# Toward a Geometric Theory of Passivity

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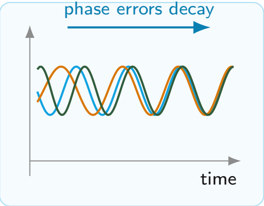
# Diffusive Networks: Why We Need Passivity and What Is Missing

# Why diffusive networks appear everywhere

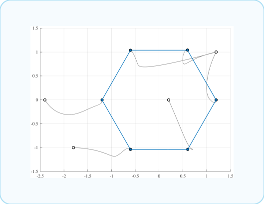
*A diffusive law uses only relative information: each unit reacts to disagreement with its neighbors.*

$$u_i = \sum_{j\in\mathcal{N}_i} a_{ij}\,\psi(y_j - y_i)$$



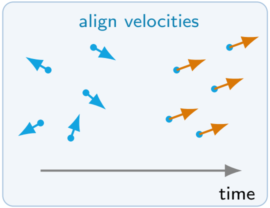

**Synchronization**

clocks, oscillators, generators, and rhythmic systems use relative phase errors to fall into step.



**Formation control**

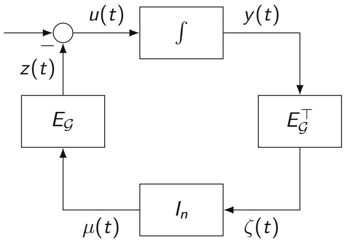
Robots regulate relative positions and distances to achieve and maintain a desired formation.




**Flocking**

Vehicles use only relative velocity measurements to achieve collective motion.

# The Linear Consensus Protocol



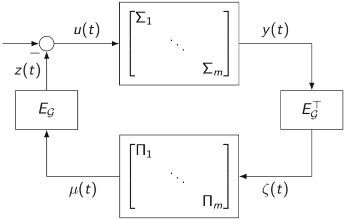

**Undirected Networks**

$$\dot{x}(t) = -E_{\mathcal{G}}\left(E_{\mathcal{G}}^{\top} y(t)\right)$$
$$y(t) = x(t)$$

*The linear consensus protocol may be analyzed using standard linear systems theory*

- the closed-loop is an autonomous linear system
$$x(t) = e^{-L_{\mathcal{G}} t} x(0)$$
- the **graph Laplacian**, $L_{\mathcal{G}} = E_{\mathcal{G}} E_{\mathcal{G}}^{T}$, creates a beautiful bridge between systems theory and graph theory

# Nonlinear Diffusive Networks




nonlinear maps:

$$\Sigma_i : u_i \mapsto y_i, \quad \Pi_j : \zeta_j \mapsto \mu_j$$

**Nonlinear Undirected Networks**

$$\dot{x}(t) = -E_{\mathcal{G}}\Pi\left(E_{\mathcal{G}}^{\top} y(t)\right)$$
$$y(t) = \Sigma(u(t))$$
$$u(t) = -E_{\mathcal{G}}\Pi(\zeta)$$

*Nonlinear networks can be significantly more challenging*

- how to characterize equilibria?
- how to assess convergence and stability properties?
- interplay between network structure and agent/controller dynamics more delicate
  - which system properties do we need to ensure convergence?
  - which graph properties do we need to ensure convergence?

# We need a compositional theory

*Network complexity comes from two sources*

- nonlinear node/edge dynamics
- arbitrary network topology

$$\text{local dynamics} \downarrow \text{network topology} \downarrow \text{closed-loop guarantees}$$

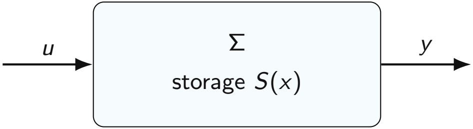
**Question**

Can we certify the components separately and let the interconnection do the rest?

**Passivity provides exactly this decomposition.**

# Passivity: storage and supplied power

$$\Sigma : \begin{cases} \dot{x} &= f(x,u), \\ y &= h(x,u) \end{cases}$$




supplied power through the port
$u^\top y$

## Storage function

A differentiable function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$S(0) = 0, \qquad S(x) \geq 0 \quad \text{for } x \neq 0.$$

- $S(x)$ measures internally stored energy
- $\dot{S}(x)$ measures the rate of energy accumulation

## Passivity

$$\dot{S}(x) \leq u^\top y$$

*Passivity says stored energy cannot increase faster than supplied energy.*

- *passive systems cannot generate energy*
- *passivity is an input-output property*

# Passivity: dissipation inequality

## Passivity

The system $\Sigma$ is passive if there exists a positive semi-definite storage function $S(x)$ and non-negative constants $\gamma, \epsilon$ such that

$$\dot{S}(x) \leq u^\top y - \gamma\|y\|_2^2 - \epsilon\|u\|_2^2 \qquad \forall(x,u).$$

| Passive | Output strictly passive | Input strictly passive |
|---|---|---|
| $\gamma = 0, \quad \epsilon = 0$ | $\gamma > 0, \quad \epsilon = 0$ | $\gamma = 0, \quad \epsilon > 0$ |
| $\dot{S} \leq u^\top y$ | $\dot{S} \leq u^\top y - \gamma\|y\|^2$ | $\dot{S} \leq u^\top y - \epsilon\|u\|^2$ |

*Strict passivity means that some supplied energy is dissipated rather than stored.*

# Passivity and stability

$$\Sigma : \begin{cases} \dot{x} = f(x, u), \\ y = h(x, u) \end{cases}$$

## Zero-input dynamics

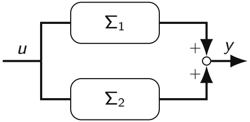
Assume $S(x)$ is *positive definite*. Under zero input ($u = 0$),

$$\dot{S} \leq -\gamma \|y\|^2.$$

- Passive: $\dot{S} \leq 0 \Rightarrow$ stored energy is non-increasing.
- If $S(x)$ is positive definite, then $S$ is a Lyapunov function for the zero-input dynamics.
- Output strict passivity: $\dot{S} \leq -\gamma \|y\|^2$ provides strict energy dissipation and stronger convergence properties.

# Passivity and common interconnections

## Parallel

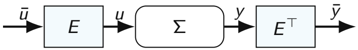



$$u_1 = u_2 = u,$$
$$y = y_1 + y_2.$$

$$u^\top y = u_1^\top y_1 + u_2^\top y_2$$

$$\boxed{S = S_1 + S_2}$$

## Adjoint port transform




$$u = E\bar{u},$$
$$\bar{y} = E^\top y.$$

$$\bar{u}^\top \bar{y} = (E\bar{u})^\top y = u^\top y$$

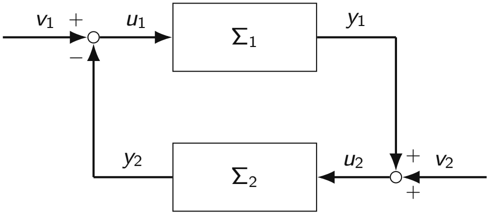
$$\boxed{\bar{S} = S}$$

### Power-balance principle

Passivity is preserved when the interconnection converts the component supply rates into the external supply rate:

$$\dot{S} \leq \sum u_i^\top y_i = u_{\text{ext}}^\top y_{\text{ext}}.$$

# Special interconnection: passivity is preserved by feedback




$$u_1 = v_1 - y_2, \qquad u_2 = v_2 + y_1$$

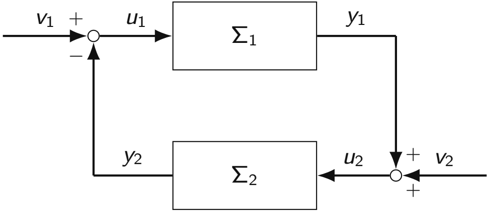
$$u_1^\top y_1 + u_2^\top y_2 = v_1^\top y_1 + v_2^\top y_2 - y_2^\top y_1 + y_1^\top y_2 = v^\top y$$

## Composite storage

If $\Sigma_1$ and $\Sigma_2$ are passive with storages $S_1, S_2$, then the feedback interconnection is passive with

$$S(x_1, x_2) = S_1(x_1) + S_2(x_2).$$

# Special interconnection: passivity is preserved by feedback




$$u_1 = v_1 - y_2, \qquad u_2 = v_2 + y_1$$

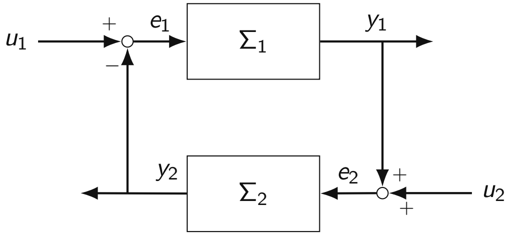
$$u_1^\top y_1 + u_2^\top y_2 = v_1^\top y_1 + v_2^\top y_2 - y_2^\top y_1 + y_1^\top y_2 = v^\top y$$

**Passivity theorem**

The negative-feedback interconnection of passive systems is passive.

*Strictness gives dissipation, and dissipation gives stability or convergence.*

# Feedback interconnection: passivity theorem




## Passivity Compensation

Consider the feedback connection above. Suppose each feedback component satisfies

$$e_i^\top y_i \geq \dot{S}_i + \varepsilon_i e_i^\top e_i + \delta_i y_i^\top y_i, \qquad i = 1, 2,$$

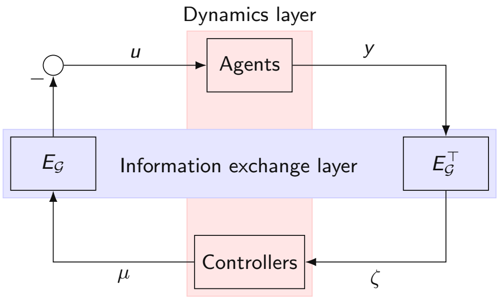
for some storage function $S_i(x_i)$. Then the closed-loop map from $u$ to $y$ is finite-gain $\mathcal{L}_2$ stable if

$$\varepsilon_1 + \delta_2 > 0, \qquad \varepsilon_2 + \delta_1 > 0.$$

# Other passivity notions

| Formulation | Related notions |
|---|---|
| **Relative to the origin**<br>$\dot{S} \leq (u-0)^\top (y-0) = u^\top y$ | (Standard) passivity |
| **Relative to specific input-output equilibria**<br>$\dot{S}_{x^\star} \leq (u-u^\star)^\top (y-y^\star)$ | Shifted passivity; Equilibrium-independent passivity (EIP); Maximal EIP (MEIP) |
| **Relative to pairs of trajectories**<br>$\dot{S}_\Delta \leq (u_1-u_2)^\top (y_1-y_2)$ | Incremental passivity; differential passivity |

# Passivity and multi-agent networks



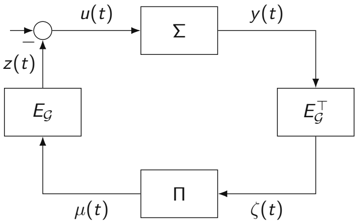

*Diffusive networks are feedback interconnections+symmetric port interconnections.*

- agent/controller dynamics can be certified locally
- graph operators route relative information
- passivity separates dynamics from topology

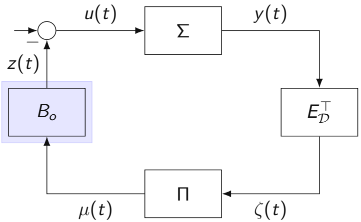
**Undirected networks**

Symmetry of the incidence interconnection gives clean passivity cancellation.

# Where the classical picture break

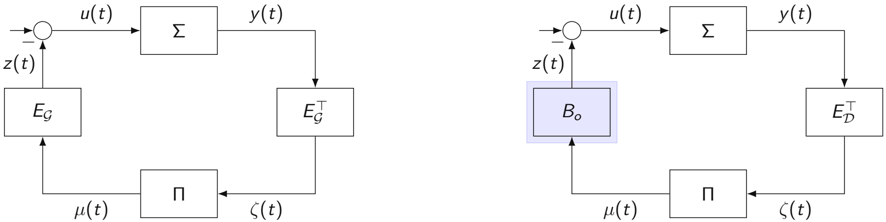



**Undirected** $(\Sigma, \Pi, \mathcal{G})$




**Directed** $(\Sigma, \Pi, \mathcal{D})$

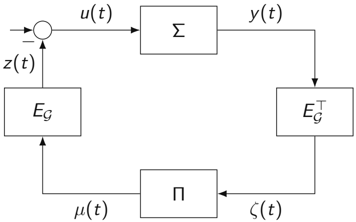
# Where the classical picture break



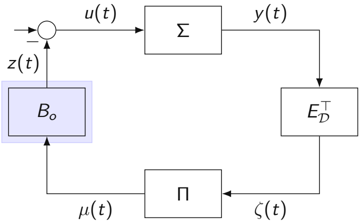

**Undirected** $(\Sigma, \Pi, \mathcal{G})$ **Directed** $(\Sigma, \Pi, \mathcal{D})$

- **Adjoint (symmetric) interconnection**
  $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$

# Where the classical picture break




**Undirected** $(\Sigma, \Pi, \mathcal{G})$




**Directed** $(\Sigma, \Pi, \mathcal{D})$

- **Adjoint (symmetric) interconnection**
  $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$
- **Passivity Analysis** ✓

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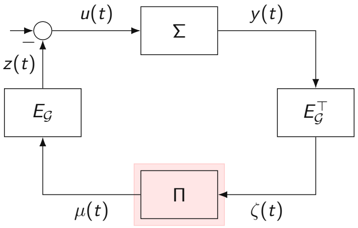


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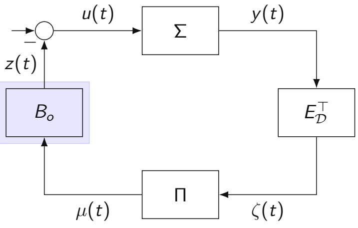
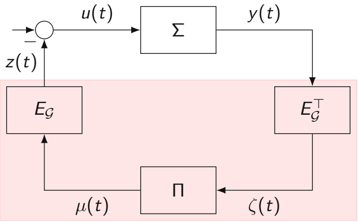



**Directed** $(\Sigma, \Pi, \mathcal{D})$

- **Adjoint (symmetric) interconnection**
  $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^\top$
- **Passivity Analysis** ✓

**Passive $\Pi$** $\quad \mu^\top(t)\zeta(t) \geq \dot{V}(\eta(t))$

# Where the classical picture break

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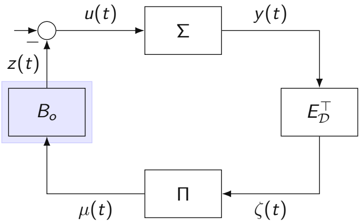



**Directed** $(\Sigma, \Pi, \mathcal{D})$

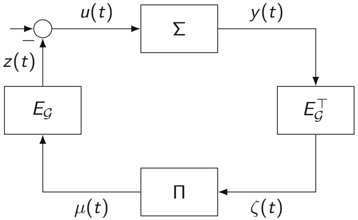
- **Adjoint (symmetric) interconnection**
  $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$
- **Passivity Analysis** ✓

**Passive** $\Pi$ $\quad \mu^{\top}(t)\zeta(t) \geq \dot{V}(\eta(t))$

$\Rightarrow$ **Passive** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$

$$z^{\top}(t)y(t) = \mu^{\top}(t)E_{\mathcal{G}}^{\top}y(t) = \mu^{\top}(t)\zeta(t)$$

# Where the classical picture break

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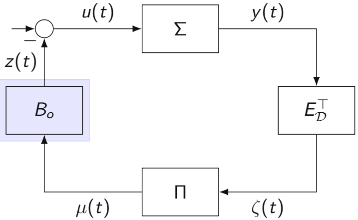
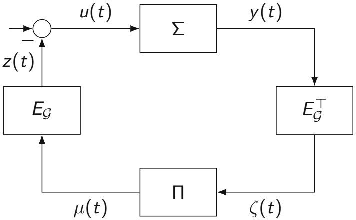



**Directed** $(\Sigma, \Pi, \mathcal{D})$

- **Adjoint (symmetric) interconnection** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$
- **Passivity Analysis** ✓
- **Passivity compensation** ✓

# Where the classical picture break

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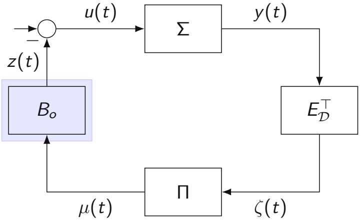


**Directed** $(\Sigma, \Pi, \mathcal{D})$

- **Adjoint (symmetric) interconnection** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$
- **Passivity Analysis** ✓
- **Passivity compensation** ✓

- **Asymmetric** operator $B_o \Pi E_{\mathcal{D}}^{\top}$

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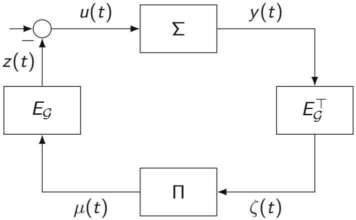
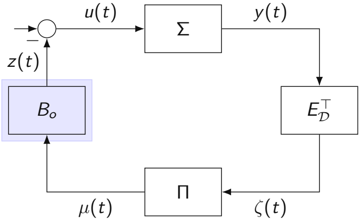


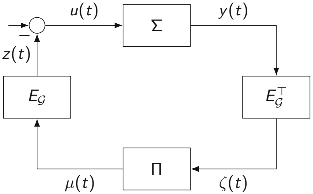

**Undirected** $(\Sigma, \Pi, \mathcal{G})$

- **Adjoint (symmetric) interconnection** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$
- **Passivity Analysis** ✓
- **Passivity compensation** ✓

**Directed** $(\Sigma, \Pi, \mathcal{D})$

- **Asymmetric** operator $B_o \Pi E_{\mathcal{D}}^{\top}$
- **Passivity Analysis?**

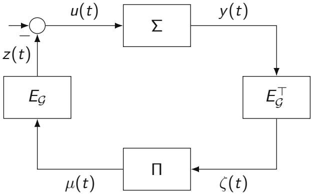
# Agreement manifold and its geometry




**Undirected** $(\Sigma, \Pi, \mathcal{G})$

- Agreement manifold $S = \text{span}\{\mathbb{1}\}$
- Disagreement manifold $S^\perp = \{y : \mathbb{1}^\top y = 0\}$
- $\text{Proj}_S(y) = \frac{1}{n}\mathbb{1}\mathbb{1}^\top y$
- $\text{Proj}_{S^\perp}(y) = y - \text{Proj}_S(y)$

# Agreement manifold and its geometry

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- Output consensus:

$$\lim_{t\to\infty} y(t) \in S \Longleftrightarrow \lim_{t\to\infty} \mathrm{Proj}_{S^\perp}(y(t)) = 0.$$

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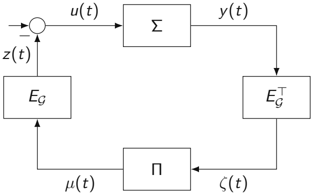


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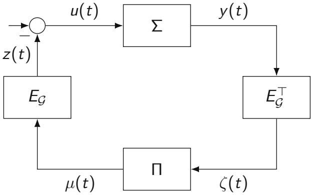
- Classical passivity is replaced by passivity with respect to the *projected disagreement output*

$$e_y = \mathrm{Proj}_{S^{\perp}}(y).$$

# Agreement manifold and its geometry

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- Output consensus:

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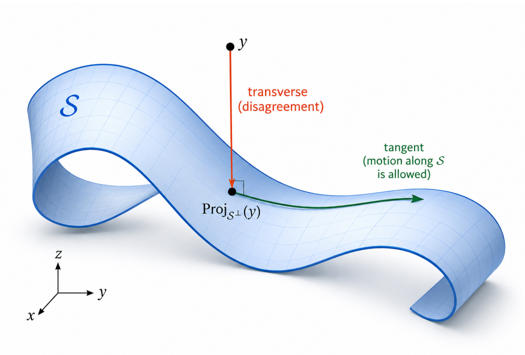
- Classical passivity is replaced by passivity with respect to the *projected disagreement output*

$$e_y = \mathrm{Proj}_{S^{\perp}}(y).$$

- The forward and feedback paths satisfy complementary passivity inequalities in the variable $e_y$, leading to convergence toward the agreement manifold.

# Consensus is manifold stabilization




Output Consensus: $\lim_{t\to\infty} \mathrm{Proj}_{\mathcal{S}^\perp}(y(t)) = \lim_{t\to\infty} y(t) - \mathrm{Proj}_{\mathcal{S}}(y(t)) = 0$

- **Desired behavior**: $y(t)$ converges to the agreement manifold $\mathcal{S}$.
- **Transverse deviations** $\boxed{y - \mathrm{Proj}_{\mathcal{S}}(y)}$ **are penalized**: We only care whether the output $y(t)$ converges to the agreement manifold $\mathcal{S}$

# The Geometry of Passivity

### Classical viewpoint

Passivity measures energy exchange relative to a point, an equilibrium, or a pair of trajectories.

### Geometric viewpoint

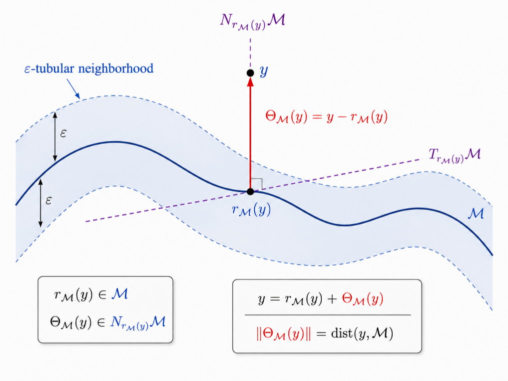
Transversal passivity measures energy exchange only in directions transverse to the desired geometry.

- errors are usually vectors like $u - \bar{u}$ and $y - \bar{y}$
- all directions away from the target enter the supply rate

- tangent motion along the target is allowed
- storage measures distance to a desired geometry

*Consensus motivates an output geometry. To obtain a feedback theory, we extend this geometry to both input and output ports.*

# Tubular retractions and manifold error maps




- **Embedded submanifold $\mathcal{M} \subset \mathbb{R}^n$**
  - Tangent space at $x \in \mathcal{M}$: $T_x\mathcal{M}$
  - Normal space at $x \in \mathcal{M}$: $N_x\mathcal{M}$
- **Tubular neighborhood $\mathcal{N}$** (region where every point has a unique closest point on the manifold)
- **Tubular retraction** (**smooth**)

$$r_{\mathcal{M}}(y) : \mathcal{N} \to \mathcal{M}$$

- **Manifold error map**

$$\Theta_{\mathcal{M}}(y) = y - r_{\mathcal{M}}(y)$$

- **Distance**

$$\|\Theta_{\mathcal{M}}(y)\| = \mathrm{dist}(y, \mathcal{M})$$

# Definition: transversal passivity

*Passivity is a relation between input and output. Therefore, we associate a target manifold with each port.*

Consider $\Sigma$ : $\dot{x} = f(x, u),\ y = h(x, u)$, with

$$\mathcal{M}_u \subset \mathcal{U}, \qquad \mathcal{M}_y \subset \mathcal{Y}.$$

$$\Theta_{\mathcal{M}_u}(u) = u - r_u(u), \qquad \Theta_{\mathcal{M}_y}(y) = y - r_y(y).$$

**Transversal passivity**

For $\Sigma$, if there exists a nonnegative storage function $V(x)$ such that

$$\dot{V}(x) \leq \langle \Theta_{\mathcal{M}_u}(u), \Theta_{\mathcal{M}_y}(y) \rangle - \varepsilon \|\Theta_{\mathcal{M}_y}(y)\|^2 - \gamma \|\Theta_{\mathcal{M}_u}(u)\|^2$$

- Passive: $\varepsilon = \gamma = 0$
- Input: $\varepsilon = 0,\ \gamma > 0$
- Output: $\varepsilon > 0,\ \gamma = 0$
- Input-output: $\varepsilon > 0,\ \gamma > 0,\ \varepsilon\gamma < \frac{1}{4}$

# Examples: transversal passivity

## Example 1: integrator to the agreement manifold

Consider the integrator

$$\dot{x} = u, \qquad y = x, \qquad x, u, y \in \mathbb{R}^n,$$

and the agreement manifold $S = \mathrm{span}\{\mathbb{1}\}$. Let

$$\Pi_S = I - \frac{1}{n}\mathbb{1}\mathbb{1}^\top, \qquad \Theta_{\{0\}}(u) = u, \qquad \Theta_S(y) = \Pi_S y.$$

## Storage and transversal passivity

$$V(x) = \frac{1}{2}x^\top \left( I - \frac{1}{n}\mathbb{1}\mathbb{1}^\top \right) x = \frac{1}{2}\|\Theta_S(y)\|^2.$$

$$\dot{V} = x^\top \Pi_S u = \langle \Theta_{\{0\}}(u), \Theta_S(y) \rangle.$$

The integrator is $(\{0\}, S)$-passive and $V$ measures disagreement energy.

# Example: a stable limit cycle as a target manifold

**Hopf oscillator in polar coordinates**

$$\dot{q} = q(\mu - q^2), \qquad q > 0,$$
$$\dot{\vartheta} = \omega_0.$$

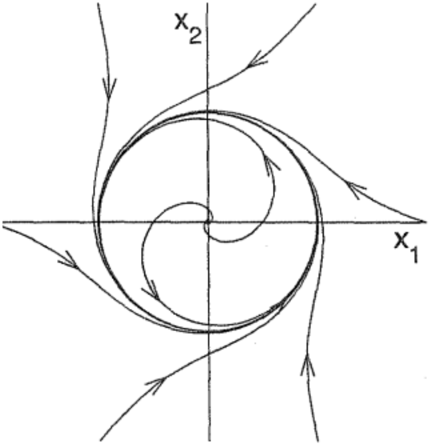
For $\mu > 0$, define

$$q_\star = \sqrt{\mu}, \qquad \Gamma = \{(q, \vartheta) : q = q_\star\}.$$

## Geometric behavior

- $q(t) \to q_\star$: trajectories converge radially to $\Gamma$;
- $\dot{\vartheta} = \omega_0 \neq 0$: motion continues along $\Gamma$;
- $\Gamma$ is an asymptotically stable limit cycle.

*The target is not an equilibrium point: stability means convergence to the orbit, while phase motion remains free.*




Trajectories approach the periodic orbit from both inside and outside.

# Hopf oscillator: transversal passivity certificate

**Controlled oscillator**

$$\dot{q} = q(\mu - q^2) + u_r,$$
$$\dot{\vartheta} = \omega_0 + u_t, \qquad y = (q, \vartheta).$$

**Port geometries**

$$\mathcal{M}_u = \{0\} \times \mathbb{R}, \qquad \Theta_{\mathcal{M}_u}(u) = (u_r, 0),$$
$$\mathcal{M}_y = \{q_\star\} \times \mathbb{R}, \qquad \Theta_{\mathcal{M}_y}(y) = (q - q_\star, 0).$$

Only the **radial components** enter the transverse supply rate!

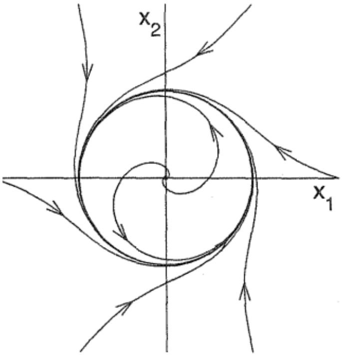
## Storage

Choose the squared radial distance to the limit cycle:

$$V(q) = \frac{1}{2}(q - q_\star)^2.$$

Since $\mu = q_\star^2$,

$$\begin{aligned} \dot{V} &= (q - q_\star)\big[q(q_\star^2 - q^2) + u_r\big] \\ &= u_r(q - q_\star) - q(q + q_\star)(q - q_\star)^2 \\ &= \langle \Theta_{\mathcal{M}_u}(u), \Theta_{\mathcal{M}_y}(y) \rangle - q(q + q_\star) \left\| \Theta_{\mathcal{M}_y}(y) \right\|^2. \end{aligned}$$




- radial deviation $q - q_\star$ is dissipated;
- radial input $u_r$ supplies transverse power;
- tangential input $u_t$ and phase $\vartheta$ do not enter the certificate.

# Hopf oscillator: transversal passivity certificate

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$\mathcal{M}_y$

Only the radi


## Storage

Choose the

$$2$$

Hopf oscillator is $(\mathcal{M}_u, \mathcal{M}_y)$-passive

Since $\mu = q_\star^2$,

$$\begin{aligned}\dot{V} &= (q - q_\star)\left[q(q_\star^2 - q^2) + u_r\right] \\ &= u_r(q - q_\star) - q(q + q_\star)(q - q_\star)^2 \\ &= \langle \Theta_{\mathcal{M}_u}(u), \Theta_{\mathcal{M}_y}(y) \rangle - q(q + q_\star)\left\|\Theta_{\mathcal{M}_y}(y)\right\|^2.\end{aligned}$$

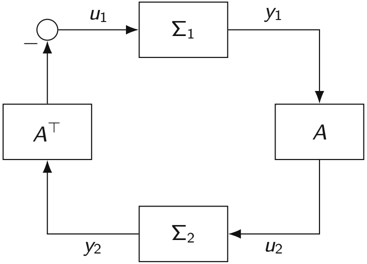
- radial deviation $q - q_\star$ is dissipated;
- radial input $u_r$ supplies transverse power;
- tangential input $u_t$ and phase $\vartheta$ do not enter the certificate.

# Classical notions as special cases

| Choice of manifolds | Transverse errors | Recovered notion |
|---|---|---|
| $\mathcal{M}_u = \{0\}, \quad \mathcal{M}_y = \{0\}$ | $\Theta_{\mathcal{M}_u} = u, \quad \Theta_{\mathcal{M}_y} = y$ | standard passivity |
| $\mathcal{M}_u = \{\bar{u}\}, \quad \mathcal{M}_y = \{\bar{y}\}$ | $u - \bar{u}, \quad y - \bar{y}$ | shifted passivity |
| diagonal subspaces in $\Sigma \times \Sigma$ | $u_1 - u_2, \quad y_1 - y_2$ | incremental passivity |
| affine input/output subspaces | projected shifted errors | projected dissipativity |

*The new behavior begins when the manifolds are curved or when feedback reshapes the feasible target set.*

# Feedback interconnection: what changes?




$$u_2 = Ay_1, \qquad u_1 = -A^\top y_2.$$

**Classical passivity**

Each subsystem satisfies

$$\dot{V}_i \leq u_i^\top y_i.$$

Adding both inequalities,

$$\dot{V}_1 + \dot{V}_2 \leq u_1^\top y_1 + u_2^\top y_2.$$

Under feedback,

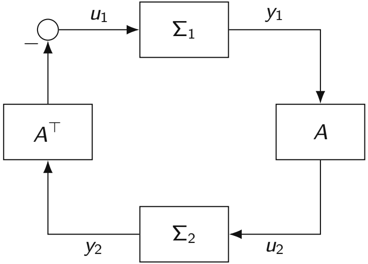
$$u_1^\top y_1 + u_2^\top y_2 = (-A^\top y_2)^\top y_1 + (Ay_1)^\top y_2 = 0.$$

**Passivity compensation**

The feedback terms cancel exactly, yielding

$$\boxed{\dot{V}_1 + \dot{V}_2 \leq 0.}$$

# Feedback interconnection: what changes?




- $\Sigma_i$ are input-output $(\mathcal{M}_{u_i}, \mathcal{M}_{y_i})$-passive

$$u_2 = Ay_1, \qquad u_1 = -A^\top y_2.$$

## Limitations when using passivity compensation

Transversal passivity uses $\Theta_{\mathcal{M}_i}(\cdot)$, and in general

$$\Theta_{\mathcal{M}_{u_2}}(Ay_1) \neq A\Theta_{\mathcal{M}_{y_1}}(y_1), \quad \Theta_{\mathcal{M}_{u_1}}(-A^\top y_2) \neq -A^\top \Theta_{\mathcal{M}_{y_2}}(y_2).$$

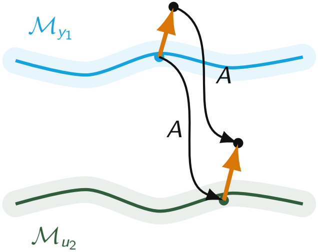
The linear channels may not commute with nonlinear retractions. The transverse supply rates generally do *not* cancel automatically.

## Question

What replaces this cancellation when the supply rate is written in terms of

$$\langle \Theta_{\mathcal{M}_u}, \Theta_{\mathcal{M}_y} \rangle?$$

# Compatibility restores cancellation




Projection and mapping commute.

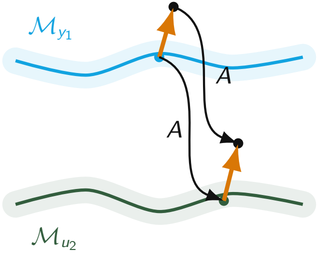
Compatibility requires

$$\boxed{\Theta_{\mathcal{M}_u}(Ay) = A\,\Theta_{\mathcal{M}_y}(y)}$$

Equivalently,

$$r_{\mathcal{M}_u}(Ay) = A\,r_{\mathcal{M}_y}(y).$$

# Compatibility restores cancellation

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Equivalently,

$$r_{\mathcal{M}_u}(Ay) = A\,r_{\mathcal{M}_y}(y).$$

## Key consequence

The transverse supply rates now cancel exactly as in the classical proof.

$$\Longrightarrow$$

The feedback interconnection remains transversally passive.

# Compatibility defects

For the feedback interconnection

$$u_1 = -A^\top y_2, \qquad u_2 = Ay_1,$$

define the channel-compatible output sets

$$\widetilde{\mathcal{M}}_1^A := \mathcal{M}_{y_1} \cap A^{-1}(\mathcal{M}_{u_2}), \qquad \widetilde{\mathcal{M}}_2^A := \mathcal{M}_{y_2} \cap (-A^\top)^{-1}(\mathcal{M}_{u_1}).$$

These are outputs that lie on their prescribed output manifold and are mapped by the channel onto the corresponding input manifold.

Define output transverse errors: $\eta_i := \Theta_{\mathcal{M}_{y_i}}(y_i), \qquad i = 1, 2.$

**Forward channel**

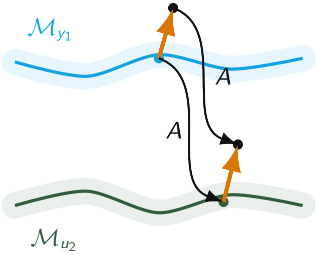
$$\delta_{12}^A(y_1) = \Theta_{\mathcal{M}_{u_2}}(Ay_1) - A\Theta_{\mathcal{M}_{y_1}}(y_1).$$

**Reverse channel**

$$\delta_{21}^A(y_2) = \Theta_{\mathcal{M}_{u_1}}(-A^\top y_2) + A^\top \Theta_{\mathcal{M}_{y_2}}(y_2).$$

The defects measure the failure of mapping and transverse projection to commute along the two feedback channels.

# Exact compatibility




Projection and mapping commute.

**Commutation condition**

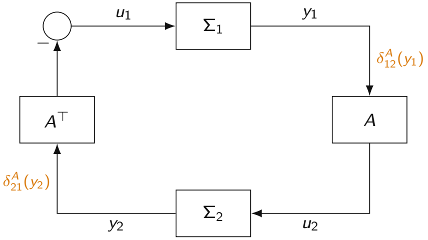
$$\Theta_{\mathcal{M}_{u_2}}(Ay) = A\Theta_{\mathcal{M}_{y_1}}(y)$$

if and only if

$$r_{u_2}(Ay) = Ar_{y_1}(y).$$

- exact compatibility means $\delta_{12}^{A} \equiv 0$
- it is naturally target-preserving
- for affine manifolds, it is an algebraic projection test

# Closed-loop dissipation with mismatch terms




Each feedback channel contributes one compatibility defect.

- $\Sigma_i$ is input-output $(\mathcal{M}_{u_i}, \mathcal{M}_{y_i})$-passive.
- $A$ is the linear interconnection channel.
- The closed-loop storage is
$$V(x_1, x_2) = V_1(x_1) + V_2(x_2).$$
- The transverse output errors are
$$\eta_i = \Theta_{\mathcal{M}_{y_i}}(y_i).$$

Mismatch terms measure the failure of the two channels to preserve the port geometries.

## Transversal passivity compensation

Along the feedback interconnection,

$$\dot V \le \underbrace{\left\langle \delta^A_{21}(y_2), \eta_1 \right\rangle}_{A^\top\text{-channel defect}} + \underbrace{\left\langle \delta^A_{12}(y_1), \eta_2 \right\rangle}_{A\text{-channel defect}} - \left( \underbrace{\epsilon_1 \|\eta_1\|^2 + \epsilon_2 \|\eta_2\|^2}_{\text{transverse output dissipation}} \right) - \left( \underbrace{\gamma_1 \left\| \Theta_{\mathcal{M}_{u_1}}(-A^\top y_2) \right\|^2 + \gamma_2 \left\| \Theta_{\mathcal{M}_{u_2}}(A y_1) \right\|^2}_{\text{transverse input dissipation}} \right)$$

# Exact compatibility restores passivity

Assume

$$\delta_{12}^{A}(y_1) = \delta_{21}^{A}(y_2) = 0.$$

Then the mismatch terms vanish, leaving

$$\boxed{\dot{V} \leq -\epsilon_1 \|\eta_1\|^2 - \epsilon_2 \|\eta_2\|^2 - \gamma_1 \|\zeta_2\|^2 - \gamma_2 \|\zeta_1\|^2.}$$

**Geometric passivity theorem**

The feedback interconnection is again

$$(\mathcal{M}_u, \mathcal{M}_y)\text{-passive},$$

with the same transverse dissipation properties as the individual subsystems.

# Exact compatibility removes the mismatch terms

## Exact channel compatibility

The forward channel $A$ is exactly compatible on $\Omega_{12} \subseteq \mathcal{D}_{12}^{A}$ if

$$\boxed{\delta_{12}^{A}(y_1) = 0, \qquad \forall\, y_1 \in \Omega_{12}.}$$

The reverse channel $-A^\top$ is exactly compatible on $\Omega_{21} \subseteq \mathcal{D}_{21}^{A}$ if

$$\boxed{\delta_{21}^{A}(y_2) = 0, \qquad \forall\, y_2 \in \Omega_{21}.}$$

## Equivalent characterization

Exact compatibility is equivalent to the retractions commuting with the channel maps:

$$r_{u_2}(Ay_1) = A\, r_{y_1}(y_1),$$

$$r_{u_1}(-A^\top y_2) = (-A^\top)\, r_{y_2}(y_2).$$

## Closed-loop consequence

The mismatch terms vanish, so the dissipation inequality becomes

$$\boxed{\dot{V} \leq -\epsilon_1\|\eta_1\|^2 - \epsilon_2\|\eta_2\|^2 - \gamma_1\|A^\top\eta_2\|^2 - \gamma_2\|A\eta_1\|^2.}$$

# Exact compatibility gives clean convergence

**Exact-compatible interconnection**

Assume both subsystems are output-strictly transversally passive, both feedback channels are exactly compatible, and the closed-loop trajectory remains in a compact forward-invariant subset of the tubular domain. Then

$$\operatorname{dist}\big(y_1(t), \widetilde{\mathcal{M}}_1^A\big) \to 0, \qquad \operatorname{dist}\big(y_2(t), \widetilde{\mathcal{M}}_2^A\big) \to 0,$$

up to the largest invariant subset of the zero-dissipation set.

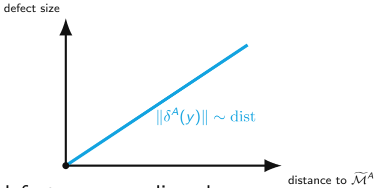
The feasible target sets are

$$\widetilde{\mathcal{M}}_1^A = \mathcal{M}_{y_1} \cap A^{-1}(\mathcal{M}_{u_2}), \qquad \widetilde{\mathcal{M}}_2^A = \mathcal{M}_{y_2} \cap (-A^\top)^{-1}(\mathcal{M}_{u_1}).$$

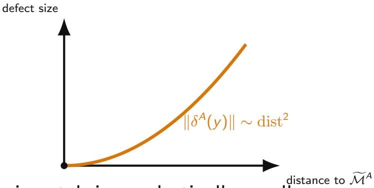
# First-order compatibility

**How rapidly does the compatibility defect grow away from the compatible manifold?**

$$\delta^A(y) = \underbrace{\delta^A(p)}_{\text{constant}} + \underbrace{\mathrm{d}_p\delta^A(p)(y-p)}_{\text{linear}} + \underbrace{O(\|y-p\|^2)}_{\text{quadratic}}, \qquad p \in \widetilde{\mathcal{M}}^A.$$




defect can grow linearly




mismatch is quadratically small

## Takeaway

First-order compatibility is weaker than exact compatibility, but still strong enough to make the geometric mismatch second order near $\widetilde{\mathcal{M}}^A$

$$\|\delta^A(y)\| = O\Big(\mathrm{dist}(y, \widetilde{\mathcal{M}}^A)^2\Big)$$

# First-order compatibility: Feedback

**Definition**

The forward channel $A$ is *first-order compatible* if

$$\boxed{\delta_{12}^{A}(p) = 0, \qquad \mathrm{d}_p\delta_{12}^{A} = 0, \qquad p \in \widetilde{\mathcal{M}}_1^A.}$$

Likewise, the reverse channel $-A^\top$ is first-order compatible if

$$\boxed{\delta_{21}^{A}(q) = 0, \qquad \mathrm{d}_q\delta_{21}^{A} = 0, \qquad q \in \widetilde{\mathcal{M}}_2^A.}$$

**Equivalent infinitesimal commutation**

Assuming the prescribed retractions are $C^1$,

$$\boxed{\mathrm{d}_{Ap} r_{u_2} \circ A = A \circ \mathrm{d}_p r_{y_1},}$$

for every $p \in \widetilde{\mathcal{M}}_1^A$,

and

$$\boxed{\mathrm{d}_{-A^\top q} r_{u_1} \circ (-A^\top) = (-A^\top) \circ \mathrm{d}_q r_{y_2},}$$

for every $q \in \widetilde{\mathcal{M}}_2^A$.

**Interpretation**

At points on the compatible manifold, the channel map and the retraction commute to first order.

# First-order compatibility: Feedback

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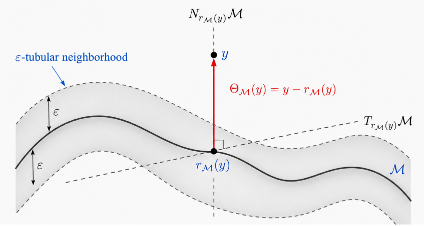
for every $q \in \widetilde{\mathcal{M}}_2^A$.

Near the compatible manifold, **strict transverse disipation** $>$ **compatibility mismatch** and we may obtain local convergence to the target manifold!

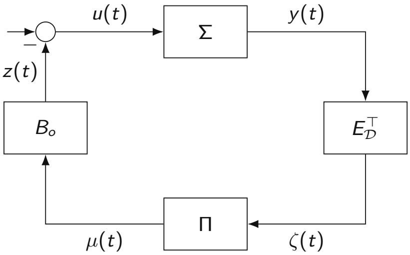
# Potential applications




- Desired behavior: $y(t) \to \mathcal{M}$
- Penalize transverse deviations $\Theta_{\mathcal{M}}(y)$
- Allow motion along the manifold

| **Task** | **Target manifold** |
|---|---|
| Setpoint regulation | $\mathcal{M}$ is a singleton |
| Consensus | $\mathcal{M}$ is the agreement manifold |
| Formation control | $\mathcal{M} = \{e \in \mathrm{Im}(E \otimes I_d) : \ \|e_k\|^2 = d_{ij}^2, \ \forall e_k = (i,j)\}$ |
| Synchronization on n-sphere | $\mathcal{M} = (S^n)^N$ |
| Path following | $\mathcal{M}$ is a curve |

# Directed agreement: same target, different channels



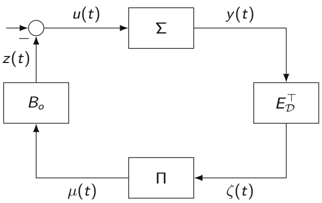

**Directed** $(\Sigma, \Pi, \mathcal{D})$

- Same agreement target:
$$S = \mathrm{span}\{\mathbb{1}\}, \qquad e_y = \mathrm{Proj}_{S^\perp}(y).$$
- Since $E_{\mathcal{D}}^\top \mathbb{1} = 0$,
$$\zeta = E_{\mathcal{D}}^\top y = E_{\mathcal{D}}^\top e_y.$$
- Undirected case:
$$E_{\mathcal{G}}^\top \quad \text{paired with} \quad E_{\mathcal{G}}.$$
- Directed case:
$$E_{\mathcal{D}}^\top \quad \text{paired with} \quad B_o, \qquad B_o \neq E_{\mathcal{D}}.$$

**Main issue**

The target is still $S$, but passivity compensation is no longer automatic.

# Directed agreement: compatible port targets




**Directed** $(\Sigma, \Pi, \mathcal{D})$

**Agent targets**

$$\mathcal{M}_y^{\Sigma} = S, \qquad \mathcal{M}_u^{\Sigma} = \{0\}.$$

**Controller targets**

$$\mathcal{M}_\zeta^{\Pi} = \{0\}, \qquad \mathcal{M}_\mu^{\Pi} = \ker B_o.$$

These choices match the channel maps:

$$E_{\mathcal{D}}^{\top} S = \{0\}, \qquad B_o(\ker B_o) = \{0\}.$$

*Agreement produces zero edge disagreement, while any $\mu \in \ker B_o$ produces zero feedback signal $z$.*

## Which output target does the closed loop select?

At a feasible closed-loop target, $\zeta = E_{\mathcal{D}}^{\top} y$, $\mu = \Pi(\zeta)$, $B_o \mu = 0$. Hence

$$y \in S_{\text{feas}} := \left\{ y : \; B_o \Pi(E_{\mathcal{D}}^{\top} y) = 0 \right\}.$$

To recover agreement, we require

$$\boxed{S_{\text{feas}} = S.}$$

# Directed agreement as a transverse-passivity problem

**Choose port targets**

$$\mathcal{M}_y^{\Sigma} = S, \qquad \mathcal{M}_u^{\Sigma} = \{0\}$$

$$\mathcal{M}_\zeta^{\Pi} = \{0\}, \qquad \mathcal{M}_\mu^{\Pi} = \ker B_o.$$

**Assume**

- agents $\Sigma$ are transversally passive w.r.t. $(\mathcal{M}_u^{\Sigma}, \mathcal{M}_y^{\Sigma})$;
- controller $\Pi$ is transversally passive w.r.t. $(\mathcal{M}_\zeta^{\Pi}, \mathcal{M}_\mu^{\Pi})$;
- the directed channels are compatible with these targets.

**Transverse dissipation**

The closed-loop storage satisfies, locally,

$$\dot{V} \leq -\alpha \|\mathrm{Proj}_{S^\perp}(y)\|^2 - \beta\|\zeta\|^2 + \text{compatibility mismatch}.$$

If the mismatch vanishes, or is dominated by strict dissipation, then

$$\mathrm{Proj}_{S^\perp}(y(t)) \to 0.$$

**Graph condition**

The rooted-outbranching condition ensures that

$$\zeta = E_{\mathcal{D}}^{\top} y = 0 \quad \Longleftrightarrow \quad y \in S.$$

Thus the only feasible transverse target is agreement.

## Summary

- Classical passivity provides a powerful compositional framework for interconnected systems.
- Many control objectives are naturally described by *manifolds* rather than isolated equilibria.
- Transverse passivity measures energy only in directions normal to the target manifold through
$$\Theta_{\mathcal{M}}(y) = y - r_{\mathcal{M}}(y).$$
- Compatibility extends the classical passivity theorem to geometric interconnections.
- The resulting framework naturally captures tasks such as synchronization, oscillations, and manifold stabilization.

**Passivity becomes a geometric theory.**

# Future Directions

## Theory

- Stronger compatibility conditions and global convergence
- Incremental and differential transverse passivity
- Infinite-dimensional and hybrid systems

## Control Design

- Transverse passivity-based controller synthesis
- Observer and output-feedback design
- Learning-compatible passivity frameworks

## Applications

- Robotics and manipulation
- Formation control and networked systems