

PASSIVITY, MONOTONICITY, AND NETWORK OPTIMIZATION: NEW PERSPECTIVES FOR NETWORK SYSTEMS ANALYSIS

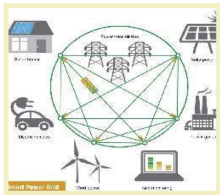
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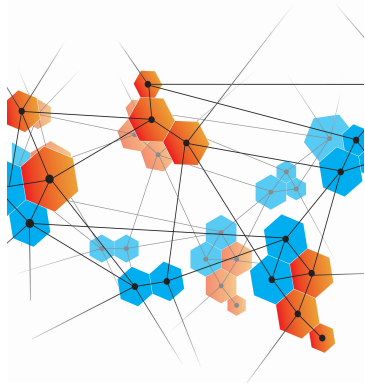
October 12, 2022



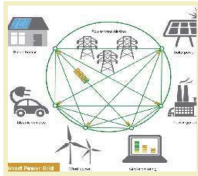
NETWORKED DYNAMIC SYSTEMS



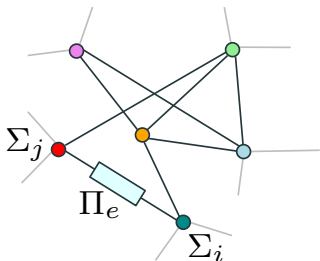
Networks of dynamical systems are one of **the** enabling technologies of the future.



NETWORKED DYNAMIC SYSTEMS



- ▶ how do we **analyze** these systems
- ▶ how do we **design** these systems

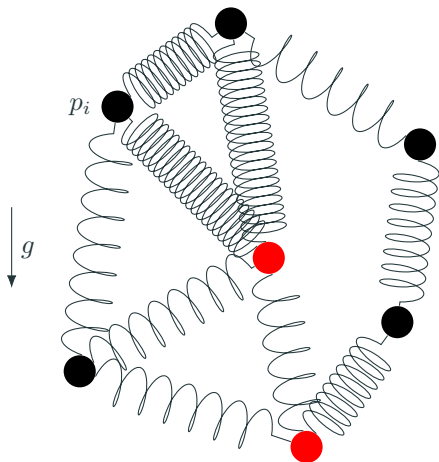


Explore the structure and mechanisms of networked systems to reveal deep connections between properties of dynamical systems and optimization theory.

- ▶ A general model of diffusively coupled networks
- ▶ Characterization of network equilibria via Network Optimization
- ▶ Convergence properties of dynamic networks via passivity theory
- ▶ Passivation, monotonicity, and equilibrium independent passive short systems

A PHYSICS WARM-UP

A MASS-SPRING NETWORK

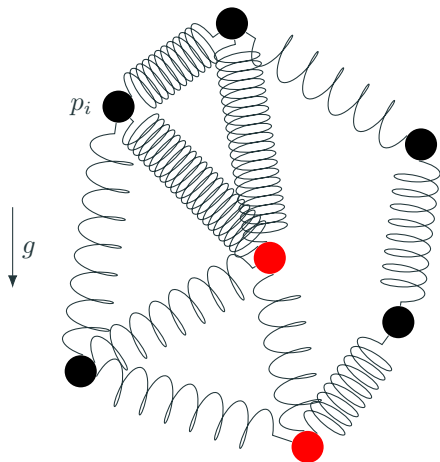


● Free

● Fixed

- ▶ A fixed network of (linear) springs
- ▶ springs connected to masses with position $p_i \in \mathbb{R}^2$ and mass m_i
- ▶ r masses have a fixed position (anchors)

A MASS-SPRING NETWORK



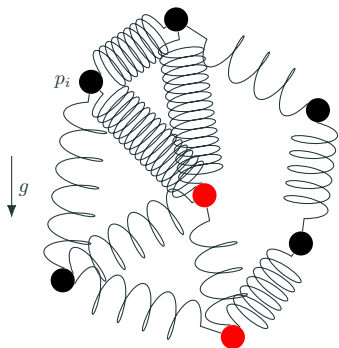
● Free

● Fixed

- ▶ A fixed network of (linear) springs
- ▶ springs connected to masses with position $p_i \in \mathbb{R}^2$ and mass m_i
- ▶ r masses have a fixed position (**anchors**)

Determine the positions of the free masses that minimize the total potential energy of the mass-spring network.

A MASS-SPRING NETWORK



- Potential Energy due to gravity

$$m_i g^T p_i$$

- Elastic Potential Energy of springs

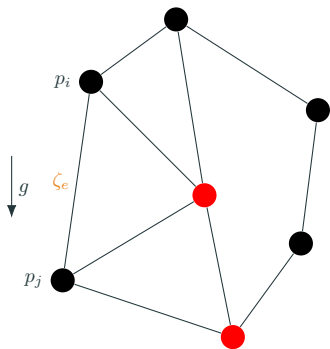
$$\frac{1}{2} k_{ij} (\|p_i - p_j\| - r_{ij})^2$$

an optimization problem (take 1)

$$\min_{p_i} \sum_i m_i g^T p_i + \sum_{i \sim j} \frac{1}{2} k_{ij} (\|p_i - p_j\| - r_{ij})^2$$

$$\text{s.t. } p_i = \mathbf{p}_i^*, i = 1, \dots, r \text{ (fixed nodes)}$$

A MASS-SPRING NETWORK



- ▶ Potential Energy due to gravity (nodes)

$$m_i g^T p_i, \quad i = 1, \dots, n$$

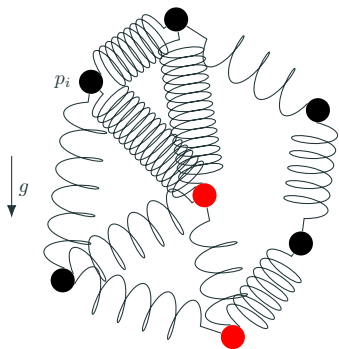
- ▶ Elastic Potential Energy of springs (edges)

$$\frac{1}{2} k_e (\underbrace{\|p_i - p_j\|}_{\zeta_e} - r_e)^2, \quad e = 1, \dots, m$$

an optimization problem (take 2)

$$\begin{aligned} \min_{p_i, \zeta_e} \quad & \sum_{i=1}^r (m_i g^T p_i + \mathbb{I}_{\mathbf{p}_i^*}(p_i)) + \sum_{i=r+1}^n m_i g^T p_i + \sum_e \frac{1}{2} k_e (\|\zeta_e\| - r_e)^2 \\ \text{s.t.} \quad & p_i - p_j = \zeta_e, \quad \forall e = (i, j) \end{aligned}$$

A MASS-SPRING NETWORK



A Convex Program!

an optimization problem (take 2)

$$\min_{p_i, \zeta_e} \sum_i^r (m_i g^T p_i + \mathbb{I}_{\mathbf{P}_i^*}(p_i)) + \sum_{i=r+1}^n m_i g^T p_i + \sum_e \frac{1}{2} k_{ij} (\|\zeta_e\| - r_e)^2$$

s. t. $p_i - p_j = \zeta_e, \forall e = (i, j)$

A MASS-SPRING NETWORK - THE DYNAMICS

► dynamic model for the masses

► springs couple masses together

$$\Sigma_i : \begin{cases} \begin{bmatrix} \dot{p}_i \\ \ddot{p}_i \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_i \\ \dot{p}_i \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_i + m_i g \\ y_i = \begin{cases} \begin{bmatrix} p_i \\ 0 \end{bmatrix}, & i = 1, \dots, r \text{ (anchors)} \\ \begin{bmatrix} p_i \\ \dot{p}_i \end{bmatrix}, & i = r + 1, \dots, n \end{cases} \end{cases} \quad \Pi_e : \begin{cases} u_i = \sum_{i \sim j} k_{ij} (\|p_i - p_j\| - r_{ij}) \frac{p_j - p_i}{\|p_j - p_i\|} + \\ \quad b_{ij} (\dot{p}_j - \dot{p}_i) \\ = \sum_{i \sim j} \kappa_{ij} (y_i - y_j) \end{cases}$$

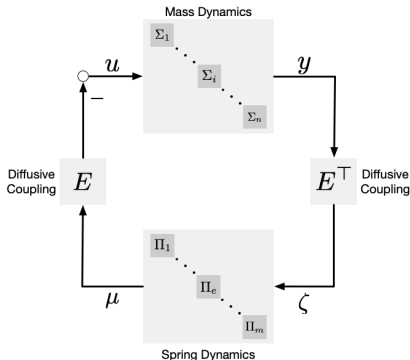
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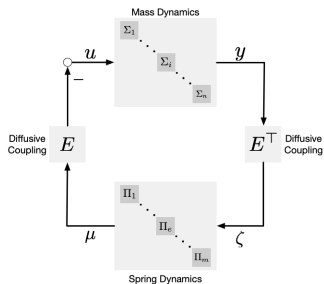


An example of a
diffusively coupled
network!

A MASS-SPRING NETWORK - THE DYNAMICS

► System Equilibrium

$$\begin{cases} 0 &= \dot{p}_i \\ 0 &= m_i g + \sum_{i \sim j} k_{ij} (\|p_i - p_j\| - r_{ij}) \frac{p_j - p_i}{\|p_j - p_i\|} \end{cases}$$

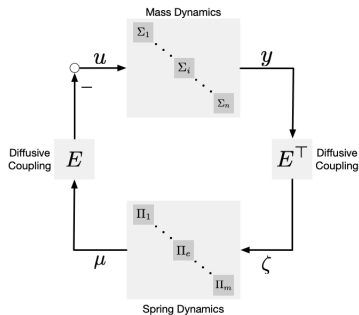


Minimum Total Potential Energy Principle (MTPE)

Equilibrium configurations extremize the total potential energy. **Stable equilibriums** correspond to **minimizers** of the total potential energy.

Dynamics

► Diffusively Coupled Network

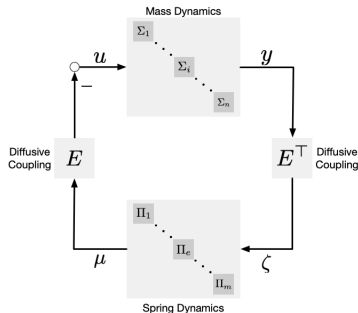


► Dissipativity Theory

$$V(x) = \frac{1}{2} \sum_i \|\dot{p}_i\|^2 + \frac{1}{2} \sum_{i \sim j} k_{ij} \|p_i - p_j\|_2^2$$

Dynamics

► Diffusively Coupled Network



► Dissipativity Theory

$$V(x) = \frac{1}{2} \sum_i \|\dot{p}_i\|^2 + \frac{1}{2} \sum_{i \sim j} k_{ij} \|p_i - p_j\|_2^2$$

Optimization

► Convex Optimization

$$\min_{p_i, \zeta_e} J(p, \zeta)$$

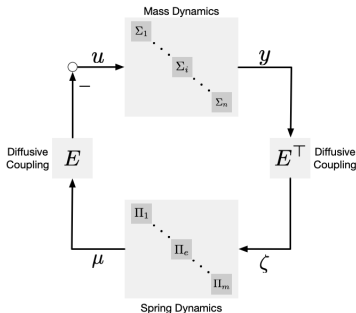
$$\text{s. t. } p_i - p_j = \zeta_e, \forall e = (i, j)$$

► Optimality Conditions

$$0 \in \partial J(p, \zeta)$$

Dynamics

► Diffusively Coupled Network



► Dissipativity Theory

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Optimization

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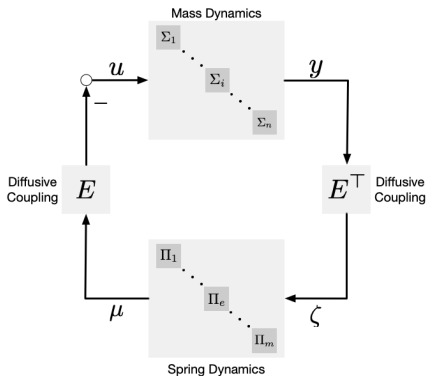
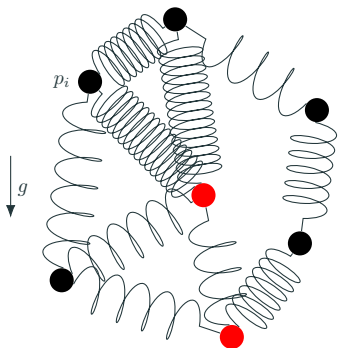
► Optimality Conditions

$$0 \in \partial J(p, \zeta)$$

MTPE Principle ensures that the dynamics of the diffusively coupled network solve the optimization problem, and vice versa.

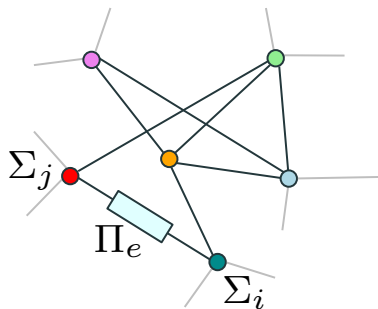
THE QUESTION

- ▶ What class of systems can be “solved” by examining a related optimization problem?
- ▶ What class of optimization problems can be be “solved” by a dynamical system?



DIFFUSIVELY COUPLED NETWORKS

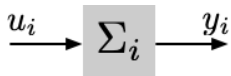
A NETWORK MODEL



A **network system** is comprised of dynamical systems that interact with each other over an information exchange network (a graph).

A NETWORK MODEL

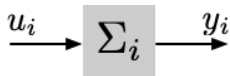
Agent dynamics:



$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i) \end{cases}$$

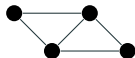
A NETWORK MODEL

Agent dynamics:



$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i) \end{cases}$$

Information Exchange Network:



$$E = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

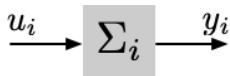
$$\mathcal{G} = (\mathbb{V}, \mathbb{E})$$

$$[E]_{ij} = \begin{cases} \pm 1 & (i, j) \in \mathbb{E} \\ 0 & \text{o.w.} \end{cases}$$

$$E^T \mathbf{1} = 0$$

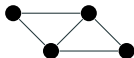
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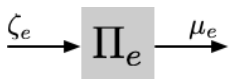
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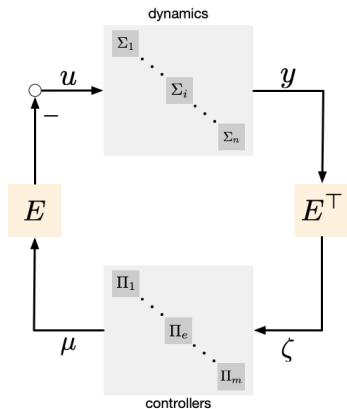
$$[E]_{ij} = \begin{cases} \pm 1 & (i, j) \in \mathbb{E} \\ 0 & \text{o.w.} \end{cases}$$

$$E^T \mathbf{1} = 0$$

Controller dynamics:



$$\Pi_e : \begin{cases} \dot{\eta}_e = \phi_e(\eta_e, \zeta_e) \\ \mu_e = \psi_e(\eta_e, \zeta_e) \end{cases}$$



$(\Sigma, \Pi, \mathcal{G})$

► Consensus Dynamics

$$\dot{x}_i = - \sum_{i \sim j} w_{ij} (x_j - x_i)$$

► Kuramoto Model

$$\dot{\theta}_i = -k \sum_{i \sim j} \sin(\theta_i - \theta_j)$$

► Traffic Dynamics

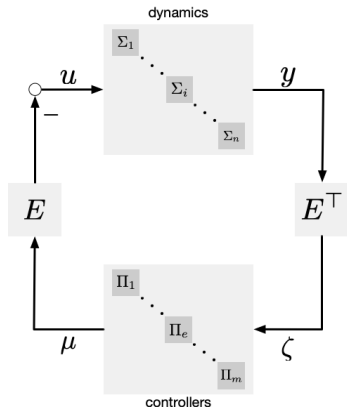
$$\dot{v}_i = \kappa_i \left(V_i^0 - v_i + V_i^1 \sum_{i \sim j} \tanh(p_j - p_i) \right)$$

► Neural Network

$$C\dot{V}_i = f(V_i, h_i) + \sum_{i \sim j} g_{ij} (V_j - V_i)$$

$$\dot{h}_i = g(V_i, h_i)$$

STEADY-STATE NETWORK SOLUTIONS



What properties must the systems Σ_i and Π_e possess such that $(\Sigma, \Pi, \mathcal{G})$ admits and converges to a steady-state solution?

$$u(t) \rightarrow u$$

$$y(t) \rightarrow y$$

$$\zeta(t) \rightarrow \zeta$$

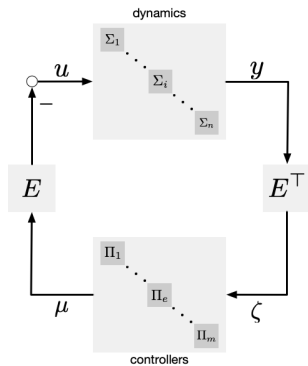
$$\mu(t) \rightarrow \mu$$

- ▶ Consensus: $y = \alpha \mathbf{1}$ ($\zeta = 0$)
- ▶ Formation: $\zeta \neq 0$ constant

All signals converge to a **constant** steady-state

NETWORK OPTIMIZATION MEETS PASSIVITY THEORY

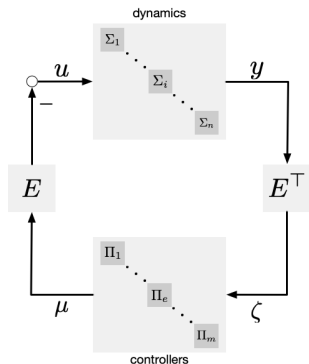
STEADY-STATE INPUT-OUTPUT MAPS



Assumption 1

Each agent Σ_i and controller Π_e admit forced steady-state solutions.

STEADY-STATE INPUT-OUTPUT MAPS



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Each agent Σ_i and controller Π_e admit forced steady-state solutions.

Input-Output Maps

The steady-state *input-output map* $k : \mathcal{U} \rightarrow \mathcal{Y}$ associated with Σ is the set consisting of all steady-state input-output pairs (u, y) of the system.

$$u_i \xrightarrow{y_i \in k_i(u_i)} y_i$$

$$u_i \rightarrow \Sigma_i \rightarrow y_i$$

$$u_i \xleftarrow{u_i \in k_i^{-1}(y_i)} y_i$$

$$\zeta_e \xrightarrow{\mu_e \in \gamma_e(\zeta_e)} \mu_e$$

$$\zeta_e \rightarrow \Pi_e \rightarrow \mu_e$$

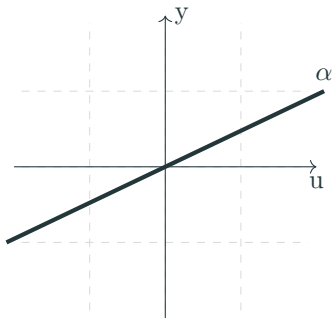
$$\zeta_e \xleftarrow{\zeta_e \in \gamma_e^{-1}(\mu_e)} \mu_e$$

INPUT-OUTPUT RELATIONS

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\Rightarrow k(u) = \{y \mid \underbrace{(-CA^{-1}B + D)}_{\alpha} u\}$$



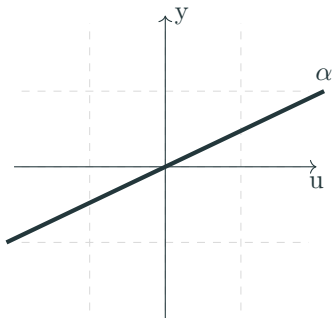
SISO and stable linear system

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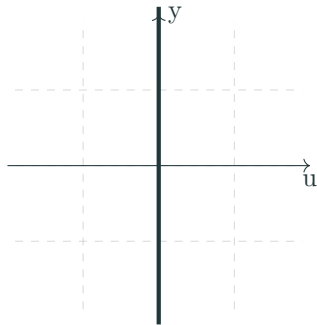


SISO and stable linear system

$$\dot{x} = u$$

$$y = x$$

$$\Rightarrow k = \{(0, y), y \in \mathbb{R}\}$$

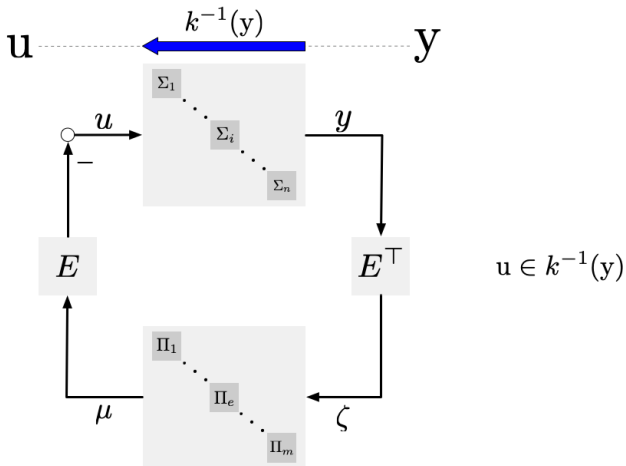


simple integrator

The network interconnection imposes constraints on allowable steady-states

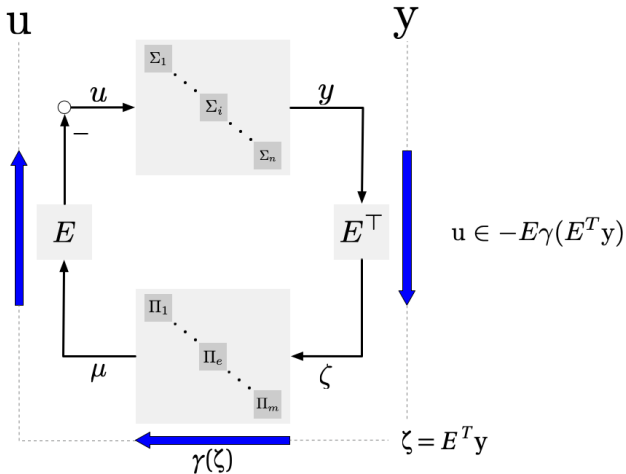
NETWORK CONSISTENCY EQUATIONS

The network interconnection imposes constraints on allowable steady-states



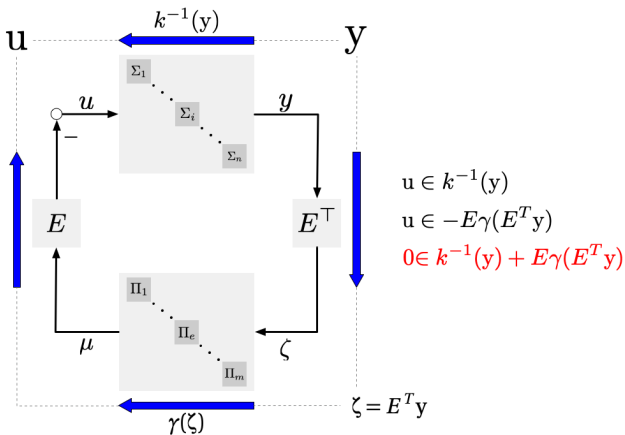
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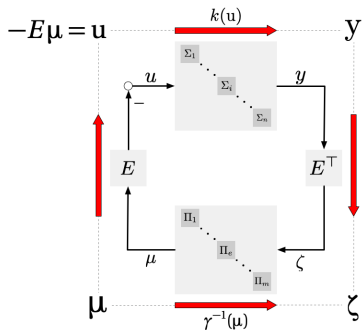
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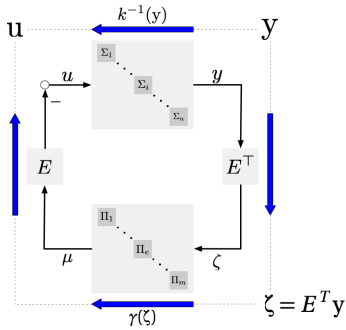


NETWORK CONSISTENCY EQUATIONS

The network interconnection imposes constraints on allowable steady-states



$$\begin{aligned} \zeta &\in \gamma^{-1}(\mu) \\ \zeta &\in E^T k(-E\mu) \\ 0 &\in \gamma^{-1}(\mu) - E^T k(-E\mu) \end{aligned}$$



$$\begin{aligned} u &\in k^{-1}(y) \\ u &\in -E\gamma(E^T y) \\ 0 &\in k^{-1}(y) + E\gamma(E^T y) \end{aligned}$$

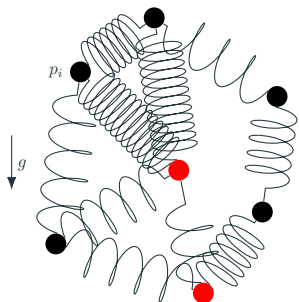
The network system $(\Sigma, \Pi, \mathcal{G})$ admits a steady-state if and only if there exists a solution to the system of non-linear inclusions

$$0 \in k^{-1}(y) + E\gamma(E^T y)$$

$$0 \in \gamma^{-1}(\mu) - E^T k(-E\mu)$$

- ▶ When do solutions exist?
- ▶ How do we find them?

A MASS-SPRING NETWORK

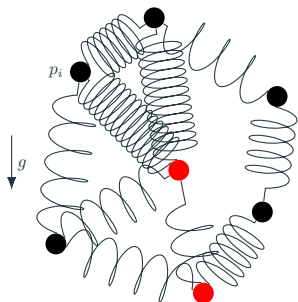


A Convex Program!

Minimum Total Potential Energy Problem

$$\begin{aligned} \min_{p_i, \zeta_e} \quad & \sum_i^r (m_i g^T p_i + \mathbb{I}_{\mathbf{P}_i^*}(p_i)) + \sum_{i=r+1}^n m_i g^T p_i + \sum_e \frac{1}{2} k_{ij} (\|\zeta_e\| - r_e)^2 \\ \text{s.t.} \quad & p_i - p_j = \zeta_e, \forall e = (i, j) \end{aligned}$$

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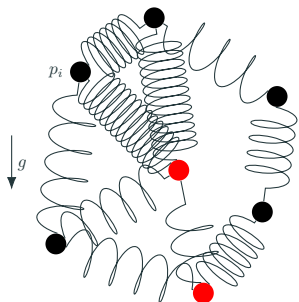


A Convex Program!

Minimum Total Potential Energy Problem

$$\begin{aligned} \min_{p_i, \zeta_e} \quad & \sum_i J_i(p_i) + \sum_e \Gamma_e(\zeta_e) \\ \text{s. t.} \quad & E^T p = \zeta \end{aligned}$$

A MASS-SPRING NETWORK



A Convex Program!

Minimum Total Potential Energy Problem

$$\min_p J(p) + \Gamma(E^T p)$$

First-order Optimality Condition:

$$0 \in \partial J(p) + E \partial \Gamma(E^T p)$$

The network system $(\Sigma, \Pi, \mathcal{G})$ admits a steady-state if and only if there exists a solution to the system of non-linear inclusions

$$0 \in k^{-1}(y) + E\gamma(E^T y)$$

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RECALL First-order Optimality Condition:

$$0 \in \partial J(p) + E\partial\Gamma(E^T p)$$

Network equations are the first-order optimality conditions of a corresponding optimization problem!

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Network equations are the first-order optimality conditions of a corresponding optimization problem!

What is it?

Definition

Let k_i be the input-output relation for system Σ_i . Define the function $K_i : \mathbb{R} \rightarrow \mathbb{R}$ such that $\partial K_i(u_i) = k_i(u_i)$ and $K = \sum_i K_i$. The function K is called the *cost function associated with the system* Σ_i .

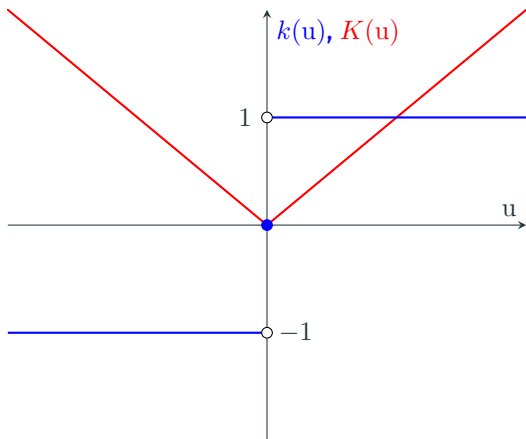
Similarly,

$$\partial K_i^*(y_i) = k_i^{-1}(y_i), K^* = \sum_i K_i^*$$

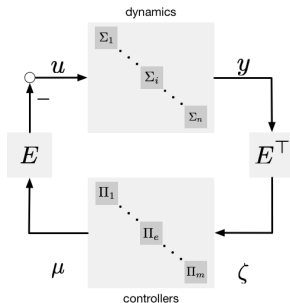
$$\partial \Gamma_e(\zeta_e) = \gamma_e(\zeta_e), \Gamma = \sum_e \Gamma_e$$

$$\partial \Gamma_e^*(\mu_e) = \gamma_e^{-1}(\mu_e) \Gamma^* = \sum_e \Gamma_e^*$$

INTEGRAL FUNCTIONS



— $K(u) = |u|$
— $y = k(u) = \text{sgn}(u)$



Steady-state values u, y, ζ and μ are the solutions of the following pair of optimization problems¹:

$$\begin{aligned} \min_{y, \zeta} \quad & \sum_i K_i^*(y_i) + \sum_e \Gamma_e(\zeta_e) \\ \text{s.t.} \quad & E^T y = \zeta. \end{aligned}$$

First-order Optimality Condition

$$0 \in k^{-1}(y) + E\gamma(E^T y)$$

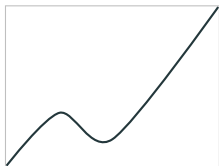
$$\begin{aligned} \min_{u, \mu} \quad & \sum_i K_i(u_i) + \sum_e \Gamma_e^*(\mu_e) \\ \text{s.t.} \quad & u = -E\mu. \end{aligned}$$

First-order Optimality Condition

$$0 \in \gamma^{-1}(\mu) - E^T k(-E\mu)$$

¹[Bürger, Z, Allgower, 2014]

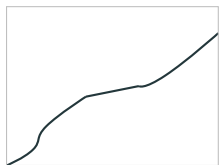
MONOTONE MAPS AND CONVEXITY



Not Monotone



Monotone but not maximal



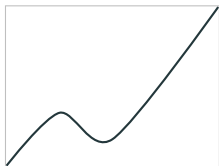
Maximal monotone function



Maximal monotone relation

A relation on \mathbb{R} is **monotone**
if they are non-decreasing curves in \mathbb{R}^2

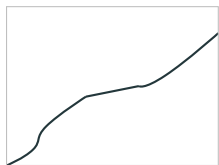
MONOTONE MAPS AND CONVEXITY



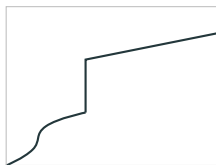
Not Monotone



Monotone but not maximal



Maximal monotone function

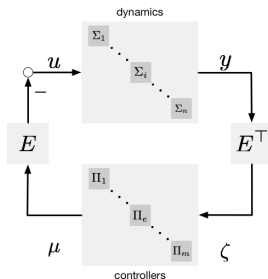


Maximal monotone relation

Theorem

The subdifferentials of convex functions on \mathbb{R} are maximally monotone relations from \mathbb{R} to \mathbb{R} .^a

^a[R. T. Rockafellar, Convex Analysis. Princeton University Press, 1997]



Theorem¹

If the input-output maps k_i and γ_e are **maximally monotone**, then the steady-state values u, y, ζ and μ are the solutions of the following pair of **convex dual optimization problems**:

Optimal Flow Problem (OFP)

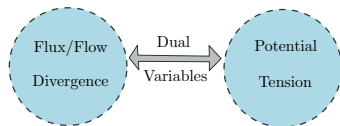
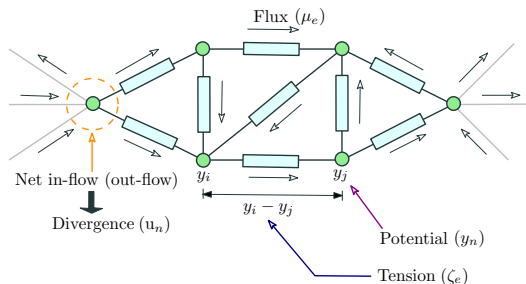
$$\begin{aligned} \min_{y, \zeta} \quad & \sum_i K_i^*(y_i) + \sum_e \Gamma_e(\zeta_e) \\ \text{s.t.} \quad & E^T y = \zeta. \end{aligned}$$

Optimal Potential Problem (OPP)

$$\begin{aligned} \min_{u, \mu} \quad & \sum_i K_i(u_i) + \sum_e \Gamma_e^*(\mu_e) \\ \text{s.t.} \quad & u = -E\mu. \end{aligned}$$

¹[Bürger, Z, Allgower, 2014]

NETWORK OPTIMIZATION



Optimal Flow Problem¹

$$\min_{u, \mu} \quad \sum_{n=1}^{|\mathcal{V}|} C_n^{\text{div}}(u_n) + \sum_{e=1}^{|\mathcal{E}|} C_e^{\text{flux}}(\mu_e)$$

$$s.t. \quad u + E\mu = 0.$$

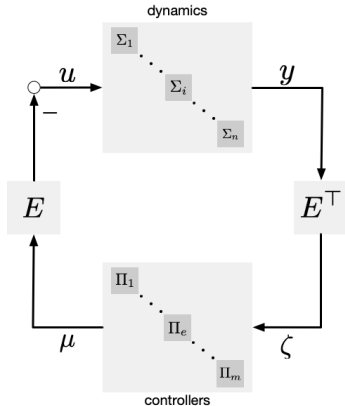
Optimal Potential Problem¹

$$\min_{y, \zeta} \quad \sum_{n=1}^{|\mathcal{V}|} C_n^{\text{pot}}(y_n) + \sum_{e=1}^{|\mathcal{E}|} C_e^{\text{ten}}(\zeta_e)$$

$$s.t. \quad E^T y = \zeta.$$

¹[R. T. Rockafellar, Network Flows and Monotropic Optimizations. John Wiley and Sons, Inc., 1984]

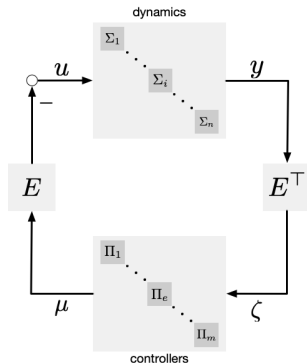
STEADY-STATE NETWORK SOLUTIONS



Diffusively coupled dynamic networks can be associated to static network optimization problems!

Monotone steady-state maps \Leftrightarrow Network Duality

MONOTONE DIFFUSIVE NETWORKS



Assumption 1

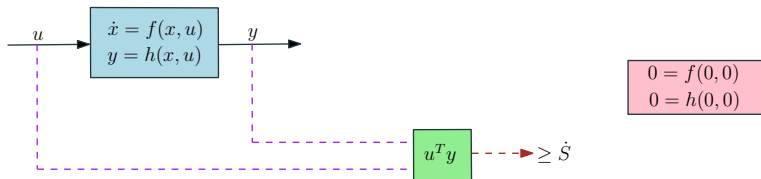
Each agent Σ_i and controller Π_e admit forced steady-state solutions.

Assumption 2

The input-output maps of each agent, k_i , and controller, γ_e , are maximally monotone.

Under what conditions does the network actually *converge* to these steady states?

PASSIVITY FOR DYNAMICAL SYSTEMS



Definition [Khalil 2002]

A system is passive if there exists a C^1 storage function $S(x)$ such that

$$u^T y \geq \dot{S} = \frac{\partial S}{\partial x} f(x, u), \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^p$$

Moreover, it is said to be

- ▶ Input-strictly passive if $\dot{S} \leq u^T y - u^T \phi(u)$ and $u^T \phi(u) > 0, \forall u \neq 0$
- ▶ Output-strictly passive if $\dot{S} \leq u^T y - y^T \rho(y)$ and $y^T \rho(y) > 0, \forall y \neq 0$

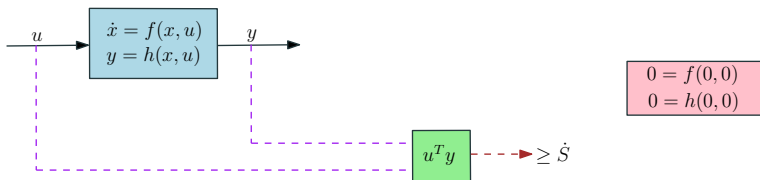
PASSIVITY FOR DYNAMICAL SYSTEMS

Definition

Let Σ be a SISO system with a constant input-output steady-state pair (u, y) . The system is said to be *input-output* (ρ, ν) -passive wrt (u, y) if there exists a storage function $S(x)$ and numbers $\rho, \nu \in \mathbb{R}$, such that $\rho\nu < 1/4$ and

$$\dot{S} = \frac{\partial S}{\partial x} f(x, u) \leq (y - y)(u - u) - \rho(y - y)^2 - \nu(u - u)^2,$$

for any trajectory u, y .



Definition

Let Σ be a SISO system with a constant input-output steady-state pair (u, y) . The system is said to be *input-output* (ρ, ν) -passive wrt (u, y) if there exists a storage function $S(x)$ and numbers $\rho, \nu \in \mathbb{R}$, such that $\rho\nu < 1/4$ and

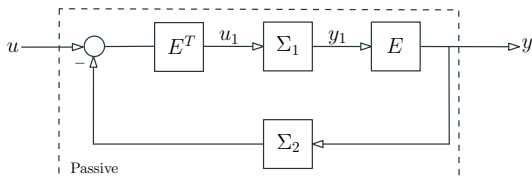
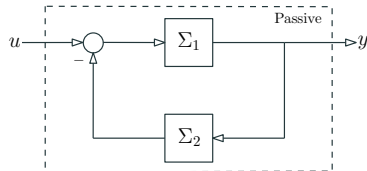
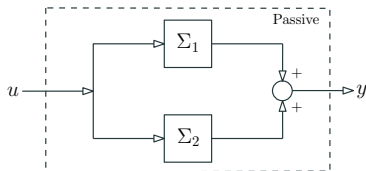
$$\dot{S} = \frac{\partial S}{\partial x} f(x, u) \leq (y - y)(u - u) - \rho(y - y)^2 - \nu(u - u)^2,$$

for any trajectory u, y .

- ▶ $\rho = \nu = 0 \Rightarrow$ **passivity**
- ▶ $\rho, \nu > 0 \Rightarrow$ **strict input/output passivity**
- ▶ $\rho, \nu < 0 \Rightarrow$ **passive short**

INTERCONNECTION OF PASSIVE SYSTEMS

- ▶ Parallel Interconnection
- ▶ Negative Feedback Interconnection
- ▶ Symmetric Interconnection



Theorem¹

Consider the network system $(\Sigma, \Pi, \mathcal{G})$ comprised of SISO agents and controllers. Suppose that there are vectors u_i, y_i, ζ_e and μ_e such that

- i) the systems Σ_i are output strictly-passive with respect to u_i and y_i ;
- ii) the systems Π_e are passive with respect to ζ_e and μ_e ;
- iii) the vectors u, y, ζ and μ satisfy $u = -\mathcal{E}\mu$ and $\zeta = \mathcal{E}^T y$.

Then the output vector $y(t)$ converges to y as $t \rightarrow \infty$.

¹[Arcak, 2007], [Bürger, Z, Allgower, 2014]

Theorem¹

Consider the network system $(\Sigma, \Pi, \mathcal{G})$ comprised of SISO agents and controllers. Suppose that there are vectors u_i, y_i, ζ_e and μ_e such that

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Then the output vector $y(t)$ converges to y as $t \rightarrow \infty$.

- requires passivity w.r.t. to specific equilibrium configuration

¹[Arcak, 2007], [Bürger, Z, Allgower, 2014]

EQUILIBRIUM-INDEPENDENT PASSIVITY (EIP)

EIP¹

A SISO system $\Sigma : u \mapsto y$ is said to be *equilibrium-independent input-output* (ρ, ν) -passive if it is input-output (ρ, ν) -passive with respect to **any** equilibrium $(u, k(u))$.

EIP systems $(\rho, \nu \geq 0)$ have monotone steady-state input-output maps!

$$\dot{S} \leq (y - y)^T (u - u) \implies k \text{ monotonically increasing function}$$

¹[G.H. Hines et al., 2011], [M. Sharf, A. Jain, Z., 2020]

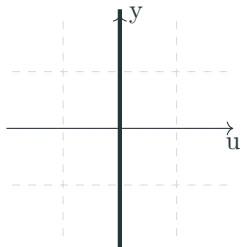
EQUILIBRIUM-INDEPENDENT PASSIVITY (EIP)

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EIP systems $(\rho, \nu \geq 0)$ have monotone steady-state input-output maps!

$$\dot{S} \leq (y - y)^T (u - u) \implies k \text{ monotonically increasing function}$$



$$\dot{x}(t) = u(t), y(t) = x(t)$$

- ▶ Passive with respect to $\mathcal{U} = \{0\}$ and any output value $y \in \mathbb{R}$ with storage function $S(x) = \frac{1}{2}(x - y)^2$.
- ▶ The equilibrium input-output map $k = \{(0, y) : y \in \mathbb{R}\}$ is not a single valued function and hence the integrator is **NOT** EIP.

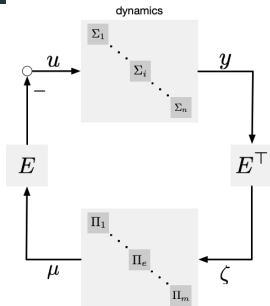
¹[G.H. Hines et al., 2011], [M. Sharf, A. Jain, Z., 2020]

MEIP¹

A dynamical SISO system Σ is *maximal equilibrium independent passive* if the following conditions hold:

- ▶ The system Σ is passive with respect to any steady-state $(u, y) \in k$.
- ▶ The relation k is maximally monotone.

¹[M. Bürger et al., 2014]



Assumption 1

Each agent Σ_i and controller Π_e admit forced steady-state solutions.

Assumption 2

The agent dynamics Σ_i are output-strictly MEIP and the controllers are MEIP.

Theorem¹

Assume Assumptions 1 and 2 hold. Then the signals $u(t), y(t), \zeta(t), \mu(t)$ converge to the solutions of the following pair of convex dual optimization problems:

Optimal Flow Problem (OFP)

$$\begin{aligned} \min_{y, \zeta} \quad & \sum_i K_i^*(y_i) + \sum_e \Gamma_e(\zeta_e) \\ \text{s.t.} \quad & E^T y = \zeta. \end{aligned}$$

Optimal Potential Problem (OPP)

$$\begin{aligned} \min_{u, \mu} \quad & \sum_i K_i(u_i) + \sum_e \Gamma_e^*(\mu_e) \\ \text{s.t.} \quad & u = -E\mu. \end{aligned}$$

¹[Bürger, Z, Allgower, 2014]

NEW PERSPECTIVES ON PASSIVATION

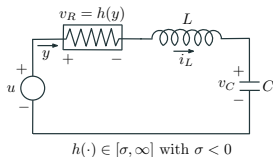
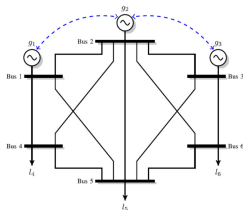
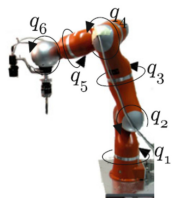


What else can we say about MEIP systems?

PASSIVITY-SHORT SYSTEMS

In practice, systems are usually passivity-short (or non-passive)!

- ▶ Generator (always generates energy) [R. Harvey , 2016]
- ▶ Oscillating systems with small or nonexistent damping [R. Harvey, 2017]
- ▶ Dynamics of robot system from torque to position [D. Babu, 2018]
- ▶ Power-system network (turbine-governor dynamics) [S. Trip, 2018]
- ▶ Electrical circuits with nonlinear components
- ▶ More general as include non-minimum phase systems and systems with relative degree greater than 1 [Z. Qu, 2014]



Passive short systems can destroy
the developed network optimization framework!

System Type	Relations	Integral Function
MEIP	k, k^{-1} max. monotone	$K(u), K^*(y)$ are convex
Input PS	k is not monotone	$K(u)$ is non-convex
Output PS	k^{-1} is not monotone	$K^*(y)$ is non-convex
Input-output PS	k, k^{-1} are not monotone	May not exist

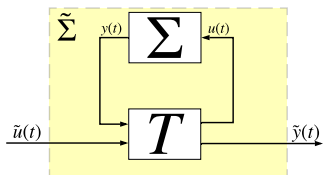
Optimal Flow Problem (OFF)

$$\begin{aligned} \min_{y, \zeta} \quad & \sum_i K_i^*(y_i) + \sum_e \Gamma_e(\zeta_e) \\ \text{s.t.} \quad & E^T y = \zeta. \end{aligned}$$

Optimal Potential Problem (OPP)

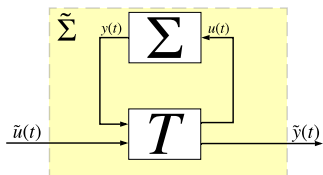
$$\begin{aligned} \min_{u, \mu} \quad & \sum_i K_i(u_i) + \sum_e \Gamma_e^*(\mu_e) \\ \text{s.t.} \quad & u = -E\mu. \end{aligned}$$

FEEDBACK PASSIVATION

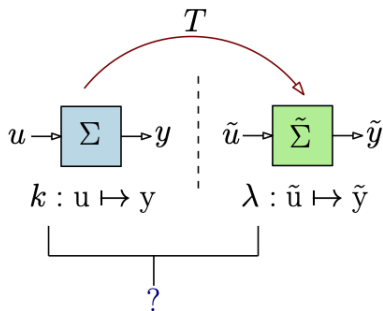


For a passive-short system $\Sigma : u \mapsto y$, we aim to find a map T such that the closed-loop system $\tilde{\Sigma} : \tilde{u} \mapsto \tilde{y}$ is passive. This is known as **feedback passivation**.

FEEDBACK PASSIVATION



For a passive-short system $\Sigma : u \mapsto y$, we aim to find a map T such that the closed-loop system $\tilde{\Sigma} : \tilde{u} \mapsto \tilde{y}$ is passive. This is known as **feedback passivation**.



how does feedback passivation affect the steady-state input/output maps?

an example

$$\dot{x} = -x + \sqrt[3]{x} + u$$

$$y = \sqrt[3]{x}$$

$$\bar{u} = k^{-1}(\bar{y}) = \bar{y}^3 - \bar{y}$$

not a monotone input-output relation!

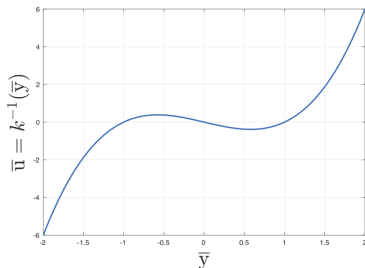
System is output passivity-short

$$S(x) = \frac{3}{4}x^{4/3} - \bar{y}x + \frac{1}{4}\bar{y}$$

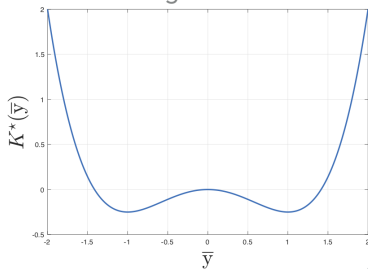
$$\dot{S} \leq (y - \bar{y})(u - \bar{u}) + (y - \bar{y})^2$$

(passivity index $\rho = -1$)

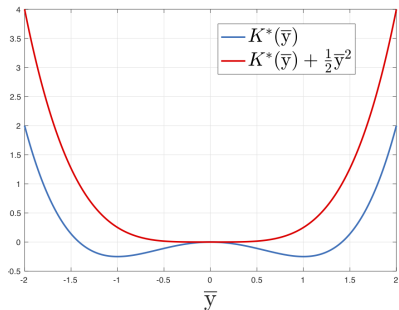
equilibrium input-output map



integral function



AN EXAMPLE



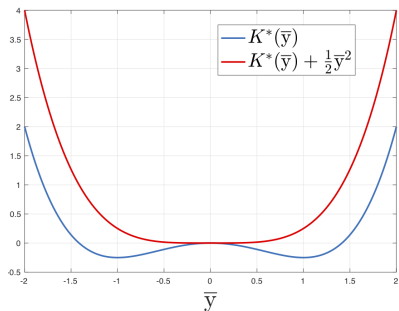
what is the system interpretation of a “convexified” integral function?

$$K^*(\bar{y}) = \frac{1}{4}\bar{y}^4 - \frac{1}{2}\bar{y}^2$$

$$\tilde{K}^*(\bar{y}) = K^*(\bar{y}) + \frac{1}{2}\bar{y}^2$$

(Tikhonov regularization term)

AN EXAMPLE



$$\partial \tilde{K}^*(\bar{y}) = \partial K^*(\bar{y}) + \bar{y}$$

$$\begin{aligned}\tilde{k}^{-1}(\bar{y}) &= k^{-1}(\bar{y}) + \bar{y} \\ &= \bar{y}^3 - \bar{y} + \bar{y} = \bar{y}^3\end{aligned}$$

a monotone function!

what is the system interpretation of a “convexified” integral function?

$$K^*(\bar{y}) = \frac{1}{4}\bar{y}^4 - \frac{1}{2}\bar{y}^2$$

$$\tilde{K}^*(\bar{y}) = K^*(\bar{y}) + \frac{1}{2}\bar{y}^2$$

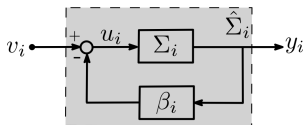
(Tikhonov regularization term)

what system yields this steady-state I/O map?

$$\dot{x} = -x + \sqrt[3]{x} - \underbrace{\sqrt[3]{y}}_u + v = -x + v$$

$$y = \sqrt[3]{x}$$

AN EXAMPLE



regularization is realized by output feedback!

$$u = v - y$$

$$\Rightarrow \dot{x} = -x + v$$

$$\Rightarrow \bar{v} = \tilde{k}^{-1}(\bar{y}) = \bar{y}^3$$

(maximally monotone!)

Theorem¹

Consider the passive-short SISO dynamical system $\Sigma : u \mapsto y$ with I/O steady-state map k and output passivity index $\rho < 0$. Then for any $\beta > |\rho|$, the feedback

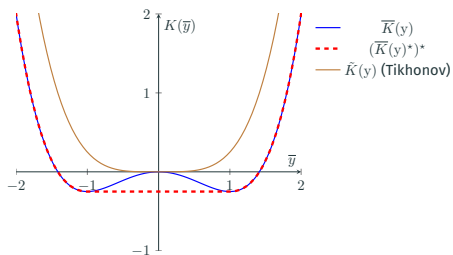
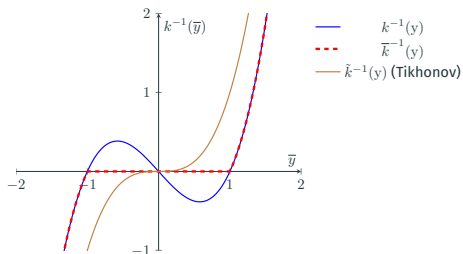
$$u = v - \beta y$$

renders the system $\tilde{\Sigma} : v \mapsto y$ output-strictly maximally monotone EIP with steady-state input map \tilde{k} satisfying

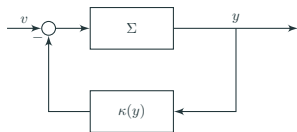
$$\tilde{k}^{-1}(\bar{y}) = k^{-1}(\bar{y}) + \beta \bar{y}.$$

¹[Jain, Sharf, Z, 2018]

MONOTONIZATION AND CONVEXIFICATION



A “better” convexification leads to different feedback passivation!



the feedback

$$\kappa(y) = \begin{cases} 0, & |x| = |y^3| > 1 \\ y^3 - y, & |x| = |y^3| \leq 1 \end{cases}$$

the closed-loop

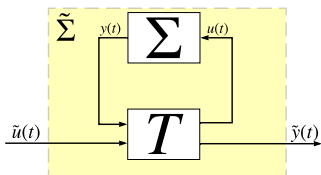
$$\dot{x} = \begin{cases} -x + \sqrt[3]{x} + v, & |x| > 1 \\ v, & |x| \leq 1 \end{cases}$$

$$y = \sqrt[3]{x}.$$

Is it possible to find a **linear transformation** $T : (u, y) \mapsto (\tilde{u}, \tilde{y})$ for a non-monotone I/O map $k : u \mapsto y$ such that $\tilde{k} : \tilde{u} \mapsto \tilde{y}$ is monotone?

MONOTONIZATION OF I/O RELATIONS

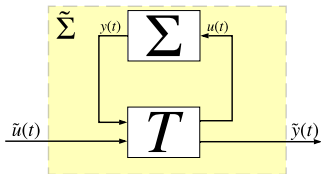
Is it possible to find a **linear transformation** $T : (u, y) \mapsto (\tilde{u}, \tilde{y})$ for a non-monotone I/O map $k : u \mapsto y$ such that $\tilde{k} : \tilde{u} \mapsto \tilde{y}$ is monotone?



For a passive-short system $\Sigma : u \mapsto y$, we aim to find a map T such that the closed-loop system $\tilde{\Sigma} : \tilde{u} \mapsto \tilde{y}$ is passive. This is known as **feedback passivation**.

MONOTONIZATION OF I/O RELATIONS

Is it possible to find a **linear transformation** $T : (u, y) \mapsto (\tilde{u}, \tilde{y})$ for a non-monotone I/O map $k : u \mapsto y$ such that $\tilde{k} : \tilde{u} \mapsto \tilde{y}$ is monotone?



For a passive-short system $\Sigma : u \mapsto y$, we aim to find a map T such that the closed-loop system $\tilde{\Sigma} : \tilde{u} \mapsto \tilde{y}$ is passive. This is known as **feedback passivation**.

Are these T maps the same?

A GEOMETRIC APPROACH

For an EI-IOP(ρ, ν) system, for any two points $(u_1, y_1), (u_2, y_2) \in k$, the following inequality holds:

$$0 \leq -\rho(y_1 - y_2)^2 + (u_1 - u_2)(y_1 - y_2) - \nu(u_1 - u_2)^2.$$

Projective Quadratic Inequalities and EI-IOP

A *projective quadratic inequality (PQI)* is an inequality with variables $\xi, \chi \in \mathbb{R}$ of the form

$$0 \leq a\xi^2 + b\xi\chi + c\chi^2 = F(\xi, \chi),$$

for some numbers a, b, c , not all zero. The inequality is called *non-trivial* if $b^2 - 4ac > 0$. The associated solution set \mathcal{A} of the PQI is the set of all points $(\xi, \chi) \in \mathbb{R}^2$ satisfying the inequality.

- ▶ passivity inequality is a PQI: $\xi = u_1 - u_2, \chi = y_1 - y_2$
- ▶ monotonicity is a PQI: $0 \leq (u_1 - u_2)(y_1 - y_2)$ with $a = c = 0$ and $b = 1$

$$0 \leq a\xi^2 + b\xi\chi + c\chi^2 = F(\xi, \chi)$$

A Recap:

- ▶ $F(u_1 - u_2, y_1 - y_2) \geq 0$ is a PQI for a EI-IOP(ρ, ν) system

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- ▶ $F(u_1 - u_2, y_1 - y_2) \geq 0$ is a PQI for a EI-IOP(ρ, ν) system
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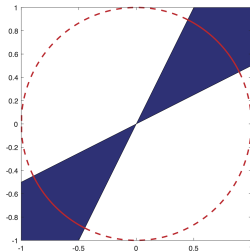
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Study the effect of the map T on the solution sets of the PQIs, $T(\mathcal{A})$

A GEOMETRIC APPROACH

The solution set of any non-trivial PQI is a symmetric double-cone. Moreover, any symmetric double-cone is the solution set of some non-trivial PQI.



Theorem¹

Let $(\xi_1, \chi_1), (\xi_2, \chi_2)$ be non-colinear solutions of $a_1\xi^2 + \xi\chi + c_1\chi^2 = 0$, and $(\tilde{\xi}_1, \tilde{\chi}_1), (\tilde{\xi}_2, \tilde{\chi}_2)$ be non-colinear solutions of $a_2\xi^2 + \xi\chi + c_2\chi^2 = 0$.

Define

$$T_1 = \begin{bmatrix} \tilde{\xi}_1 & \tilde{\xi}_2 \\ \tilde{\chi}_1 & \tilde{\chi}_2 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ \chi_1 & \chi_2 \end{bmatrix}^{-1}, T_2 = \begin{bmatrix} \tilde{\xi}_1 & -\tilde{\xi}_2 \\ \tilde{\chi}_1 & -\tilde{\chi}_2 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ \chi_1 & \chi_2 \end{bmatrix}^{-1}.$$

Then one of T_1, T_2 transforms the PQI $a_1\xi^2 + \xi\chi + c_1\chi^2 \geq 0$ to the PQI $\tau a_2\xi^2 + \tau\xi\chi + \tau c_2\chi^2 \geq 0$ for some $\tau > 0$.

¹[Sharf, Jain, Z, 2021]

EXAMPLE

Consider the system

$$\Sigma : \dot{x} = -\sqrt[3]{x} + .5x + .5u, \quad y = .5x - .5u$$

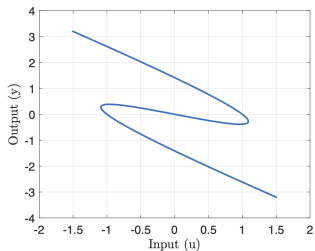
Using $S(x) = \frac{1}{6}(x - x)^2$ we have

$$\dot{S}(x) \leq (u-u)(y-y) + \frac{1}{3}(u-u)^2 + \frac{2}{3}(y-y)^2$$

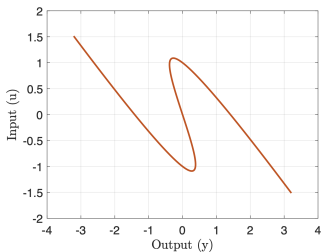
System is EI-IOP(ρ, ν) with

$$\rho = -2/3, \quad \nu = -1/3$$

Passive-short system with
non-monotone input-output
relations (not even a function!)



(a) k .



(b) k^{-1}

EXAMPLE

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System is EI-IOP(ρ, ν) with $\rho = -2/3, \nu = -1/3$

Corresponding PQI:

$$0 \leq \frac{1}{3}\xi^2 + \xi\chi + \frac{2}{3}\chi^2$$

Find a linear map T that monotonizes the input-output relations, i.e., leads to the PQI

$$\tilde{\xi}\tilde{\chi} \geq 0$$

EXAMPLE

non-colinear solutions to
PQI

$$\tilde{\xi}\tilde{\chi} = 0$$

$$\begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\chi}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \tilde{\xi}_2 \\ \tilde{\chi}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The map

$$T_1 = \begin{bmatrix} \tilde{\xi}_1 & \tilde{\xi}_2 \\ \tilde{\chi}_1 & \tilde{\chi}_2 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ \chi_1 & \chi_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

can be used to monotinize the relation! Indeed, for $(\xi, \chi) = T^{-1}(\tilde{\xi}, \tilde{\chi})$

$$\begin{aligned} 0 &\leq \frac{1}{3}\xi^2 + \xi\chi + \frac{2}{3}\chi^2 \\ &= \frac{1}{3}(2\tilde{\xi} - \tilde{\chi})^2 + (2\tilde{\xi} - \tilde{\chi})(-\tilde{\xi} + \tilde{\chi}) + \frac{2}{3}(-\tilde{\xi} + \tilde{\chi})^2 = \frac{1}{3}\tilde{\xi}\tilde{\chi}, \end{aligned}$$

non-colinear solutions to original
PQI

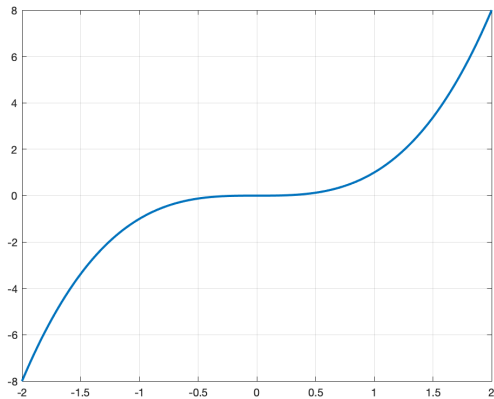
$$0 = \frac{1}{3}\xi^2 + \xi\chi + \frac{2}{3}\chi^2$$

$$\begin{bmatrix} \xi_1 \\ \chi_1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} \xi_2 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

EXAMPLE

Steady-state input-output maps under T_1 ,

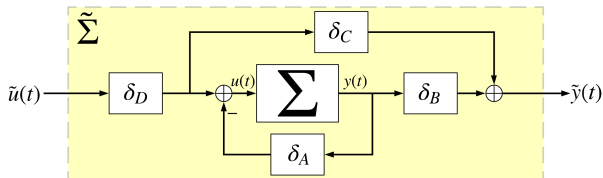
$$\begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix} = T_1 \begin{bmatrix} u \\ y \end{bmatrix}$$



MONOTONIZATION TO PASSIVATION

Theorem¹

Let Σ be EI-IOP(ρ, ν). If the map T monotizes the input-output relation k , then it passivizes the system Σ .



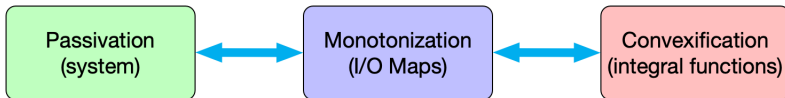
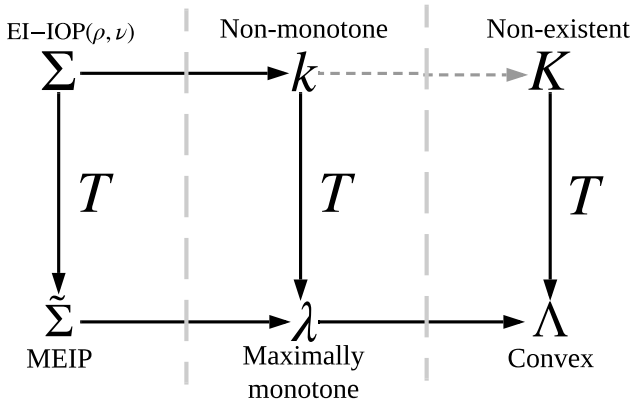
$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} \delta_D & 0 \\ 0 & 1 \end{bmatrix}}_{L_D} \underbrace{\begin{bmatrix} 1 & 0 \\ \delta_C & 1 \end{bmatrix}}_{L_C} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \delta_B \end{bmatrix}}_{L_B} \underbrace{\begin{bmatrix} 1 & \delta_A \\ 0 & 1 \end{bmatrix}}_{L_A},$$

¹[Sharf, Jain, Z, 2020]

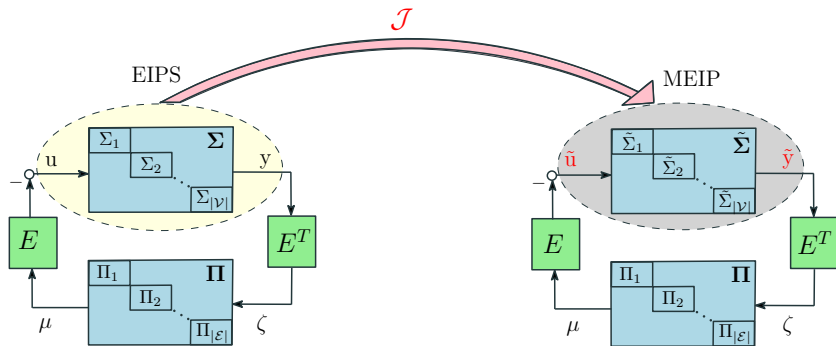
MONOTIZATION AND PASSIVATION

Elementary Transformation	Relation between I/O of Σ and $\tilde{\Sigma}$	Effect on Steady-State Relations	Realization	Effect on Integral Functions
$L_A = \begin{bmatrix} 1 & \delta_A \\ 0 & 1 \end{bmatrix}$	$\tilde{u} = u + \delta_A y$ $\tilde{y} = y$	$\lambda_A^{-1}(\tilde{y}) = k^{-1}(\tilde{y}) + \delta_A \tilde{y}$	output-feedback	$\Lambda^*(y) = K^*(y) + \frac{1}{2} \delta_A y^2$
$L_B = \begin{bmatrix} 1 & 0 \\ 0 & \delta_B \end{bmatrix}$	$\tilde{u} = u$ $\tilde{y} = \delta_B y$	$\lambda_B(u) = \delta_B k(u)$ or $\lambda_B^{-1}(\tilde{y}) = k^{-1}(\frac{1}{\delta_B} \tilde{y})$	post-gain	$\Lambda^*(y) = \frac{1}{\delta_B} K^*(\frac{1}{\delta_B} y)$ or $\Lambda(u) = \delta_B K(u)$
$L_C = \begin{bmatrix} 1 & 0 \\ \delta_C & 1 \end{bmatrix}$	$\tilde{u} = u$ $\tilde{y} = y + \delta_C u$	$\lambda_C(\tilde{u}) = k(\tilde{u}) + \delta_C \tilde{u}$	input-feedthrough	$\Lambda(u) = K(u) + \frac{1}{2} \delta_C u^2$
$L_D = \begin{bmatrix} \delta_D & 0 \\ 0 & 1 \end{bmatrix}$	$\tilde{u} = \delta_D u$ $\tilde{y} = y$	$\lambda_D^{-1}(y) = \delta_D k^{-1}(y)$ or $\lambda_D(\tilde{u}) = k(\frac{1}{\delta_D} \tilde{u})$	pre-gain	$\Lambda^*(y) = \delta_D K^*(\frac{y}{\delta_D})$ or $\Lambda(u) = \frac{1}{\delta_D} K(\frac{1}{\delta_D} u)$

PASSIVATION, MONOTONIZATION AND CONVEXIFICATION



PASSIVATION OF DIFFUSIVELY-COUPLED NETWORKS OF EIPS SYSTEMS



- ▶ Without loss of generality assume that the systems at nodes are EIPS (applicable if some of the systems are EIPS)
- ▶ Loop Transformation results in a pair of **regularized** network optimization problems

$$\mathcal{J} = \text{diag}(T_i)$$

CONCLUDING REMARKS



New perspectives on networks and passivity

- ▶ networks of EIP agents can be understood through solutions of a pair of static dual optimization problems
- ▶ passivity and monotonicity of input-output maps are essential
- ▶ passivation means monotonization - monotonization means convexification

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Dr. Mathias Bürger



Prof. Dr.-Ing.
Frank Allgöwer



QUESTIONS?

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- ▶ M. Sharf and D. Zelazo, “A Network Optimization Approach to Cooperative Control Synthesis,” *IEEE Control Systems Letters*, 1(1):86-91, 2017.
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- ▶ M. Sharf, A. Jain and D. Zelazo, “A Geometric Method for Passivation and Cooperative Control of Equilibrium-Independent Passivity-Short Systems”, *IEEE Transactions on Automatic Control*, 66(12):5877-5892, 2021.
- ▶ M. Sharf and D. Zelazo, “Passivity-Based Network Identification Algorithm with Minimal Time Complexity,” arXiv preprint, 2019.
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