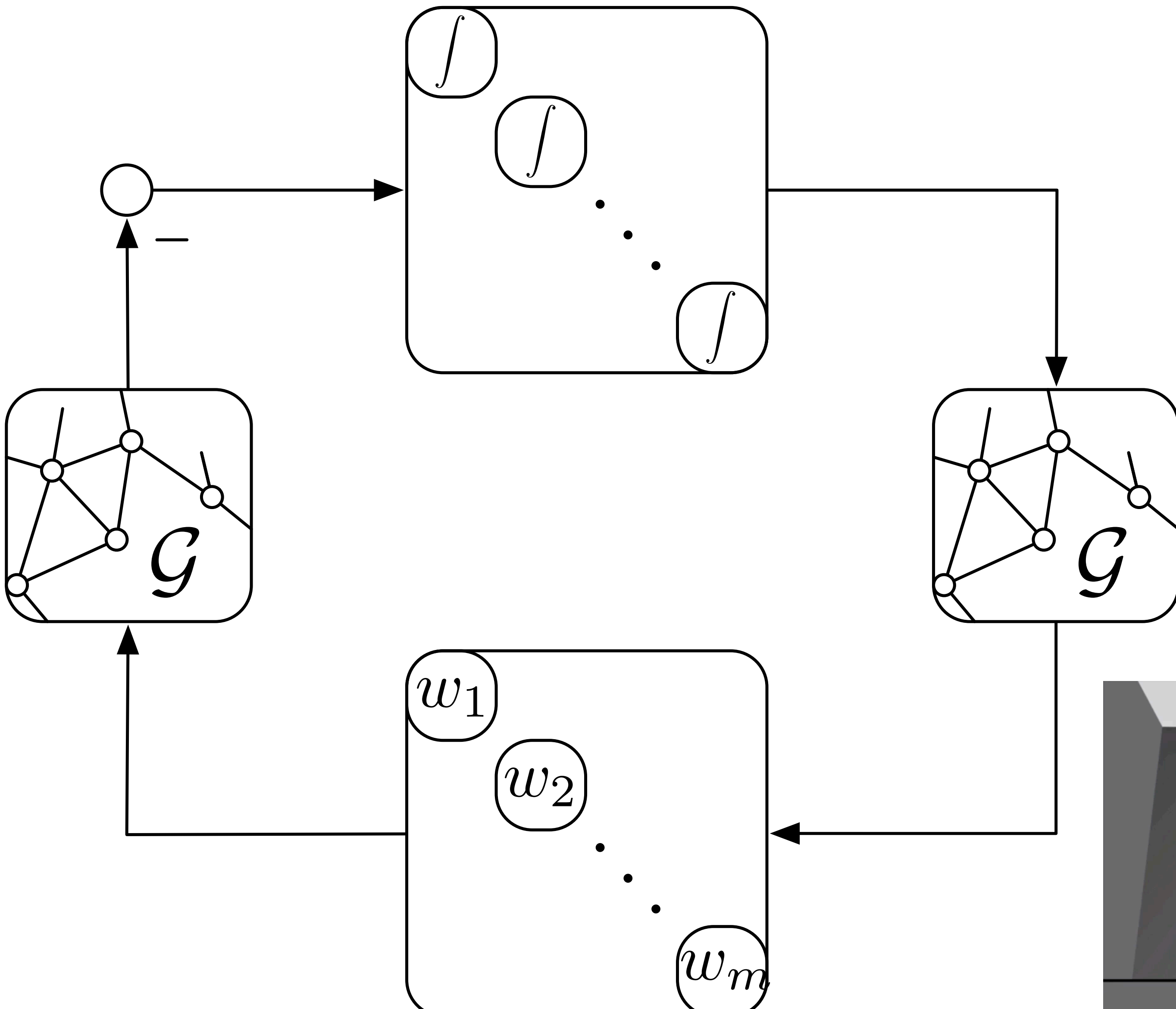


Consensus Protocol

$$\dot{x}_i = \sum_{j \sim i} w_{ij}(x_j - x_i)$$

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

Diffusively Coupled Networks



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The Consensus Protocol

Consensus Protocol

$$u_i(t) = \sum_{i\sim j} w_{ij}(x_j(t) - x_i(t))$$

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

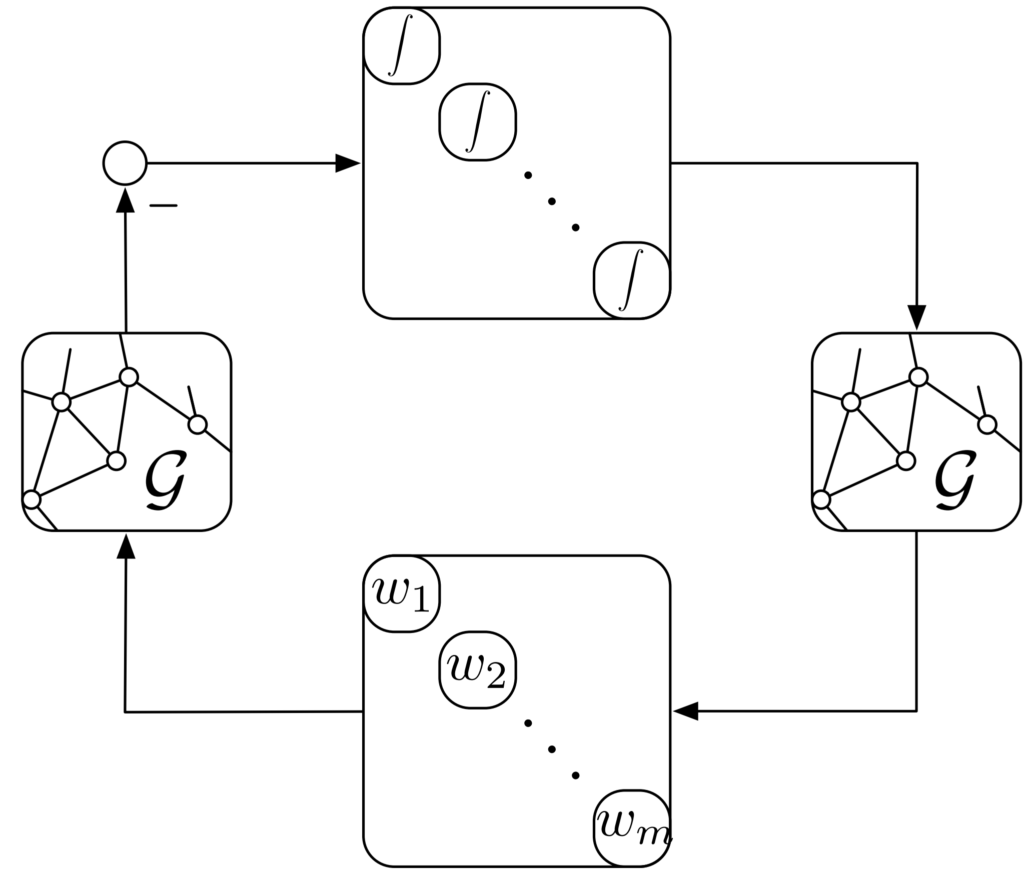
Theorem *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ *be a weighted and connected graph with positive edge weights* $\mathcal{W}(k) > 0$ *for* $k = 1, \ldots, |\mathcal{E}|$. *Then the consensus dynamics synchronizes; i.e.,* $\lim_{t\to\infty} x_i(t) = \beta$ *for* $i = 1, \ldots, |\mathcal{V}|$.

Mesbahi & Egerstedt, Olfati-Saber, Ren

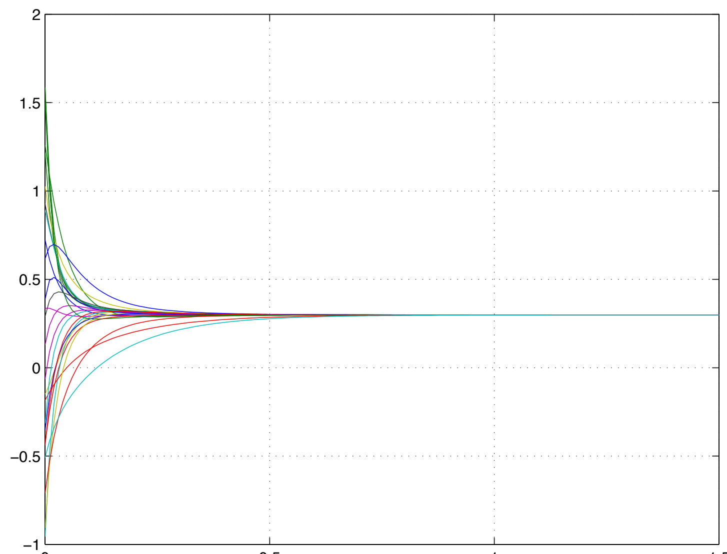
Synchronization and the Laplacian

The Linear Weighted Consensus Protocol

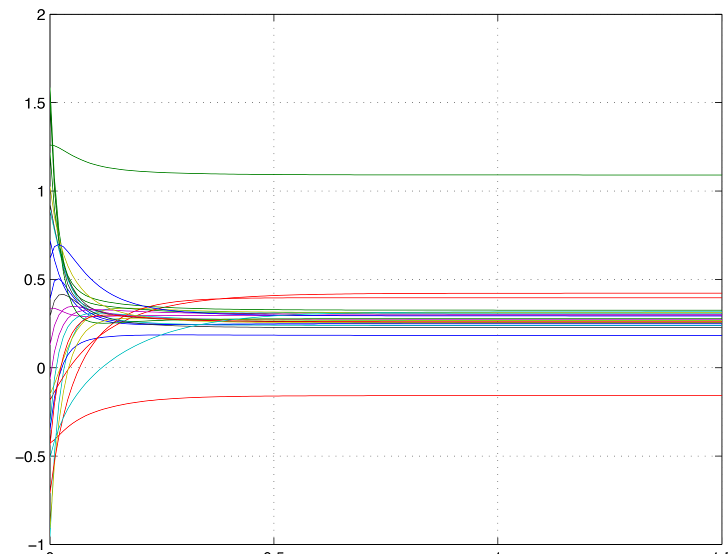
$$\dot{x}_i(t) = \sum_{i \sim j} w_{ij}(x_j(t) - x_i(t))$$



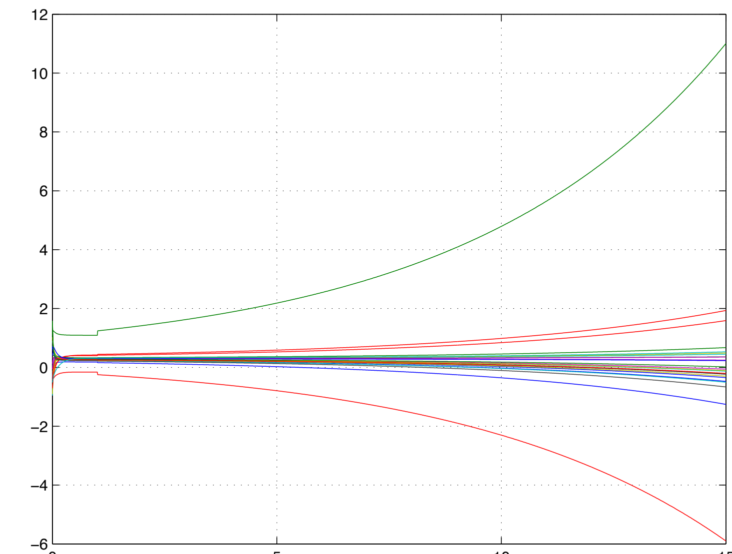
$\mathcal{G}$ 25 nodes
98 edges



$w^* > -10.1911$



$w^* = -10.1911$

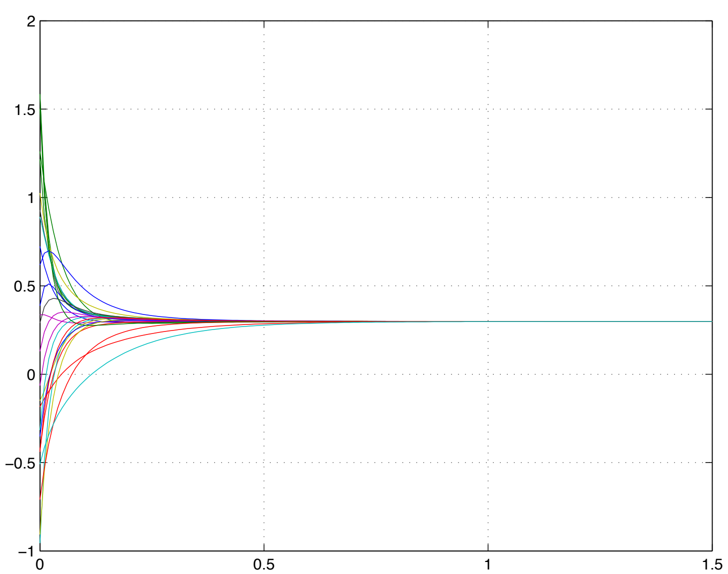


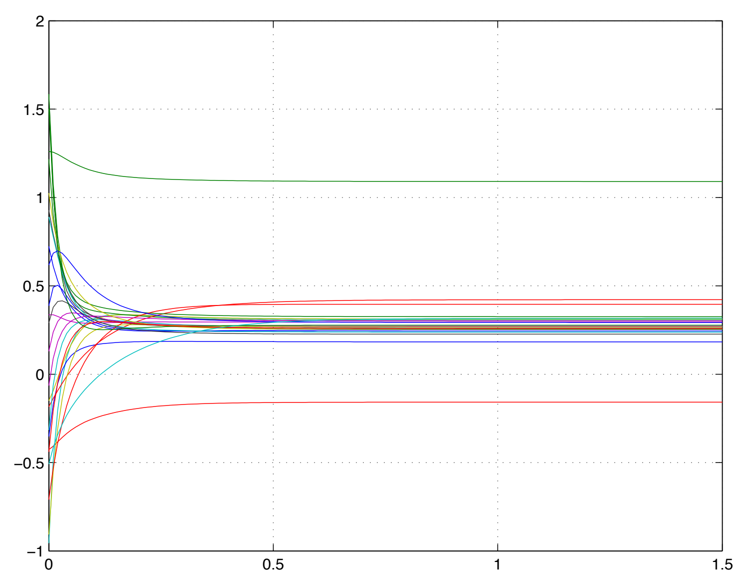
$w^* < -10.1911$

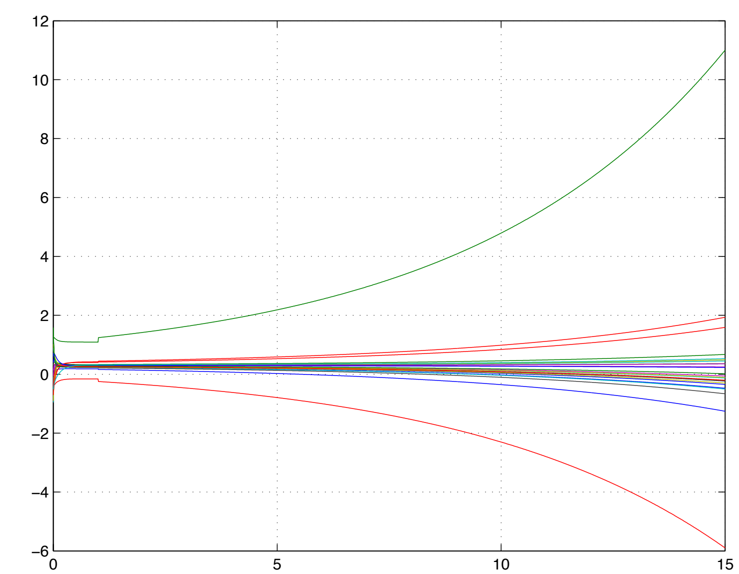
Synchronization and the Laplacian

$$x(t) = e^{-L(\mathcal{G})t}x_0$$

$$\lim_{t\to\infty} x(t) = \beta\mathbb{1} \Leftrightarrow L(\mathcal{G})$$ has only **one** eigenvalue at the origin







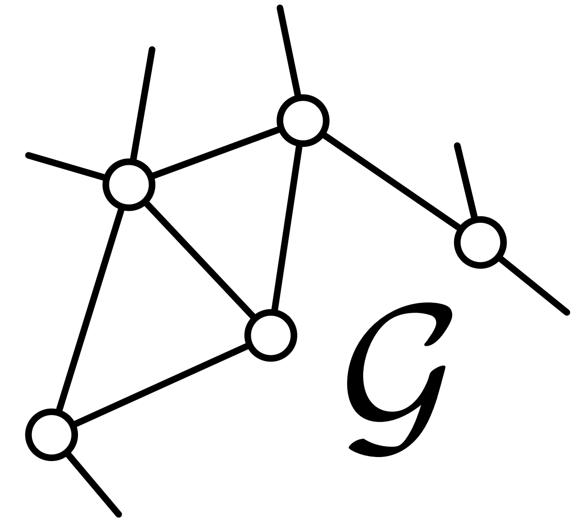
$L(\mathcal{G}) \geq 0$ has only **one** eigenvalue at the zero

$L(\mathcal{G}) \geq 0$ has **more than one** eigenvalue at the zero

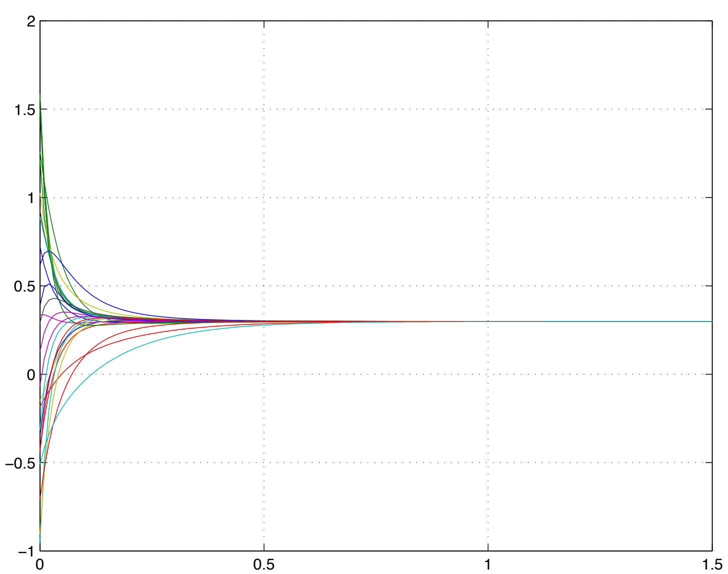
$L(\mathcal{G})$ has **at least one** negative eigenvalue (indefinite)

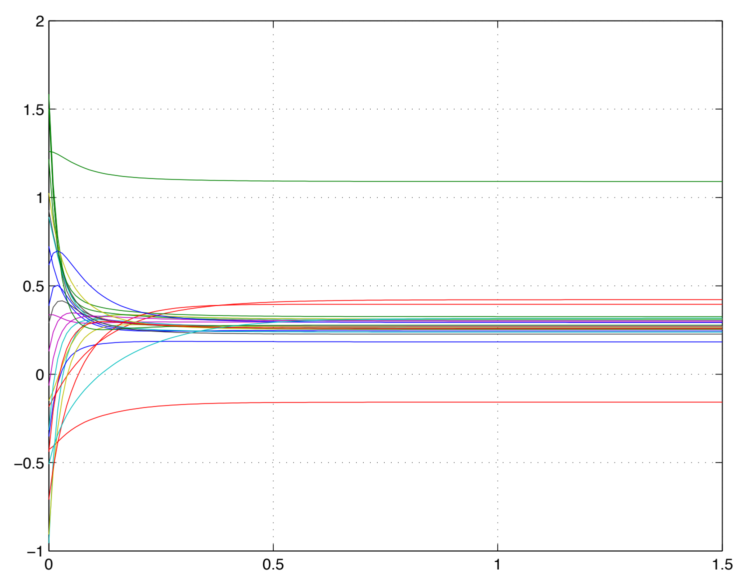
Synchronization and the Laplacian

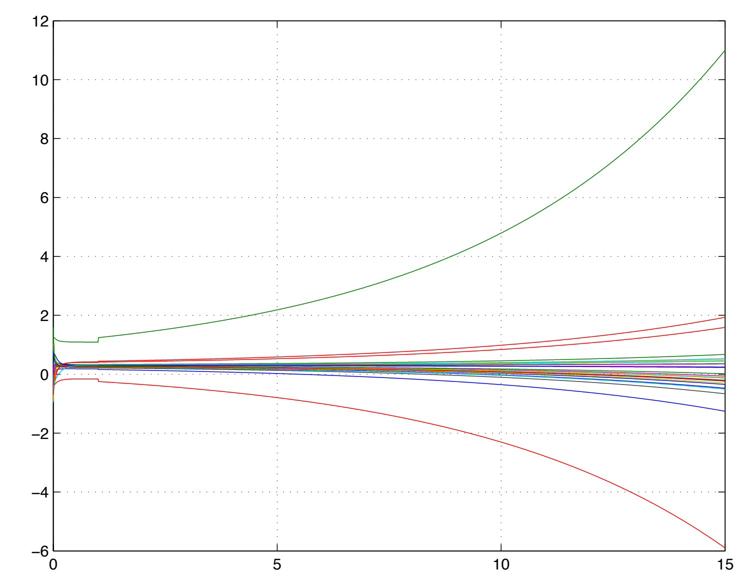
$$\dot{x}(t) = -L(\mathcal{G})x(t)$$



system behavior depends on the spectral properties of the graph Laplacian







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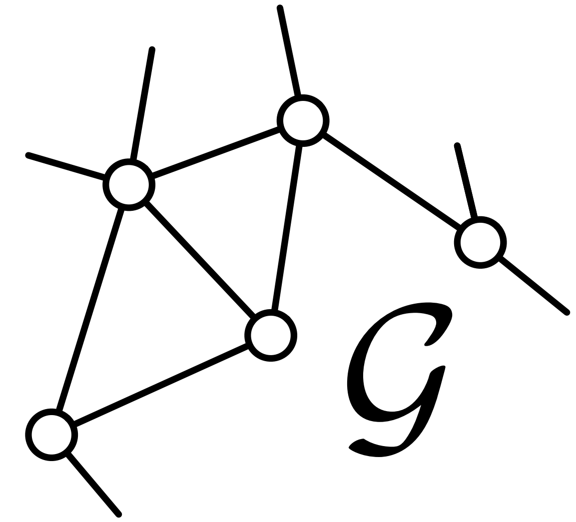
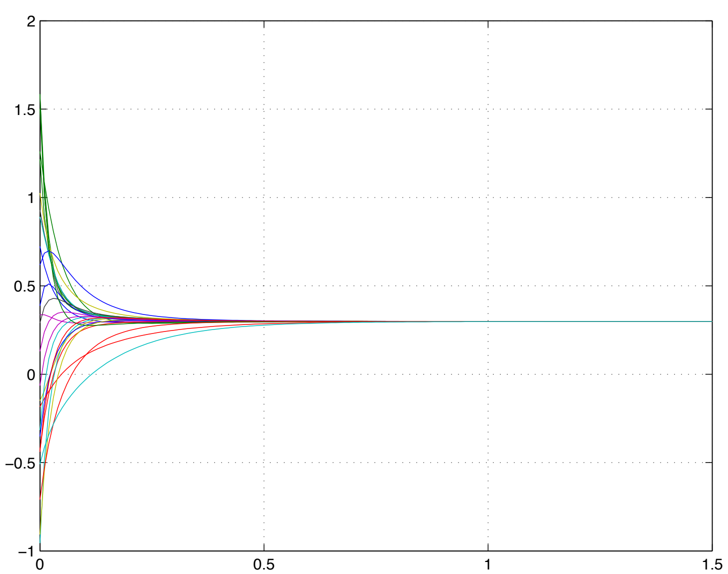
$$\dot{x}(t) = -L(\mathcal{G})x(t)$$



can we understand spectral properties of the Laplacian from the structure of the graph?



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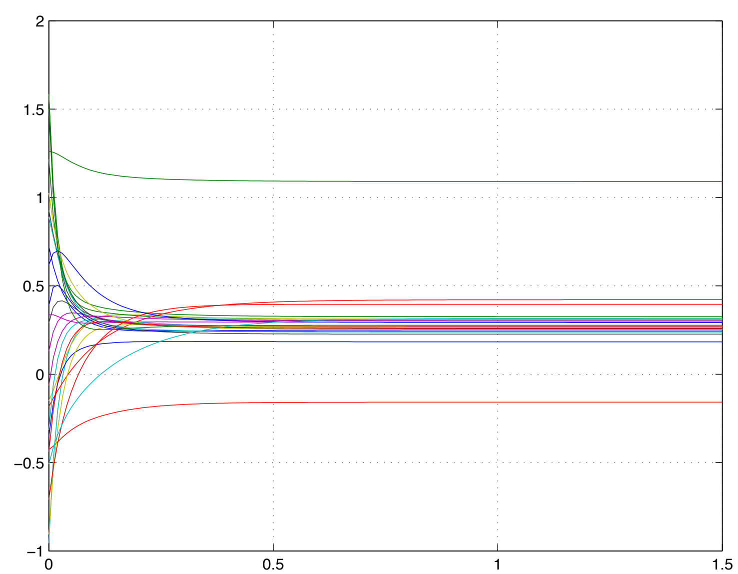

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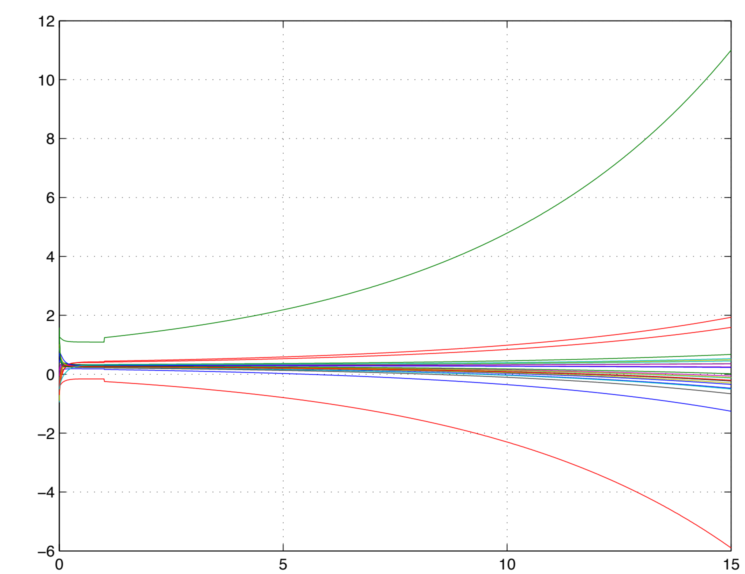

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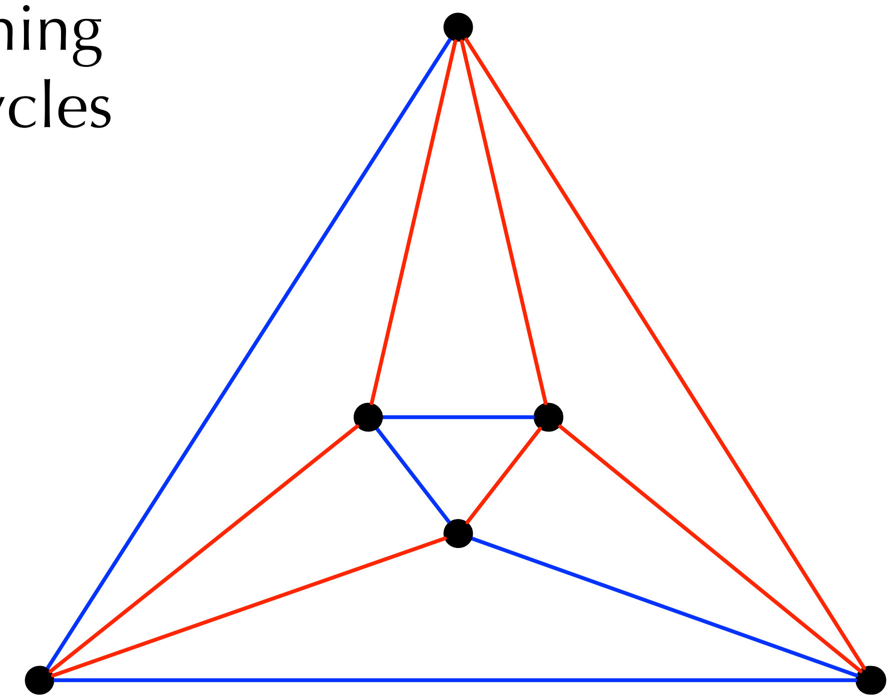
Spanning Trees and Cycles

A graph as the union of a spanning tree and edges that complete cycles



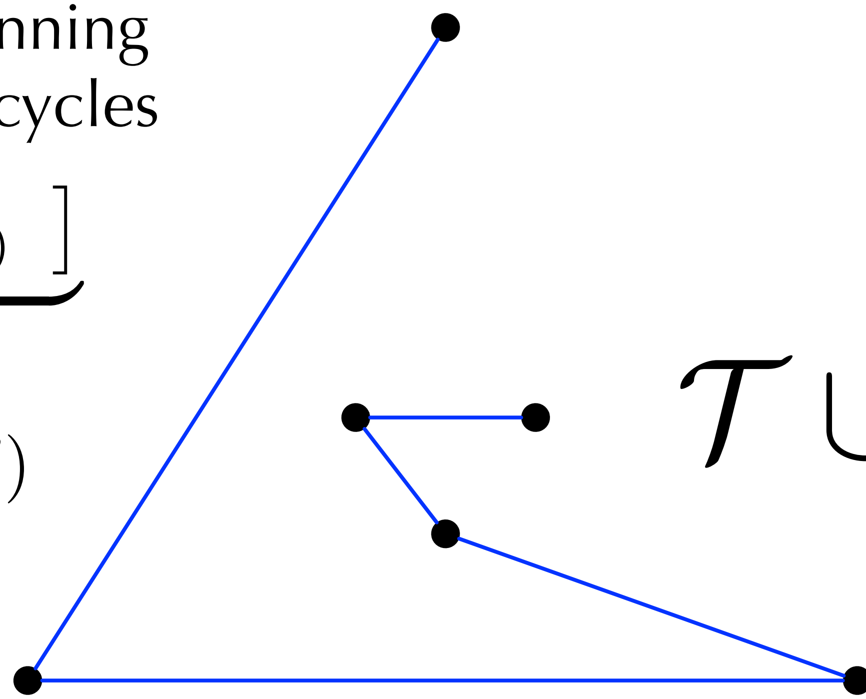
Spanning Trees and Cycles

A graph as the union of a spanning tree and edges that complete cycles

$$E(\mathcal{G}) = E(\mathcal{T})\underbrace{\left[\begin{array}{cc} I & T_{(\mathcal{T},\mathcal{C})}\end{array}\right]}_{\mathcal{R}_{(\mathcal{T},\mathcal{C})}}$$

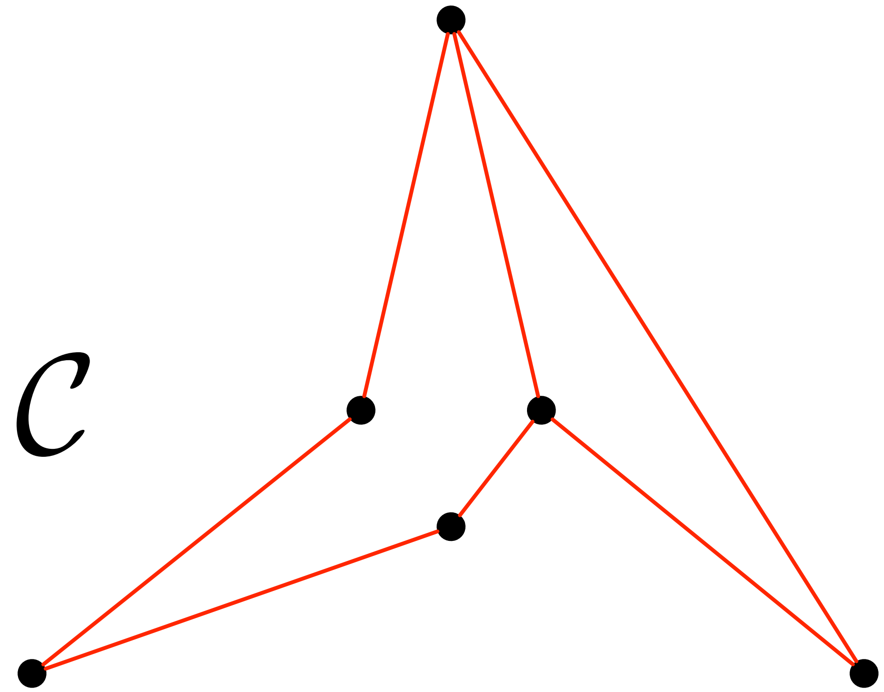
$$T_{(\mathcal{T},\mathcal{C})} = \underbrace{(E_{\mathcal{T}}^T E_{\mathcal{T}})^{-1} E_{\mathcal{T}}^T}_{E_{\mathcal{T}}^L} E(\mathcal{C})$$

$$L(\mathcal{G}) = E(\mathcal{G})E(\mathcal{G})^T$$



a spanning tree

$$\mathcal{T} \cup \mathcal{C}$$



remaining edges
"complete cycles"

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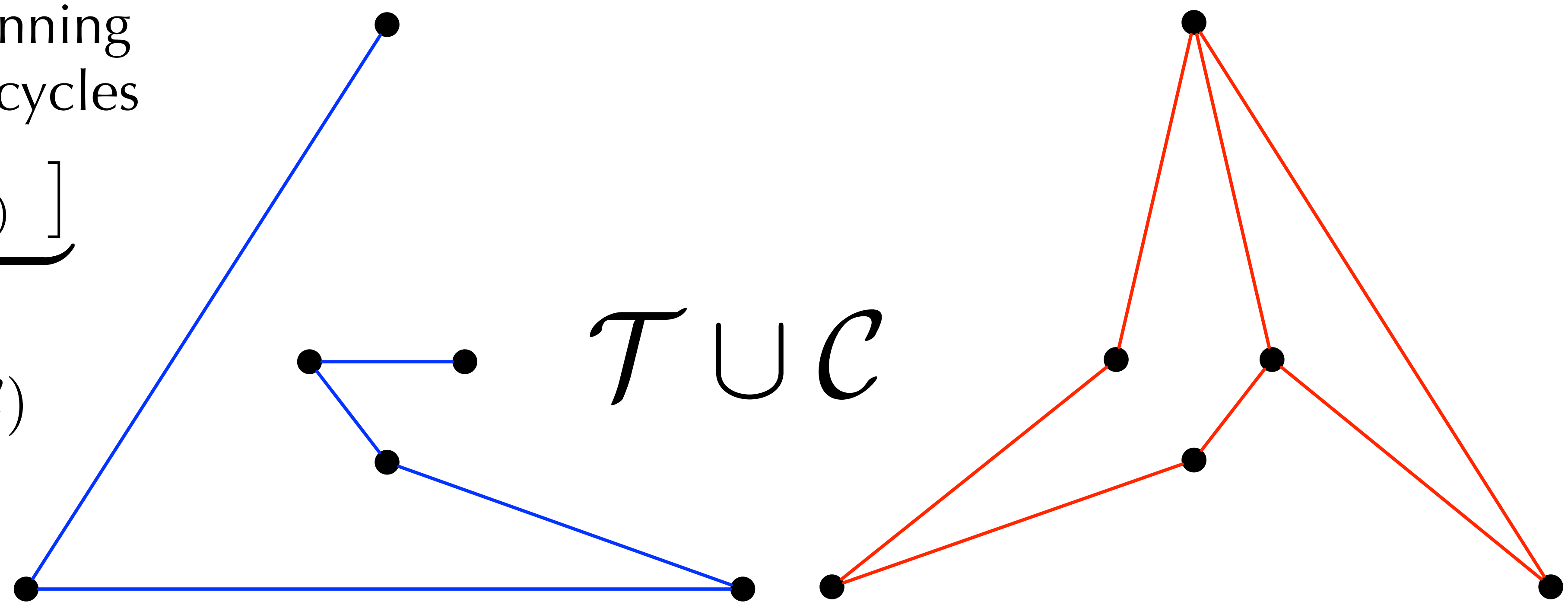

a spanning tree

remaining edges "complete cycles"

Weighted Edge Laplacian

$$L_e(\mathcal{G}) = W^{\frac{1}{2}} E(\mathcal{G})^T E(\mathcal{G}) W^{\frac{1}{2}}$$

Essential Edge Laplacian

$$L_e(\mathcal{T}) R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T := L_{ess}(\mathcal{G})$$

$\mathcal{R}_{(\mathcal{T},\mathcal{C})}$ rows form a basis for the *cut space* of the graph

similarity between edge and graph Laplacians

$$L(\mathcal{G}) \rightleftharpoons L_e(\mathcal{G})$$

Some Properties of $L_e(\mathcal{G})$

Proposition *The matrix* $L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T$ *has the same inertia as* $R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T$. *Similarly, the matrix* $(L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T)^{-1}$ *has the same inertia as* $(R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T)^{-1}$.

Recall: The *inertia* of a matrix is the number of negative, 0, and positive eigenvalues

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Recall: The *inertia* of a matrix is the number of negative, 0, and positive eigenvalues

Proof:

$$L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})} \sim L_e(\mathcal{T})^{\frac{1}{2}}R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})}L_e(\mathcal{T})^{\frac{1}{2}}$$

$L_e(\mathcal{T})^{\frac{1}{2}}R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})}L_e(\mathcal{T})^{\frac{1}{2}}$ is congruent to $R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})}$

congruent matrices have the same inertia

Some Properties of $L_e(\mathcal{G})$

Proposition

$$L(\mathcal{G}) \geq 0 \Leftrightarrow R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T \geq 0$$

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The definiteness of the graph Laplacian can be studied through another matrix!

$$R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T$$

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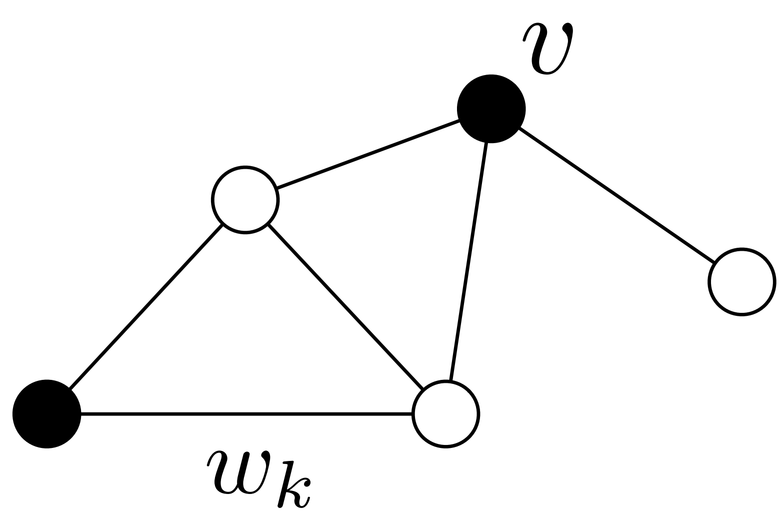
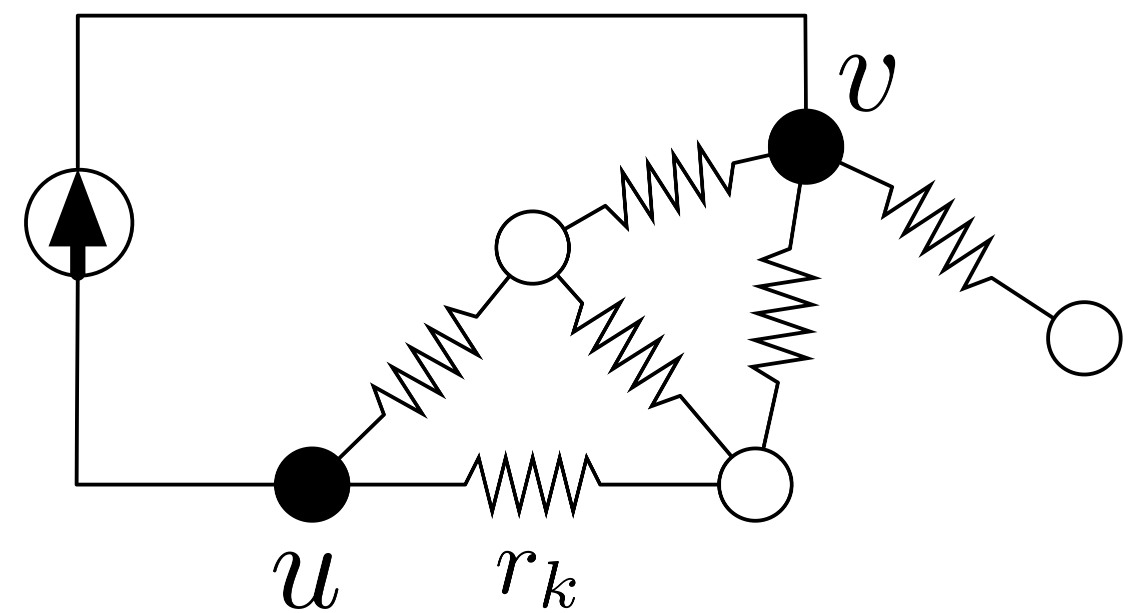
$$R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T$$

intimately related to the notion of **effective resistance** of a network

Effective Resistance of a Graph

The **effective resistance** between two nodes *u* and *v* is the electrical resistance measured across the nodes when the graph represents an electrical circuit with each edge a resistor





$$r_k = \frac{1}{w_k}$$ edge weights are the *conductance* of *each resistor*

$$\begin{aligned} r_{uv} &= (\mathbf{e}_u - \mathbf{e}_v)^T L^\dagger(\mathcal{G})(\mathbf{e}_u - \mathbf{e}_v) \\ &= \left[L^\dagger(\mathcal{G})\right]_{uu} - 2\left[L^\dagger(\mathcal{G})\right]_{uv} + \left[L^\dagger(\mathcal{G})\right]_{vv} \end{aligned}$$

Klein and Randić 1993

Effective Resistance of a Graph

Proposition

$$\begin{aligned} L^{\dagger}(\mathcal{G}) &= (E_{\mathcal{T}}^{L})^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} E_{\mathcal{T}}^{L} \\ &= \quad (E_{\mathcal{T}}^{L})^T L_{ess}(\mathcal{T})^{-1} E_{\mathcal{T}}^{T} \end{aligned}$$

Effective Resistance of a Graph

Proposition

$$L^{\dagger}(\mathcal{G}) = (E_{\mathcal{T}}^{L})^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} E_{\mathcal{T}}^{L}$$
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$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T L^{\dagger}(\mathcal{G})(\mathbf{e}_u - \mathbf{e}_v)$$

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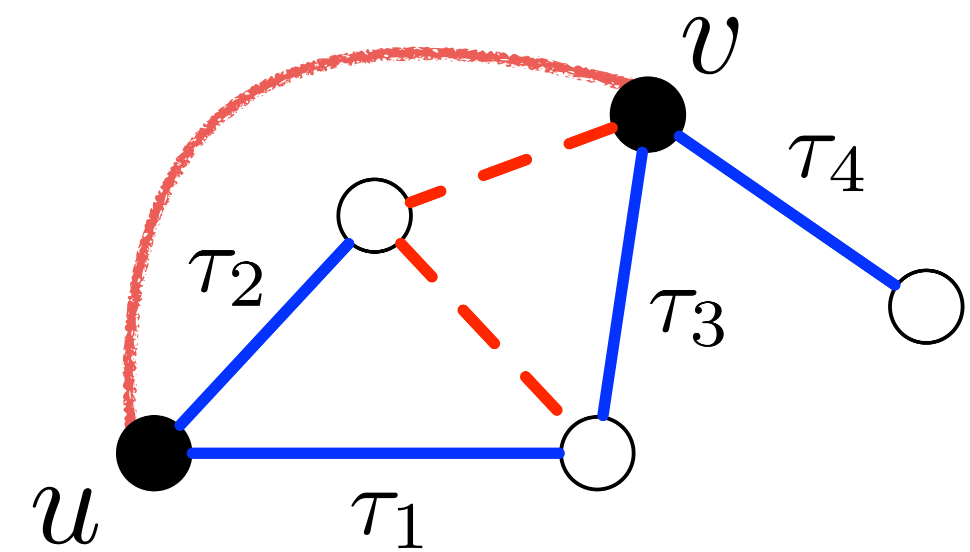


$$\mathcal{G} = \mathcal{T} \cup \mathcal{C}$$

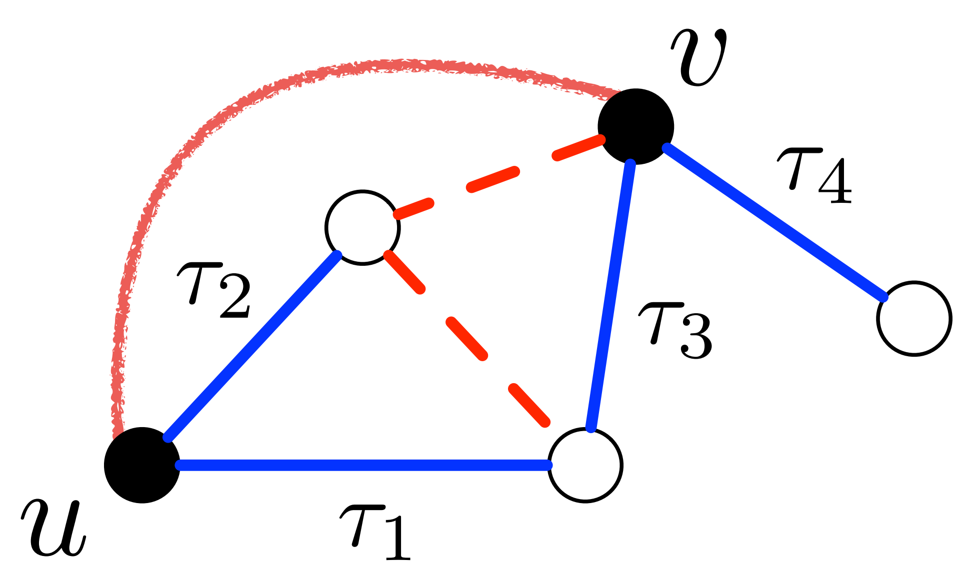
Effective Resistance of a Graph

Proposition

$$L^{\dagger}(\mathcal{G}) = (E_{\mathcal{T}}^{L})^{T}\left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^{T}\right)^{-1} E_{\mathcal{T}}^{L}$$
$$= (E_{\mathcal{T}}^{L})^{T} L_{ess}(\mathcal{T})^{-1} E_{\mathcal{T}}^{T}$$

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T L^{\dagger}(\mathcal{G})(\mathbf{e}_u - \mathbf{e}_v)$$

$$E_{\mathcal{T}}^{L}(\mathbf{e}_u - \mathbf{e}_v) = \begin{array}{cc} \left[\begin{array}{c} \pm 1 \\ 0 \\ \pm 1 \\ 0 \end{array}\right] & \begin{array}{c} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{array} \end{array}$$



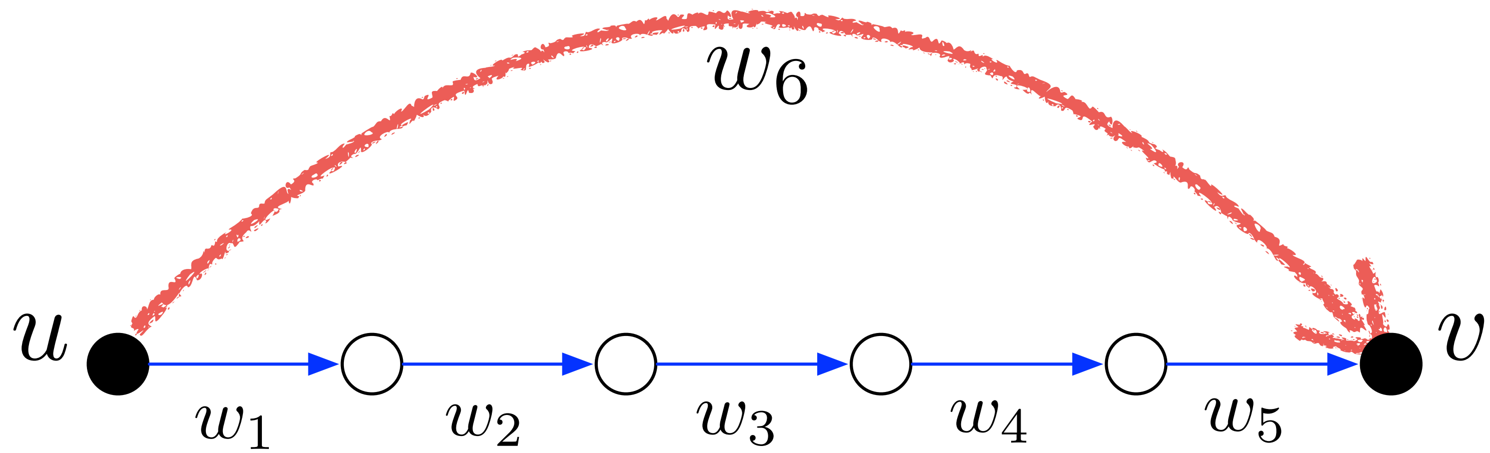
$$\mathcal{G} = \mathcal{T} \cup \mathcal{C}$$

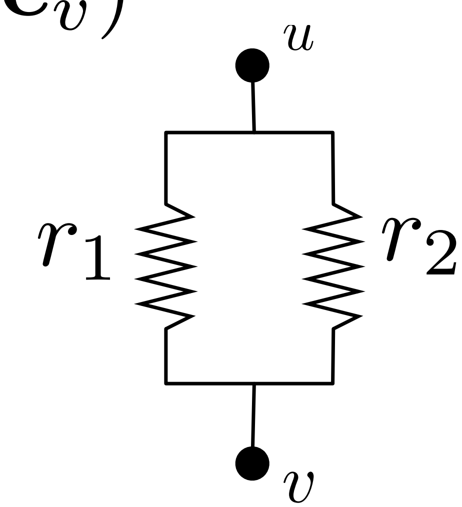
indicates a path from node u to v using only edges in the spanning tree

$$T_{(\mathcal{T},\mathcal{C})} = \underbrace{(E_{\mathcal{T}}^{T} E_{\mathcal{T}})^{-1} E_{\mathcal{T}}^{T}}_{E_{\mathcal{T}}^{L}} E(\mathcal{C})$$

Effective Resistance of a Graph

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T (E_{\mathcal{T}}^L)^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v)$$





$$r_{uv} = \frac{r_1 r_2}{r_1 + r_2}$$

$$R_{(\mathcal{T},\mathcal{C})} = \left[\begin{array}{cc} I & \mathbb{1} \end{array}\right]$$

$$E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v) = \mathbb{1}$$

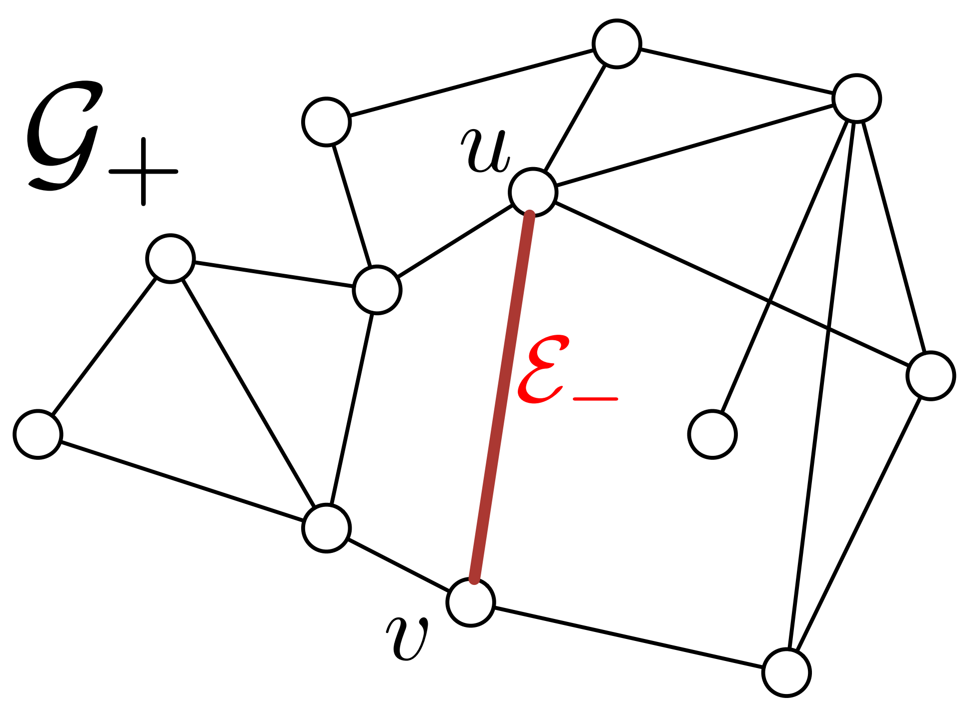
$$r_k = \frac{1}{w_k}$$

$$W_{\mathcal{T}} = \mathrm{diag}\{w_1, \ldots, w_5\}$$

$$\begin{aligned} r_{uv} &= \mathbb{1}^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} \mathbb{1} \\ &= \mathbb{1}^T \left(W_{\mathcal{T}} + w_6 \mathbb{1}\mathbb{1}^T\right)^{-1} \mathbb{1} \\ &= \frac{\left(\mathbb{1}^T W_{\mathcal{T}}^{-1} \mathbb{1}\right) w_6^{-1}}{\mathbb{1}^T W_{\mathcal{T}}^{-1} \mathbb{1} + w_6^{-1}} \end{aligned}$$

Spectral Properties of Signed Graphs

Theorem *Assume that* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ *has one edge with a negative weight,* $e_- = (u, v) \in \mathcal{E}$*. Let* $\mathcal{G}_+ = (\mathcal{V}, \mathcal{E} \setminus \{e_-\}, \mathcal{W})$ *and* $\mathcal{G}_- = (\mathcal{V}, e_-, \mathcal{W})$ *and assume* $\mathcal{G}_+$ *is connected. Furthermore, let* $\mathcal{R}_{uv}(\mathcal{G}_+)$ *denote the effective resistance between nodes* $u, v \in \mathcal{V}$ *over the graph* $\mathcal{G}_+$*. Then* $L(\mathcal{G})$ *is positive semi-definite if and only if* $|\mathcal{W}(e_-)| \leq \mathcal{R}_{uv}^{-1}(\mathcal{G}_+)$.



Spectral Properties of Signed Graphs

Proof:

$$L(\mathcal{G}) = E_{\mathcal{T}_+} R_{(\mathcal{T}_+,\mathcal{C}_+)} W_+ R^T_{(\mathcal{T}_+,\mathcal{C}_+)} E^T_{\mathcal{T}_+} - E_- |\mathcal{W}(e_-)| E_-^T$$

$$\begin{bmatrix} |\mathcal{W}(e_-)|^{-1} & E_-^T \\ E_- & E_{\mathcal{T}_+} R_{(\mathcal{T}_+,\mathcal{C}_+)} W_+ R^T_{(\mathcal{T}_+,\mathcal{C}_+)} E^T_{\mathcal{T}_+} \end{bmatrix} \geq 0$$

$$\begin{bmatrix} |\mathcal{W}(e_-)|^{-1} & E_-^T (E^L_{\mathcal{T}_+})^T \\ E^L_{\mathcal{T}_+} E_- & R_{(\mathcal{T}_+,\mathcal{C}_+)} W_+ R^T_{(\mathcal{T}_+,\mathcal{C}_+)} \end{bmatrix} \geq 0$$

$$|W_-|^{-1} - \underbrace{E_-^T (E^L_{\mathcal{F}_+})^T (R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R^T_{(\mathcal{F}_+,\mathcal{C}_+)})^{-1} E^L_{\mathcal{F}_+} E_-}_{r_{uv}(\mathcal{G}_+)} \geq 0$$

Spectral Properties of Signed Graphs

Proof:

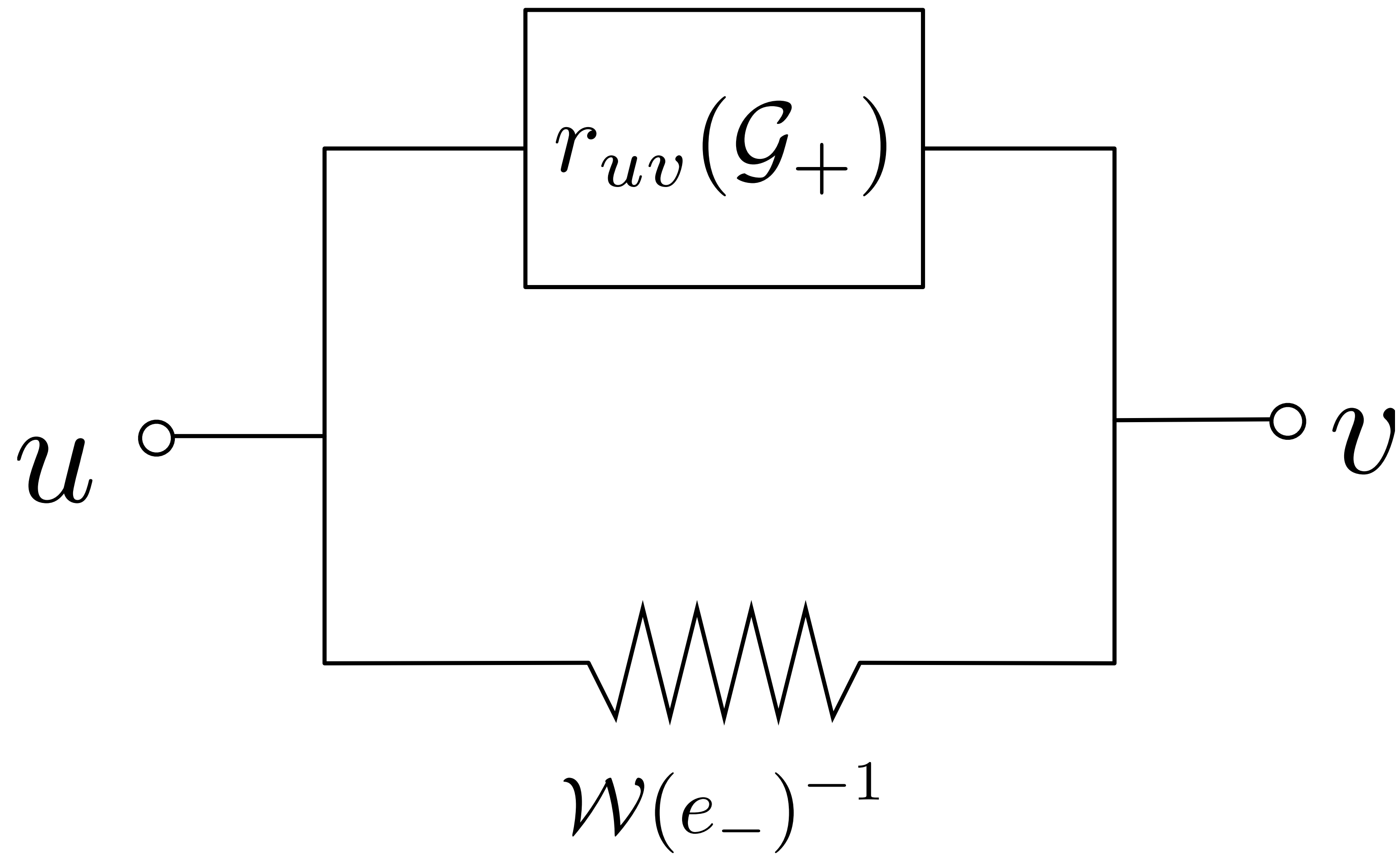
$$L(\mathcal{G}) = E_{\mathcal{T}_+} R_{(\mathcal{T}_+,\mathcal{C}_+)} W_+ R^T_{(\mathcal{T}_+,\mathcal{C}_+)} E^T_{\mathcal{T}_+} - E_- |\mathcal{W}(e_-)| E^T_-$$

Schur Complement
$$\begin{bmatrix} |\mathcal{W}(e_-)|^{-1} & E^T_- \\ E_- & E_{\mathcal{T}_+} R_{(\mathcal{T}_+,\mathcal{C}_+)} W_+ R^T_{(\mathcal{T}_+,\mathcal{C}_+)} E^T_{\mathcal{T}_+} \end{bmatrix} \geq 0$$

Congruent Transformation
$$\begin{bmatrix} |\mathcal{W}(e_-)|^{-1} & E^T_- (E^L_{\mathcal{T}_+})^T \\ E^L_{\mathcal{T}_+} E_- & R_{(\mathcal{T}_+,\mathcal{C}_+)} W_+ R^T_{(\mathcal{T}_+,\mathcal{C}_+)} \end{bmatrix} \geq 0$$

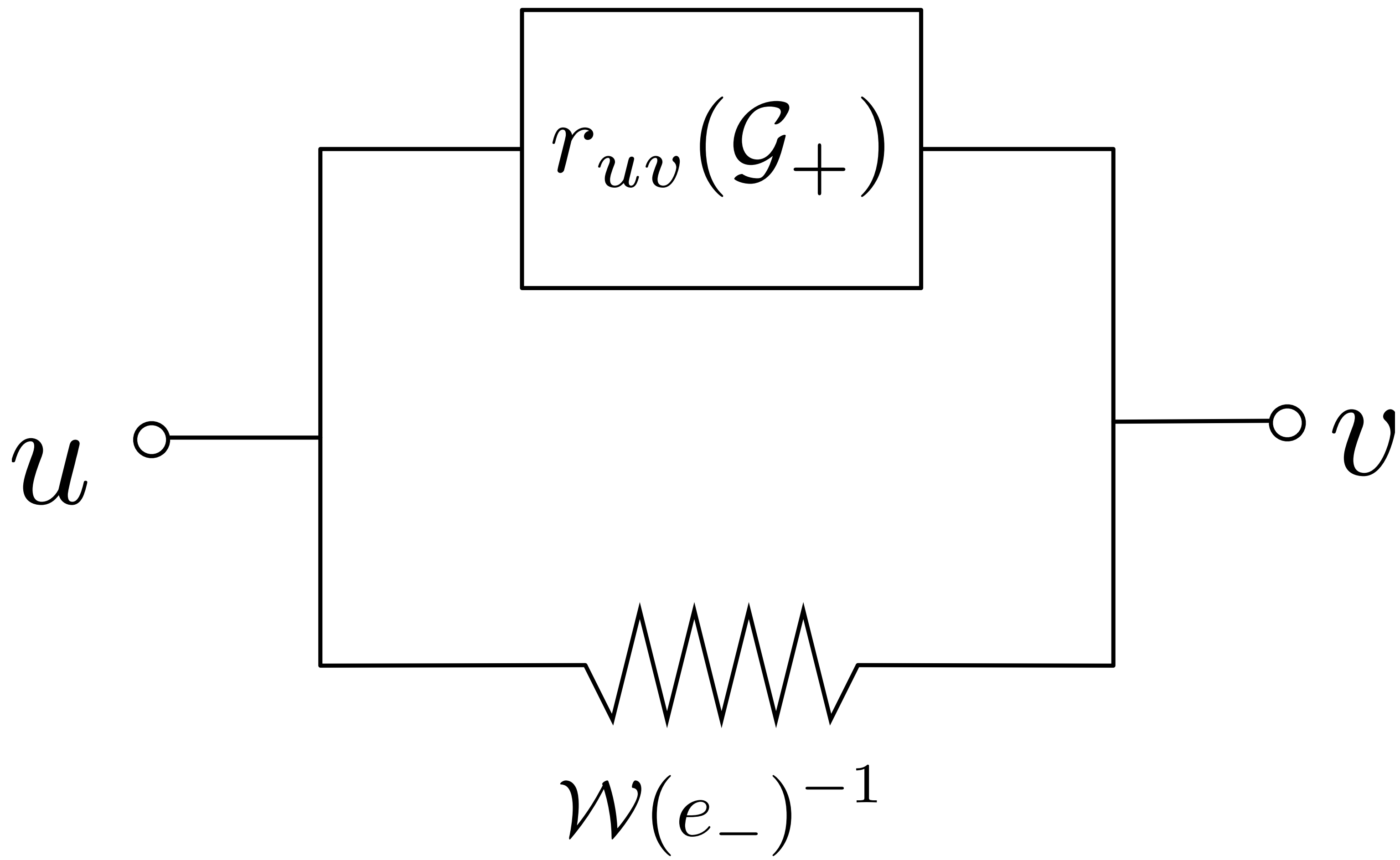
Schur Complement
$$|W_-|^{-1} - \underbrace{E^T_- (E^L_{\mathcal{F}_+})^T (R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R^T_{(\mathcal{F}_+,\mathcal{C}_+)})^{-1} E^L_{\mathcal{F}_+} E_-}_{r_{uv}(\mathcal{G}_+)} \geq 0$$

Spectral Properties of Signed Graphs



$$r_{uv}(\mathcal{G}) = \frac{r_{uv}(\mathcal{G}_+)\mathcal{W}(e_-)^{-1}}{r_{uv}(\mathcal{G}_+) + \mathcal{W}(e_-)^{-1}}$$

Spectral Properties of Signed Graphs



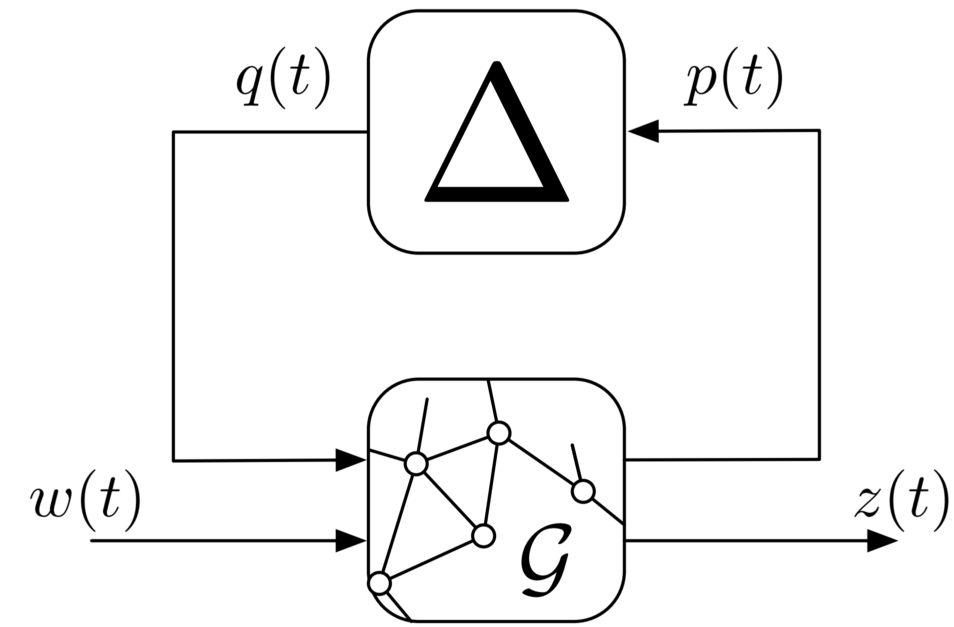
$$r_{uv}(\mathcal{G}) = \frac{r_{uv}(\mathcal{G}_+)\mathcal{W}(e_-)^{-1}}{r_{uv}(\mathcal{G}_+) + \mathcal{W}(e_-)^{-1}}$$

a single negative weight edge can create an *open-circuit*

Uncertain Consensus Networks

a consensus network with an *uncertain* edge weight

$$w = w_0 + \delta$$



$$\overline{S}(\Sigma_{\mathcal{F}}(\mathcal{G}), \Delta) = M_{22} + M_{21}\Delta\left(I - M_{11}\Delta\right)^{-1} M_{12}$$

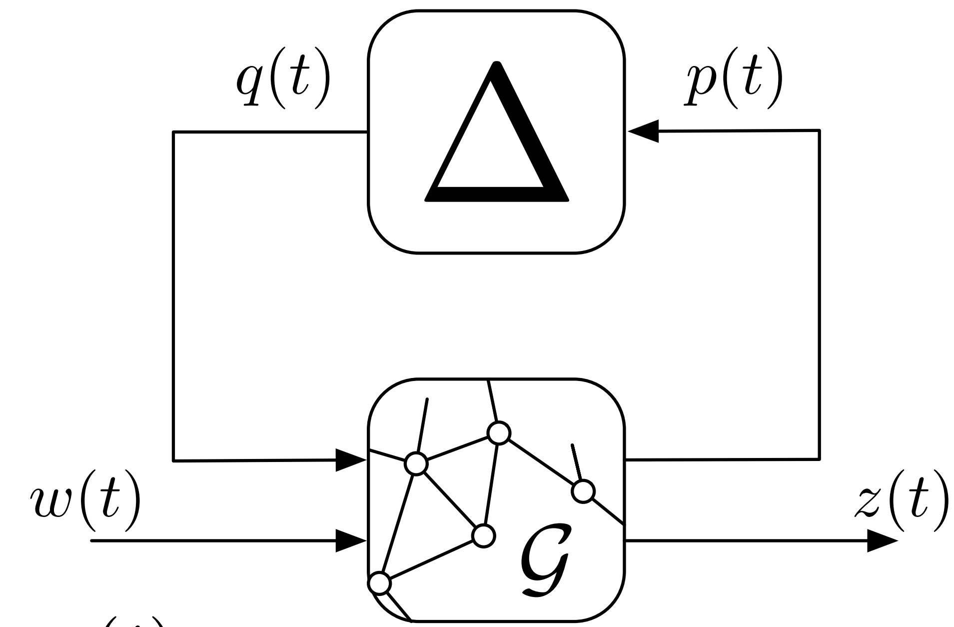
Theorem

- $\|M_{11}(s)\|_\infty = \mathcal{R}_{uv}$
- The uncertain consensus network is stable for any $\|\Delta\|_\infty < \mathcal{R}_{uv}^{-1}$

Uncertain Consensus Networks

a consensus network with an *uncertain* edge weight

$$w = w_0 + \delta$$



$$\Sigma(\mathcal{G},\Delta) \, : \left\{ \begin{array}{rcl} \dot{x}(t) & = & -E(\mathcal{G})(W+\Delta)E(\mathcal{G})^T x(t) + \mathrm{w}(t) \\ z(t) & = & E(\mathcal{G}_o)^T x(t) \end{array} \right.$$

$$\overline{S}(\Sigma_{\mathcal{F}}(\mathcal{G}),\Delta) = M_{22} + M_{21}\Delta\left(I - M_{11}\Delta\right)^{-1} M_{12}$$

Theorem

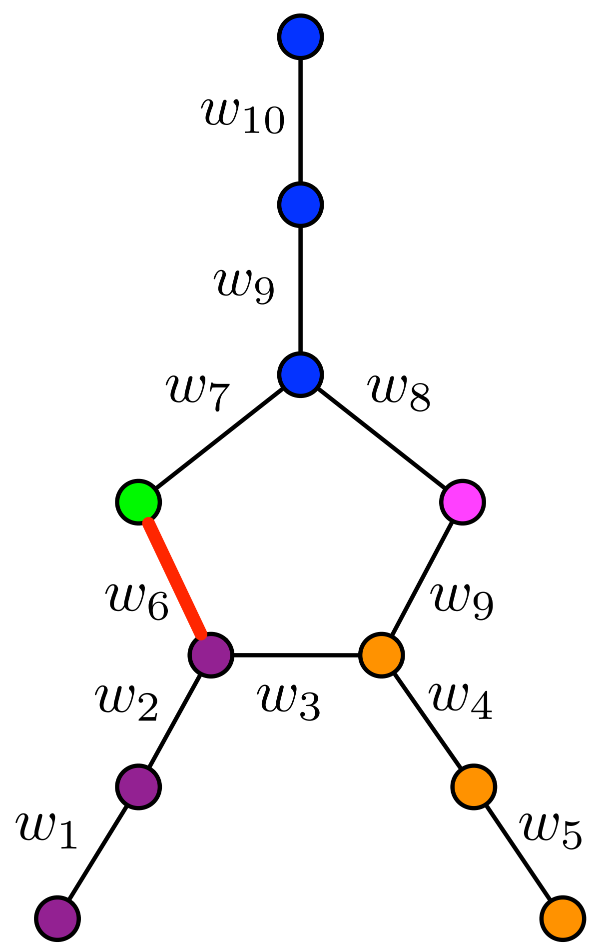
- $\|M_{11}(s)\|_\infty = \mathcal{R}_{uv}$
- The uncertain consensus network is stable for any $\|\Delta\|_\infty < \mathcal{R}_{uv}^{-1}$

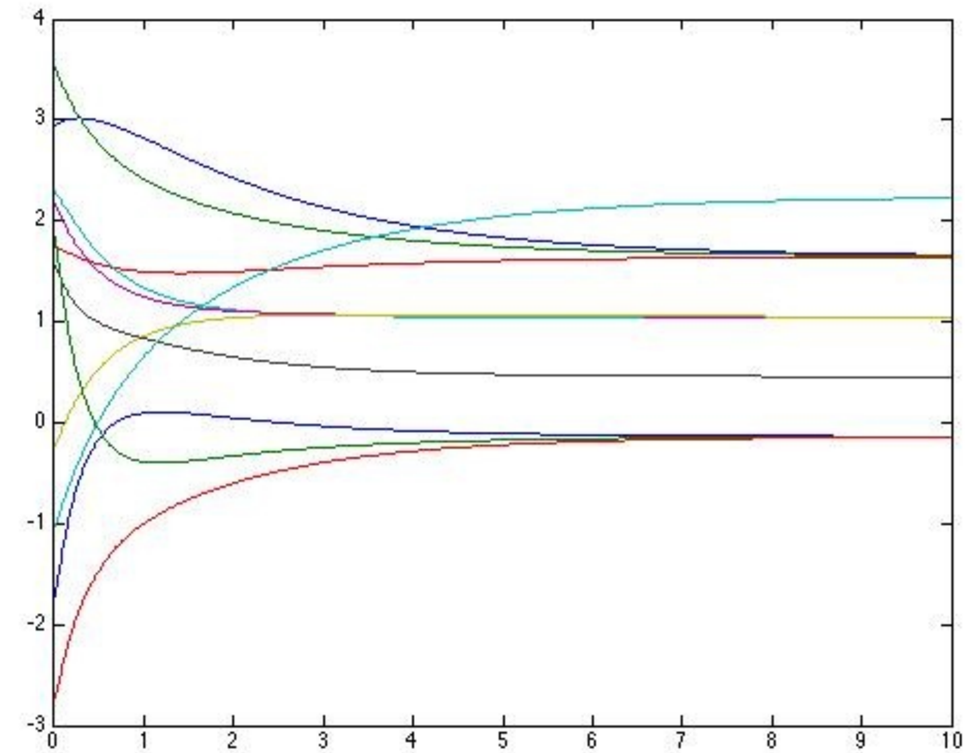
An Illustrative Example

any single edge in the cycle can make the Laplacian indefinite

$$w_6 = -\frac{1}{r_6} = -\frac{1}{4}$$

$L(\mathcal{G})$ has two eigenvalues at the origin

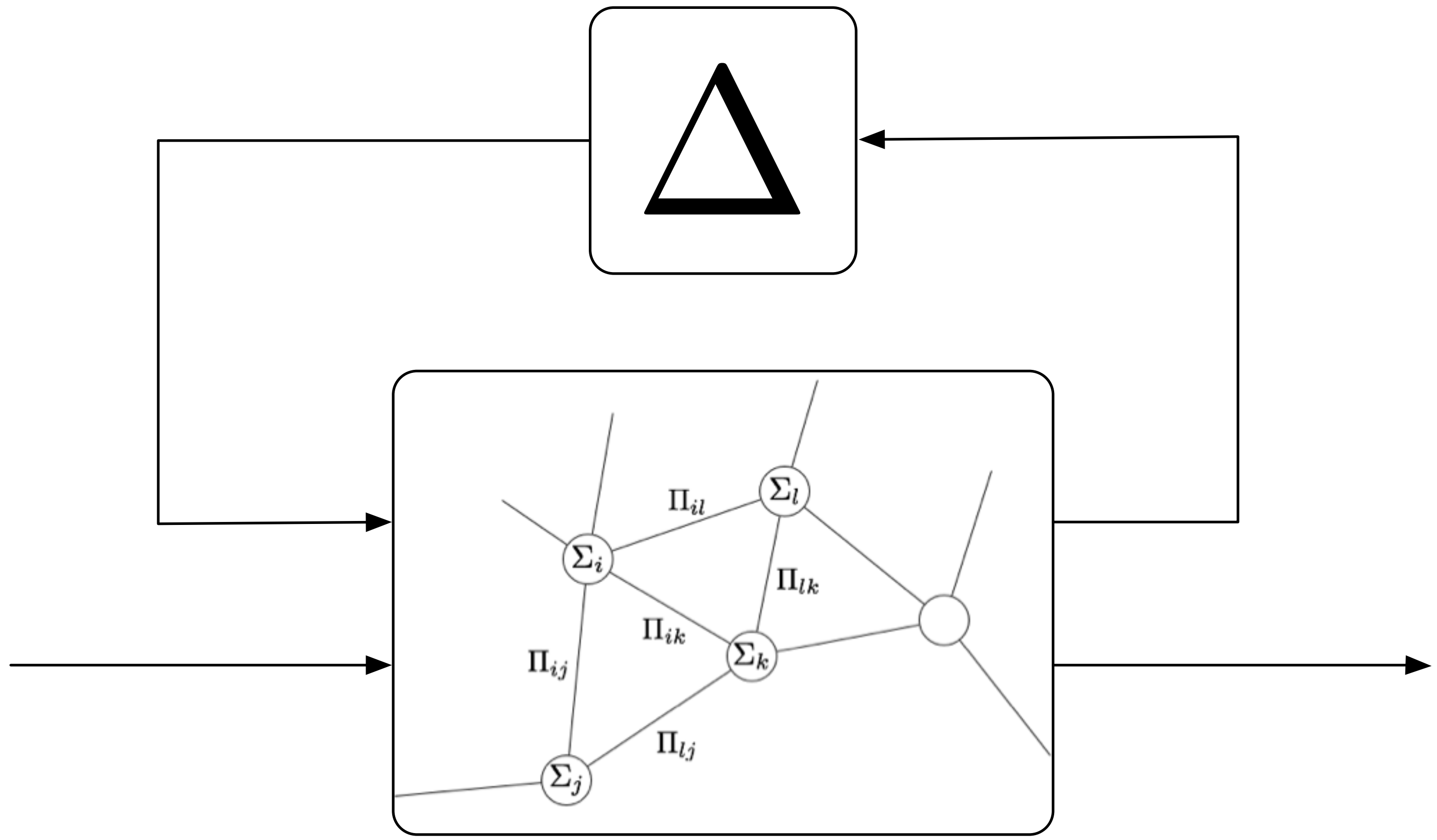




Proposition

Consider a graph with only one cycle and one negative weight edge contained in the cycle. Then the number of clusters equals the number of components in the graph obtained by removing all the edges in the cycle.

Concluding Remarks



- implications for robustness of consensus networks
- explore the robust performance and robust synthesis problems
- how can one *measure* the effective resistance in a multi-agent system?
- *combinatorial uncertainties*

Acknowledgements

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Thank-you!

Questions?

[1] D. Zelazo and M. Bürger, “On the Definiteness of the Weighted Laplacian and its Connection to Effective Resistance,” IEEE CDC, Los Angeles, CA, 2014.

[2] D. Zelazo and M. Bürger, “On the Robustness of Uncertain Consensus Networks,” submitted to IEEE Transactions on Control of Network Systems, 2014 (preprint on arXiv)