DIODES AND THE IMPORTANCE OF NETWORK ORIENTATIONS IN DIFFUSIVELY-COUPLED NETWORKS

IACAS-63

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DEPARTMENT OF AEROSPACE ENGINEERING
TECHNION
Israel Institute of Technology

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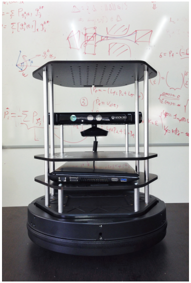
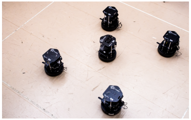
Cooperative Networks and Controls Lab

MULTI-AGENT SYSTEMS AND COOPERATIVE CONTROL





- Multi-agent systems:
  - Highly complex networked systems arising from the large-scale interconnection of many nonlinear dynamical systems
  - An enabling technology for a diverse range of application domains
- Cooperative control:
  - Formation flying
  - Cooperative surveillance
  - Synchronization

SYNCHRONIZATION PROBLEM

Rendezvous

Formation flying

- Objective:
  - Control the dynamics of each agent to generate the same trajectories or reach an agreement on some quantity or state.
  - The control strategies are distributed

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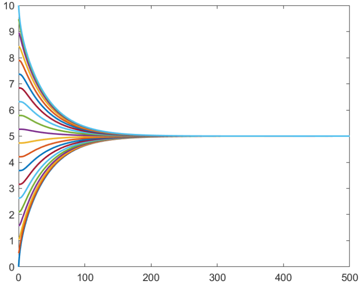
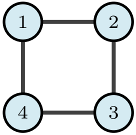
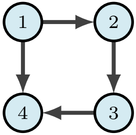


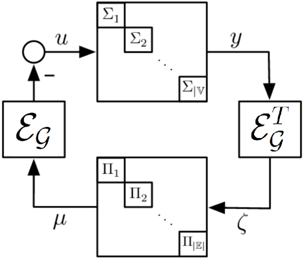


An Undirected graph

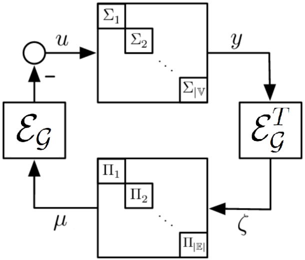


A Directed graph

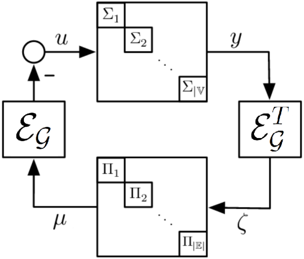
- Nodes: agents
  Edges: information exchange protocol
- Graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$
  - Vertex set $\mathbb{V} = \{1, 2, 3, 4\}$
  - Edge set $\mathbb{E} = \{e_{12}, e_{23}, e_{34}, e_{41}\}$
- Undirected graphs and directed graphs



- Diffusively-coupled network $(\Sigma, \Pi, \mathcal{G})$



- Diffusively-coupled network $(\Sigma, \Pi, \mathcal{G})$
  - Defined on graph $\mathcal{G}$
  - Agents Σ_i
  - Edge controllers Π_e
  - information exchange protocol: Incidence matrix $\mathcal{E}_\mathcal{G}$



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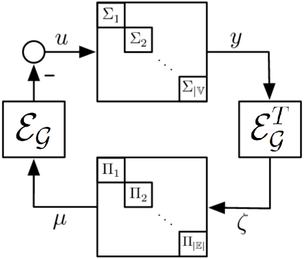
- Difference operators: $\zeta_e = (\mathcal{E}_\mathcal{G}^\top y)_e$



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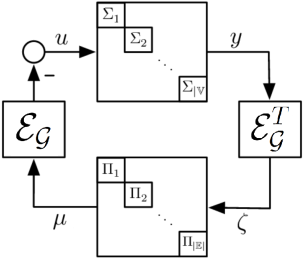

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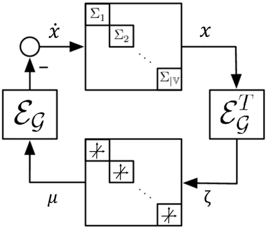
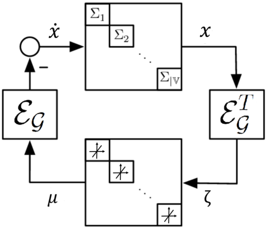
- Divergence operators: $u_i = -(\mathcal{E}_\mathcal{G}\Pi(\mathcal{E}_\mathcal{G}^\top y))_i$



- Classic setup $\rightarrow$ Linear consensus protocol
  - $\mathcal{G}$, $\mathcal{E}_\mathcal{G}$
  - $\Sigma_i : \dot{x}_i(t) = u_i(t)$ (integrators)
  - $\Pi_e : \mu_e(t) = w_e \zeta_e(t)$ ($W = \mathrm{diag}\{(\mathrm{w_e})_{\mathrm{e} \in \mathbb{E}}\}$)
  - closed-loop: $\dot{x} = -\mathcal{E} W \mathcal{E}^\top x$

LINEAR CONSENSUS PROTOCOL (UNDIRECTED GRAPH)



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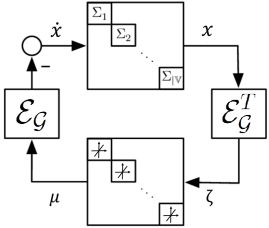
Theorem

Let $\mathcal{G}$ be a connected graph. Then the undirected agreement protocol $\dot{x}(t) = -\mathcal{E} W \mathcal{E}^\top x(t)$ converges to the agreement set.



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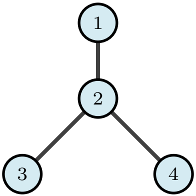
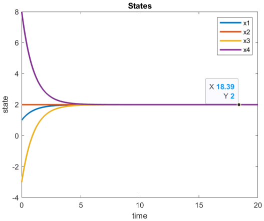
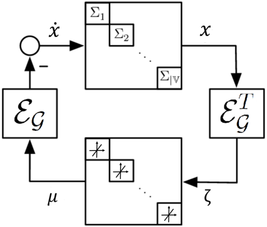
Initial condition $x_0 = [1, 2, -3, 8]^\top$
Average: 2







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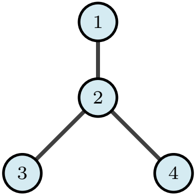

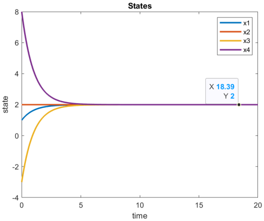

1. Bürger, M., Zelazo, D., and Allgöwer, F. "Duality and network theory in passivity-based cooperative control." Automatica.

2. Sharf, M., and Zelazo, D. "Analysis and synthesis of MIMO multi-agent systems using network optimization." IEEE TAC.

What about consensus over directed graphs?

LINEAR CONSENSUS PROTOCOL (DIRECTED GRAPH)

What about consensus over directed graphs?

Even if the agents are linear systems, it is difficult to analyze the network dynamic system. Problem is much harder when agents are nonlinear systems.

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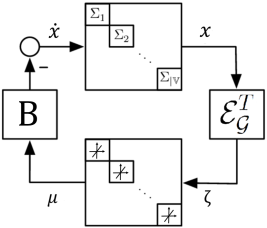
Even if the agents are linear systems, it is difficult to analyze the network dynamic system. Problem is much harder when agents are nonlinear systems.

- main point: the linear consensus protocol for directed graphs is not a diffusively-coupled network.



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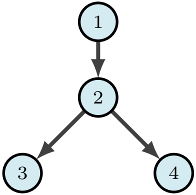

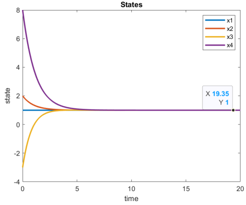

How can we model the directed graphs in diffusively coupled networks?

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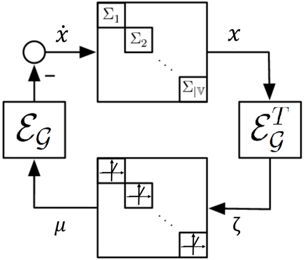

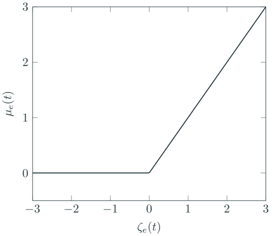

- Single conductance property of a diode

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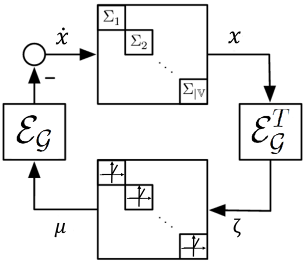

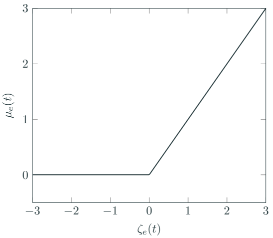

- Single conductance property of a diode
- Diffusive diode network:
  - $\mathcal{G}$, $\mathcal{E}_\mathcal{G}$
  - $\Sigma_i : \dot{x}_i(t) = u_i(t)$ (integrators)
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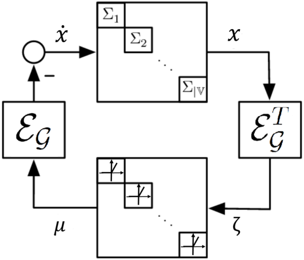

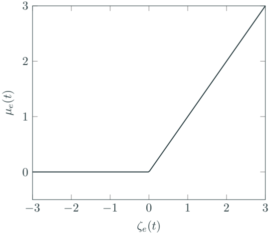

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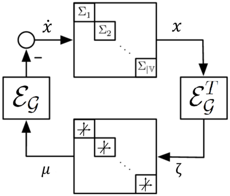
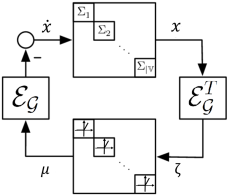
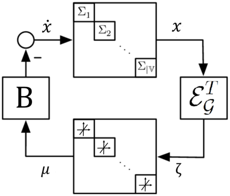
  - Edge controllers: nonlinear



Undirected



Diode

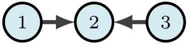


Directed

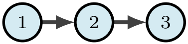
OBSERVATION

Graph $\mathcal{G}$

Graph $\mathcal{G}_1$



Graph $\mathcal{G}_2$

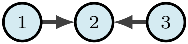


Graph $\mathcal{G}_3$

Bridge the gap?

Graph $\mathcal{G}$

Graph $\mathcal{G}_1$



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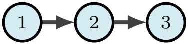

Graph $\mathcal{G}_3$

Bridge the gap?
Can the systems achieve (average) consensus?

- Different orientations, different protocols
- Different initial conditions
- Each agent has a different initial state

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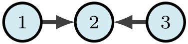
Graph $\mathcal{G}_1$



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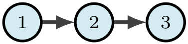

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	G	G1	G2	G3
Diffusively-Coupled	Avg	\	\	\
Directed Protocol	\	Yes	Yes	Yes
Diffusive Diodes	\	No	No	IC-Avg

NOTE*: Avg(average consensus); Yes(consensus); IC(depends on initial conditions)

OBSERVATION

$x_2(t) > x_1(t)$, $x_2(t) > x_3(t)$
$\mu_{21} = \zeta_{21} = x_2(t) - x_1(t) > 0$,
$\mu_{23} = \zeta_{23} = x_2(t) - x_3(t) > 0$

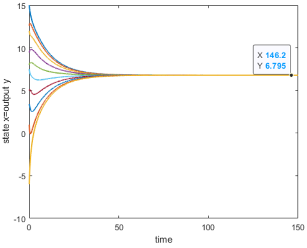
$x_2(t) < x_1(t)$, $x_2(t) > x_3(t)$
$\mu_{21} = \Pi(\zeta_{21}) = \Pi(x_2(t) - x_1(t)) = 0$,
$\mu_{23} = \zeta_{23} = x_2(t) - x_3(t) > 0$

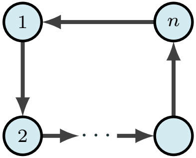
MAIN RESULTS: DIRECTED PATHS

Proposition

Let $\mathcal{G}$ be a directed path graph. If $x_1(0) > x_2(0) > \cdots > x_i(0) > \cdots > x_n(0)$, then the network diode dynamics achieves average consensus.

Directed path graph.

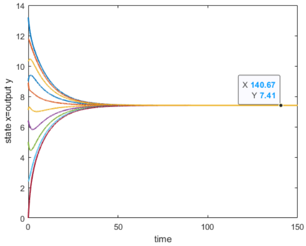


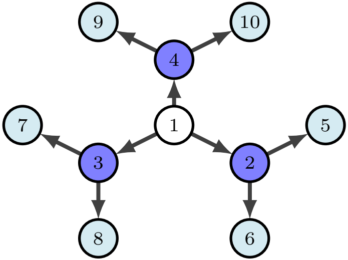


Directed cycle graph.

Proposition

Let $\mathcal{G}$ be a directed cycle graph. If there is at most one edge $e_k = (k, k+1)$ such that $x_k(0) - x_{k-1}(0) < 0$, then the network diode dynamics achieves average consensus.

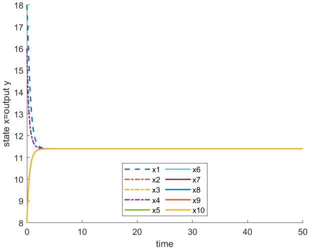


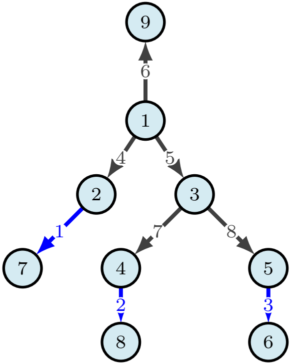


Radially symmetric rooted out-branching.

Proposition

Let $\mathcal{G}$ be a radially symmetric rooted out-branching. If all the edges are active when $t = 0$ and the nodes of the same depth have the same initial conditions, then the network diode dynamics achieves average consensus.

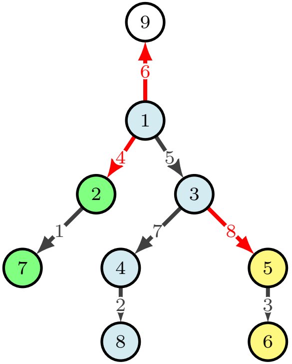




Rooted out-branching.

Proposition

Let $\mathcal{G}$ be a root out-tree and all the edges are active at the initial time. The network cannot reach an agreement only if there exists time T such that some of the edges displayed in black become inactive.



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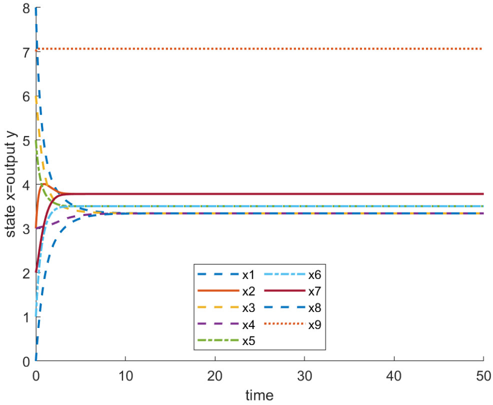

CONCLUSION AND FUTURE DIRECTION

- Conclusion
  - Properties of networked diode dynamics;
  - Sufficient condition on when the diffusive diode networks can achieve average consensus;
  - Sufficient conditions on the graphs (orientations) and the initial conditions of the network that lead to consensus.
  - A necessary condition that graphs containing rooted out-branchings can not achieve consensus.
- Future directions:
  - Generalize to more complicated graphs and (agent) dynamics
  - More general conditions for rooted out-branchings to achieve consensus.
  - Are there situations where an inactive edge becomes an active edge?

Thank-You!

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