SYMMETRY-FORCED FORMATION CONTROL

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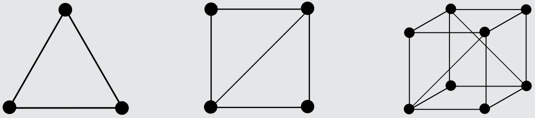
Formation Control Objective

Given a team of robots endowed with the ability to sense/ communicate with neighboring robots, design a control for each robot using only local information that moves the team into a desired spatial configuration.



FORMATION CONSTRAINTS

- The desired formation is characterized by a set of M constraints, encoded in the function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, and a configuration $\mathbf{p}^\star$ satisfying the constraints.
- The set of all feasible formations is

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \,|\, F(p) = F(\mathbf{p}^\star)\}$$

Formation Control Objective

For an ensemble of n agents with dynamics

$$\dot{p}_i = u_i,$$

with $p_i(t) \in \mathbb{R}^d$, an information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and formation constraint function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, design a distributed control law for each agent $i \in \{1, \ldots, n\}$ such that the set

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \,|\, F(p) = F(\mathbf{p}^\star)\},$$

is asymptotically stable.

Theorem - Distance Constrained Formation Control [Krick 2009]

Consider the potential function

$$F_f(p) = \frac{1}{4}\sum_{ij\in\mathcal{E}}\left(\|p_i(t)-p_j(t)\|^2-(d_{ij}^\star)^2\right)^2$$

and assume the desired distances $d_{ij}^\star$ correspond to a feasible formation. Then the gradient dynamical system

$$u_i = -\nabla_{p_i}F_f(p) = \sum_{ij\in\mathcal{E}}\left(\|p_i-p_j\|^2-(d_{ij}^\star)^2\right)(p_j-p_i)$$

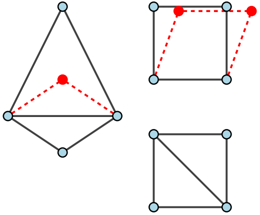
$$\dot{p} = -\nabla_p F_f(p) = -R^T(p)R(p)p + R^T(p)(d^\star)^2$$

asymptotically converges to the critical points of the potential function, i.e., $\frac{\partial F_f(p)}{\partial p} = 0$.

- $R(p)$ is the *rigidity matrix* for the framework $(\mathcal{G}, p)$
- rigidity theory used to understand more about the equilibrium sets

Rigidity theory helps us understand

- how many constraints are required to ensure uniqueness of formation shape (modulo translations, rotations, and flip ambiguities)
- how the constraints should be distributed in the network



A widely accepted architectural requirement for distance constrained formation control is that minimally infinitesimally rigid frameworks are required. Equivalent to:

$$\operatorname{rk} R(p) = 2|\mathcal{V}| - 3 \text{ and } |\mathcal{E}| = 2|\mathcal{V}| - 3 \quad (\text{in } \mathbb{R}^2)$$

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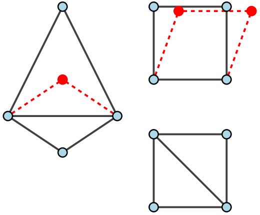

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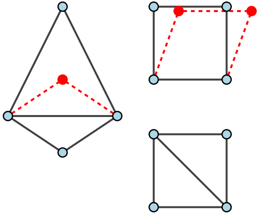

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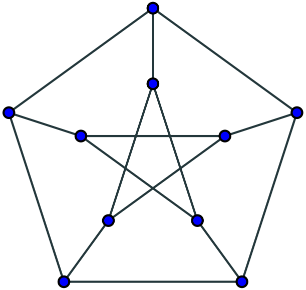
Q: is this a necessary condition? (can we solve the problem with fewer edges?)

A: Impose additional symmetry constraints without requiring more information exchange (in fact, less!)

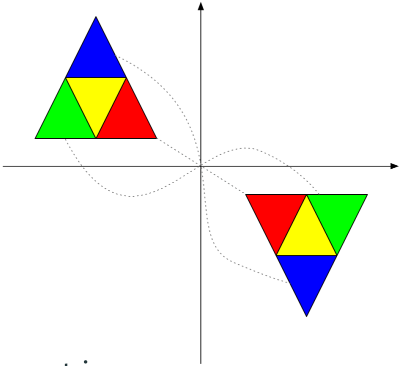
GRAPH SYMMETRIES AND POINT GROUPS

Graph Symmetries



- graph automorphisms

Point Groups

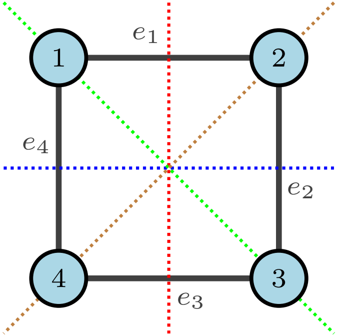


- isometries

Graph Automorphism

An automorphism of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of of its vertex set such that

$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$



Automorphisms encode graph symmetries

- identity: $\mathrm{Id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$
- $90°$ rotation: $\psi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
- $180°$ rotation: $\psi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
- $270°$ rotation: $\psi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

Graph Automorphism

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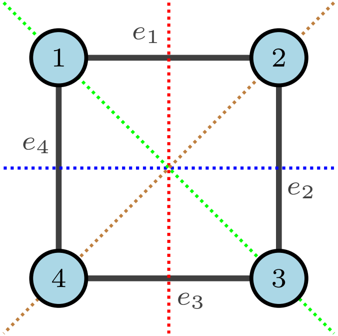

Automorphisms encode graph symmetries

- reflection: $\psi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
- reflection: $\psi_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
- reflection: $\psi_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$
- reflection: $\psi_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

Definition

Let X be a set, and let Γ be a collection of invertible functions $X \to X$. Then Γ is called a group if the identity map, Id, belongs to Γ, and for any $\Gamma \ni f, g : X \to X$, both the composite function $f \circ g$ and the inverse function f^{-1} belong to Γ.

Automorphisms of a graph form a *group* - $\mathrm{Aut}(\mathcal{G})$

- $\mathrm{Aut}(\mathcal{G}) = \{\mathrm{Id}, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\}$

A subgroup is a subset of a group, and also satisfies all properties of a group

- $\{\mathrm{Id}, \psi_1, \psi_2, \psi_3\}$
- $\{\mathrm{Id}, \psi_2, \psi_4, \psi_5\}$
- $\{\mathrm{Id}, \psi_2\}$
- $\{\mathrm{Id}, \psi_6\}$
- $\{\mathrm{Id}, \psi_7\}$

Γ-SYMMETRIC GRAPHS

- Subgroups of $\mathrm{Aut}(\mathcal{G})$ define specific symmetries in $\mathcal{G}$
- for any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$, we say that $\mathcal{G}$ is Γ-symmetric

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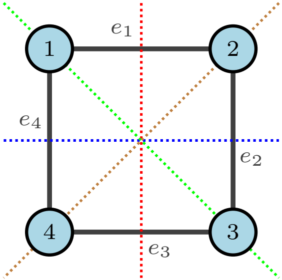
Definition

For a Γ-symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and vertex $i \in \mathcal{V}$, the set $\Gamma_i = \{\gamma(i) \,|\, \gamma \in \Gamma\}$ is called the vertex orbit of i. Similarly, for an edge $e = ij \in \mathcal{E}$, the set $\Gamma_e = \{\gamma(i)\gamma(j) \,|\, \gamma \in \Gamma\}$ is termed the edge orbit of e.



Consider $\Gamma = \{\mathrm{Id}, \psi_2\}$ (ψ_2 is the $180°$ rotation)

- **Vertex Orbit:**
  $\Gamma_1 = \Gamma_3 = \{1, 3\},\ \Gamma_2 = \Gamma_4 = \{2, 4\}$

  vertices inside a vertex orbit are equivalent

  representative vertex set: $\mathcal{V}_0 = \{1, 2\}$

- **Edge Orbit:**
  $\Gamma_{e_1} = \Gamma_{e_3} = \{e_1, e_3\}$,
  $\Gamma_{e_2} = \Gamma_{e_4} = \{e_2, e_4\}$
  representative edge set: $\mathcal{E}_0 = \{e_1, e_2\}$

combine notions of graph symmetries with point groups

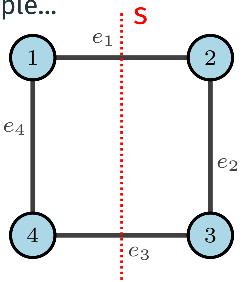
- let $\mathcal{G}$ be a Γ-symmetric graph
- Γ also represented as a *point group*
  - a set of isometries that preserve symmetries
  - homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$
  - τ assigns an orthogonal matrix (describing an isometry of $\mathbb{R}^d$ such as a rotation or reflection) to each element of Γ

Definition

A framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called $\tau(\Gamma)$-symmetric if

$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}.$$

example...



- consider $\Gamma = \{\mathrm{Id}, \psi_4\} \subseteq \mathrm{Aut}(\mathcal{G})$
- $\gamma = \psi_4 \in \Gamma$ (reflection about mirror S)
- isometry $\tau(\gamma) : (a, b) \mapsto (-a, b)$

satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$.

- note: for a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ and for every $j \in \Gamma_i$, there is a $\gamma_j \in \Gamma$ such that $\tau(\gamma_j)p_j = p_i$ for all $j \in \Gamma_i$

isometries of configuration p coincide with symmetries of the automorphisms of $\mathcal{G}$

- in $\tau(\Gamma)$-symmetric frameworks, the configurations p are in a special geometric position (not necessarily generic)
- symmetry can lead to unexpected infinitesimal flexibility/rigidity

Definition

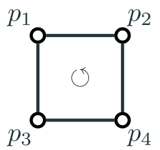
An infinitesimal motion u of a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric* if

$$\tau(\gamma)(u_i) = u_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}. \tag{1}$$

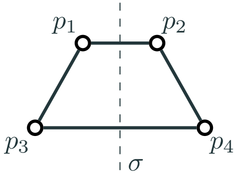
We say that $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric infinitesimally rigid* if every $\tau(\Gamma)$-symmetric infinitesimal motion is trivial.

- recall that infinitesimal motions are in the kernel of the rigidity matrix
- we can find a subspace of the kernel that is isomorphic to the space of 'fully-symmetric' infinitesimal motions
- velocity assignments to the points of $(\mathcal{G}, p)$ that exhibit exactly the same symmetry as the configuration p

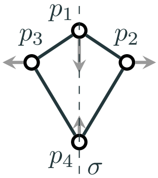
SYMMETRIC RIGIDITY



(a)



(b)



(c)

- $\mathcal{C}_{4v}$-symmetric (and hence $\tau(\Gamma)$-symmetric for any subgroup $\tau(\Gamma)$ of $\mathcal{C}_{4v}$)
- $\tau(\Gamma)$-symmetric infinitesimally rigid

- $\mathcal{C}_s$-symmetric (with respect to the reflection σ)
- $\tau(\Gamma)$-symmetric infinitesimally rigid

- $\mathcal{C}_s$-symmetric (with respect to the reflection σ) with a non-trivial $\mathcal{C}_s$-symmetric infinitesimal motion
- $\tau(\Gamma)$-symmetric infinitesimally flexible

Symmetric Formation Control Objective

Consider a group of n integrator agents that interact over the Γ-symmetric sensing graph $\mathcal{G}$. Let $p^\star \in \mathbb{R}^{dn}$ be a configuration such that $(\mathcal{G}, p^\star)$ is $\tau(\Gamma)$-symmetric for some desired point group $\tau(\Gamma)$, and let $\mathcal{V}_0$ be a set of representatives of the vertex orbits of $\mathcal{G}$ under Γ. Design a control $u_i(t)$ for each agent i such that

(i) $\lim\limits_{t\to\infty} \|p_i(t) - p_j(t)\| = \|p_i^\star - p_j^\star\| = d_{ij}^\star$ for all $ij \in \mathcal{E}$; (distance constraints)

(ii) $\lim\limits_{t\to\infty} \|p_u(t) - \tau(\gamma_{vu}) p_v(t)\| = 0$ for all $u, v \in \Gamma_i$, $i \in \mathcal{V}_0$. (symmetry constraints)

A GRADIENT APPROACH

- the formation potential

$$F_f(p(t)) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2 \right)^2$$

A GRADIENT APPROACH

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- the symmetry potential

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu}) p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the formation potential

$$F_f(p(t)) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2 \right)^2$$

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Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the symmetric formation potential

$$F(p(t)) = F_f(p(t)) + F_s(p(t))$$

A GRADIENT APPROACH

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

A GRADIENT APPROACH

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- closed-loop dynamics

$$\dot{p}(t) = -R(p(t))^T \left(R(p(t))p(t) - (d^\star)^2\right) - Qp(t)$$

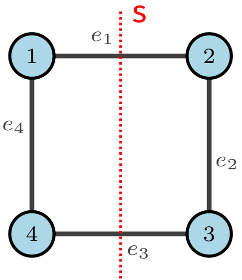
where Q is symmetric and a block-diagonal matrix with

$$[Q_i]_{uv} = \begin{cases} d_{\Gamma_i}(u)I, & u = v,\, u \in \Gamma_i \\ -\tau(\gamma_{uv}), & uv \in \mathcal{E}, u, v \in \Gamma_i \\ 0, & \text{o.w.} \end{cases} .$$

- $Q_i \in \mathbb{R}^{|\Gamma_i|d \times |\Gamma_i|d}$
- $[Q]_{uv} \in O(\mathbb{R}^d)$ (orthogonal group)
- $\tau(\gamma_{uv})^{-1} = \tau(\gamma_{uv})^T$

"NICE" GRAPHS

- symmetric formation potential makes no assumption on relation between the graph $\mathcal{G}$ and the point group $\tau(\Gamma)$
- we restrict our study to graphs where communication required by symmetric potential use same edges as $\mathcal{G}$



- $\Gamma = \{\mathrm{Id}, \psi_4\} \subseteq \mathrm{Aut}(\mathcal{G})$
- $\Gamma_1 = \Gamma_2 = \{1, 2\},\ \Gamma_3 = \Gamma_4 = \{3, 4\}$
- $\mathcal{V}_0 = \{1, 4\}$
- isometry $\tau(\gamma) : (a, b) \mapsto (-a, b)$

satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$ and
for each $i \in \mathcal{V}_0$ and $j \in \Gamma_i \setminus \{i\}$,
the edge ij is in $\mathcal{E}$ (i.e. $\mathcal{G}(\Gamma_i)$ is connected)

A GRADIENT APPROACH

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- closed-loop dynamics

$$\dot{p}(t) = -R(p(t))^T \left(R(p(t))p(t) - (d^\star)^2\right) - Qp(t)$$

- dynamics at for each agent

$$\dot{p}_i(t) = \sum_{ij \in \mathcal{E}} (\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2)(p_j(t) - p_i(t)) + \sum_{\substack{ij \in \mathcal{E} \\ i,j \in \Gamma_u}} (\tau(\gamma_{ij})p_j(t) - p_i(t)).$$

Theorem

Consider a team of n integrator agents interacting over a Γ-symmetric graph $\mathcal{G}$ satisfying Assumption 1 that can be drawn with maximum point group symmetry $\mathcal{S}$ in $\mathbb{R}^d$, and let

$$\mathcal{F}_f = \{p \in \mathbb{R}^{dn} \mid \|p_i - p_j\| = d^\star_{ij}\; ij \in \mathcal{E}\}, \text{ and } \mathcal{F}_s = \{p \in \mathbb{R}^{dn} \mid \tau(\gamma)(p_i) = p_{\gamma(i)}\; \forall \gamma \in \Gamma,\; i \in \mathcal{V}\}.$$

Then for initial conditions $p_i(0)$ satisfying

$$\sum_{ij\in\mathcal{E}} (\|p_i(0) - p_j(0)\| - d^\star_{ij})^2 \le \epsilon_1, \text{ and } \|p_i(0) - \tau(\gamma_{ij})p_j(0)\|^2 \le \epsilon_2$$

for all $i, j \in \Gamma_u$ and $u \in \mathcal{V}_0$, for a sufficiently small and positive constant ϵ_1 and ϵ_2, the control

$$u = -\nabla F(p(t)),$$

renders the set $\mathcal{F}_f \cap \mathcal{F}_s$ exponentially stable, i.e.

$$\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = d^\star_{ij} \text{ and } \lim_{t\to\infty} \tau(\gamma)(p_i(t)) = \lim_{t\to\infty} p_{\gamma(i)}(t) \quad \text{for all } \gamma \in \Gamma, i \in \mathcal{V}.$$

Proof Sketch

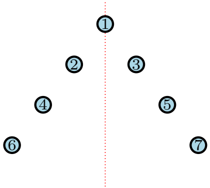
- observe the invariant quantity (group average)

$$z(t) = \sum_{v \in \mathcal{V}} \sum_{\gamma \in \Gamma} \tau(\gamma) p_v(t)$$

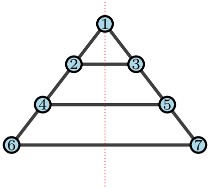
- combine with stability properties of gradient dynamical systems

EXAMPLE: THE VIC FORMATION

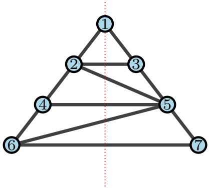
- formation flight for aircraft originated in WWI
- Vic formation used by pilots to improve visual communication and defensive advantages



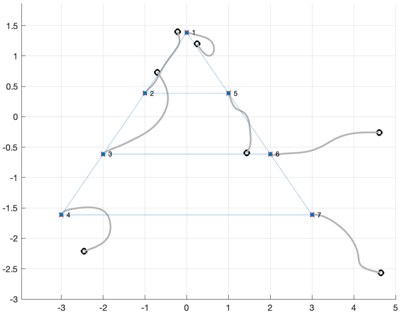
Vic formation with symmetry mirror

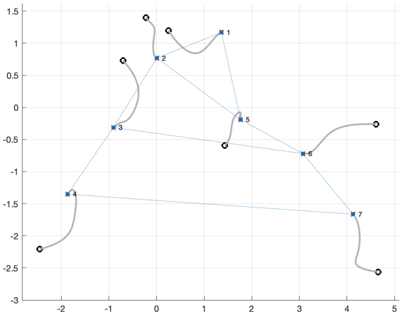


Flexible framework (9 edges; satisfies Assumption 1)



Minimally Rigid framework (11 edges)





- symmetry constraints force agents to correct formation
- requires less agent communication than standard formation control with MIR requirement

- with flexible framework and only formation potential can not guarantee convergence to correct shape

- proposed strategy does not take advantage of the full power of symmetry

EXPLOIT MORE SYMMETRY

- proposed strategy does not take advantage of the full power of symmetry
- can we find redundant information between the symmetry constraints and the distance constraints?

Definition

An infinitesimal motion u of a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric* if

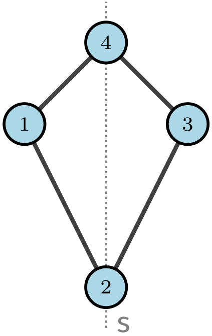
$$\tau(\gamma)(u_i) = u_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}. \tag{2}$$

We say that $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric infinitesimally rigid* if every $\tau(\Gamma)$-symmetric infinitesimal motion is trivial.

infinitesimal motions can also be studied in this framework

- $\tau(\gamma)(u_i) = u_{\theta(\gamma)(i)}$
- understanding symmetry structure means we only need to find infintesimal motion for one representative vertex in each vertex orbit

EXAMPLE



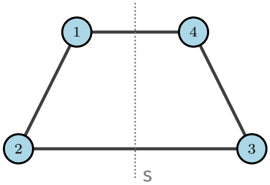
$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (0, c)^T$
- $p_3 = (-a, b)^T$
- $p_4 = (0, d)^T$

$$R(p) = \begin{bmatrix} (a \ b-c) & (-a \ c-b) & (0 \ 0) & (0 \ 0) \\ (a \ b-d) & (0 \ 0) & (0 \ 0) & (-a \ d-b) \\ (0 \ 0) & (a \ c-b) & (-a \ b-c) & (0 \ 0) \\ (0 \ 0) & (0 \ 0) & (-a \ b-d) & (a \ d-b) \end{bmatrix}$$

- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by
$(-1 \ 0 \ 0 \ \frac{a}{c-b} \ 1 \ 0 \ 0 \ \frac{a}{d-b})^T$
flex is symmetric! with respect to s
($\tau(\gamma) : (a, b) \mapsto (-a, b)$)



$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-c, d)^T$
- $p_4 = (-a, b)^T$

Rigidity matrix

$$R(p) = \begin{bmatrix} (a-c \;\; b-d) & (c-a \;\; d-b) & (0 \;\; 0) & (0 \;\; 0) \\ (2a \;\; 0) & (0 \;\; 0) & (0 \;\; 0) & (-2a \;\; 0) \\ (0 \;\; 0) & (2c \;\; 0) & (-2c \;\; 0) & (0 \;\; 0) \\ (0 \;\; 0) & (0 \;\; 0) & (a-c \;\; d-b) & (c-a \;\; b-d) \end{bmatrix}$$

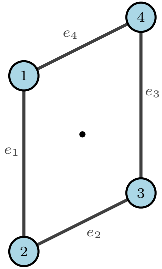
- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by

$\begin{pmatrix} -1 & -1 & -1 & \frac{2(c-a)+b-d}{d-b} & -1 & -\frac{2(c-a)+b-d}{d-b} & 1 & 1 \end{pmatrix}^T$

flex is not symmetric with respect to s

EXAMPLE



$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-a, -b)^T$
- $p_4 = (-c, -d)^T$

$$R(p) = \begin{bmatrix} (a-c \;\; b-d) & (c-a \;\; d-b) & (0 \;\; 0) & (0 \;\; 0) \\ (a+c \;\; b+d) & (0 \;\; 0) & (0 \;\; 0) & (-a-c \;\; -b-d) \\ (0 \;\; 0) & (0 \;\; 0) & (c-a \;\; d-b) & (a-c \;\; b-d) \\ (0 \;\; 0) & (a+c \;\; b+d) & (-a-c \;\; -b-d) & (0 \;\; 0) \end{bmatrix}$$

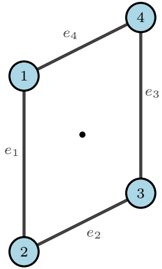
- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by

$(-1 \;\; 0 \;\; \frac{cd-ab}{ad-bc} \;\; \frac{a^2-c^2}{ad-bc} \;\; 1 \;\; 0 \;\; -\frac{cd-ab}{ad-bc} \;\; -\frac{a^2-c^2}{ad-bc})^T$

flex is symmetric with respect to $180°$ rotation ($\mathcal{C}_2$)

EXAMPLE



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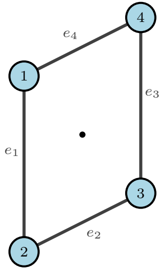
- 180° rotation of points corresponds to $\psi_2 \in \mathrm{Aut}(\mathcal{G})$
- recall: vertex orbits : $\{1, 3\}$, $\{2, 4\}$, edge orbits: $\{e_1, e_3\}$, $\{e_2, e_4\}$

EXAMPLE



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symmetries make certain rows and columns of the rigidity matrix redundant

ORBIT RIGIDITY MATRIX

symmetries make certain rows and columns of the rigidity matrix redundant

$$R(p) = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 = \mathcal{C}_2(1) & 4 = \mathcal{C}_2(2) \end{array} \\ \begin{array}{c} e_1 \\ e_4 \\ \mathcal{C}_2(e_1) \\ \mathcal{C}_2(e_4) \end{array} & \begin{pmatrix} (a-c \;\; b-d) & (c-a \;\; d-b) & (0 \;\; 0) & (0 \;\; 0) \\ (a+c \;\; b+d) & (0 \;\; 0) & (0 \;\; 0) & (-a-c \;\; -b-d) \\ (0 \;\; 0) & (0 \;\; 0) & (c-a \;\; d-b) & (a-c \;\; b-d) \\ (0 \;\; 0) & (a+c \;\; b+c) & (-a-c \;\; -b-d) & (0 \;\; 0) \end{pmatrix} \end{array}$$

symmetries make certain rows and columns of the rigidity matrix redundant

$$R(p) = \begin{array}{c} \\ e_1 \\ e_4 \\ \mathcal{C}_2(e_1) \\ \mathcal{C}_2(e_4) \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3=\mathcal{C}_2(1) & 4=\mathcal{C}_2(2) \end{array} \\ \begin{pmatrix} (a-c\ \ b-d) & (c-a\ \ d-b) & (0\ \ 0) & (0\ \ 0) \\ (a+c\ \ b+d) & (0\ \ 0) & (0\ \ 0) & (-a-c\ \ -b-d) \\ (0\ \ 0) & (0\ \ 0) & (c-a\ \ d-b) & (a-c\ \ b-d) \\ (0\ \ 0) & (a+c\ \ b+c) & (-a-c\ \ -b-d) & (0\ \ 0) \end{pmatrix} \end{array}$$

Orbit Rigidity Matrix

$$\begin{array}{c} \\ e_1 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} (p_1-p_2)^T & (p_2-p_1)^T \\ (p_1-\mathcal{C}_2(p_2))^T & (p_2-\mathcal{C}_2^{-1}(p_1))^T \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} (a-c,\ b-d) & (c-a,\ d-b) \\ (a+c,\ b+d)) & (c+a,\ d+b) \end{pmatrix} \end{array}$$

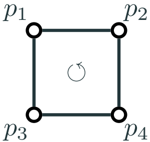
- 2 rows - one for each representative of edge orbits under action of $\mathcal{C}_2$
- 4 columns - nodes p_1, p_2 each have two dof; nodes $p_3 = \mathcal{C}_2(p_1)$ and $p_4 = \mathcal{C}_2(p_2)$ are uniquely determined by the symmetries

- relation between vertices within vertex orbits and between vertex orbits (through edge orbits) captured by quotient gain graph of a Γ-symmetric graph
  - node set is representative vertex set $\mathcal{V}_0$
  - edge set is representative edge set $\mathcal{E}_0$: choose edge of form $i\gamma(j)$ with $i, j \in \mathcal{V}_0$

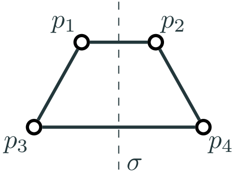
    it is ok for $i = j$

    edges are directed with 'edge gain' being the group action $\gamma \in \Gamma$

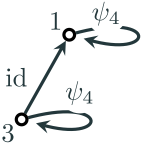
QUOTIENT GAIN GRAPHS

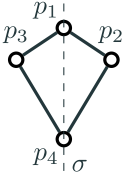


- $\Gamma = \{\mathrm{Id}, \psi_1\}$ (rotation)
- $\Gamma_i = \{1, 2, 3, 4\}$
- $\mathcal{V}_0 = \{1\}$, $\mathcal{E}_0 = \{e_1\}$

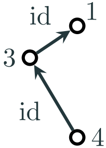


- $\Gamma = \{\mathrm{Id}, \psi_4\}$ (reflection)
- $\Gamma_{1,2} = \{1, 2\}$, $\Gamma_{3,4} = \{3, 4\}$
- $\mathcal{V}_0 = \{1, 3\}$,
  $\mathcal{E}_0 = \{12,\ 13,\ 24\}$





- $\Gamma = \{\mathrm{Id}, \psi_6\}$ (reflection)
- $\Gamma_1 = \{1\}$, $\Gamma_4 = \{4\}$,
  $\Gamma_{2,3} = \{2, 3\}$
- $\mathcal{V}_0 = \{1, 3, 4\}$, $\mathcal{E}_0 = \{13,\ 14\}$



Definition [Shulze 2011]

For a Γ-symmetric framework $(\mathcal{G}, p)$ with quotient gain Γ-gain graph $(\mathcal{G}_0, w)$, the orbit rigidity matrix, $\mathcal{O}(\mathcal{G}_0, w, p)$, is the $|\mathcal{E}_0| \times d|\mathcal{V}_0|$ matrix defined as follows. Choose a representative vertex $\tilde{i}$ for each vertex Γ_i in $\mathcal{V}_0$. The row corresponding to the edge $\tilde{e} = (\tilde{i}, \tilde{j})$ with gain $w(\tilde{e})$ in $\mathcal{E}_0$ is given by

$$(0 \;\cdots\; 0 \;\underbrace{p(\tilde{i}) - \tau(w(\tilde{e}))p(\tilde{j})}_{\tilde{i}}\; 0 \;\cdots\; 0 \;\underbrace{p(\tilde{j}) - \tau(w(\tilde{e}))^{-1}p(\tilde{j})}_{\tilde{i}}\; 0 \;\cdots\; 0).$$

If $\tilde{e} = (\tilde{i}, \tilde{i})$ is a loop at $\tilde{i}$, then the row corresponding to $\tilde{e}$ is given by

$$(0 \;\cdots\; 0 \;\underbrace{2p(\tilde{i}) - \tau(w(\tilde{e}))p(\tilde{i}) - \tau(w(\tilde{e}))^{-1}p(\tilde{i})}_{\tilde{i}}\; 0 \;\cdots\; 0\; 0\; 0 \;\cdots\; 0).$$

ORBIT RIGIDITY MATRIX

Theorem [Shulze 2011]

The kernel of the orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, w, p)$ is the space of (w, Γ)-symmetric infinitesimal motions of $(\mathcal{G}, p)$ restricted to the set of vertex orbits Γ_i of $\mathcal{G}$.

- Orbit rigidity matrix can be used to identify symmetric infinitesimal flexes
- full-rank $\mathcal{O}(\mathcal{G}_0, w, p)$ implies none exist
- size of $\mathcal{O}(\mathcal{G}_0, w, p)$ does not depend on p, but only the graph and symmetry constraints

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key point: quotient gain graph and orbit rigidity matrix suggests a further way to exploit symmetry in formation control

- representative edges used to maintain distances
- symmetry within vertex orbits have no need for distance constraints

A MODIFIED FORMATION POTENTIAL

- the representative edge formation potential

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij\in\mathcal{E}_0} \left(\|p_i - \tau(\gamma_{ij})p_j\|^2 - (d_{ij}^{\star})^2\right)^2 .$$

- the representative edge formation potential

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij \in \mathcal{E}_0} \left(\|p_i - \tau(\gamma_{ij}) p_j\|^2 - (d_{ij}^\star)^2 \right)^2 .$$

- the symmetry potential

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu}) p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

A MODIFIED FORMATION POTENTIAL

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$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij \in \mathcal{E}_0} \left(\|p_i - \tau(\gamma_{ij}) p_j\|^2 - (d_{ij}^\star)^2 \right)^2 .$$

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Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the symmetric formation potential

$$F(p(t)) = F_e(p(t)) + F_s(p(t))$$

A MODIFIED FORMATION CONTROL

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

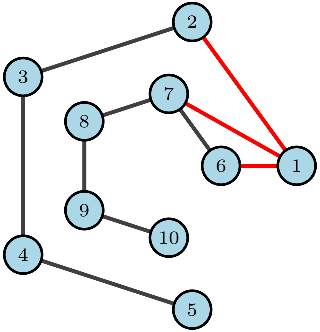
- closed-loop dynamics

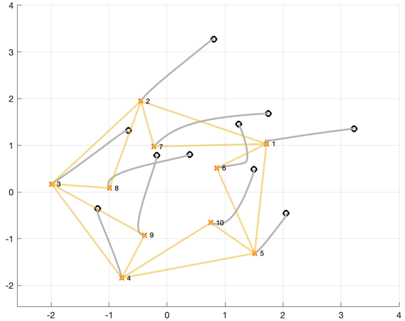
$$\dot{p}(t) = -\mathcal{O}^T \left(\mathcal{O}p(t)_{|\mathcal{V}_0} - (d^\star_{|\mathcal{E}_0})^2 \right) - Qp(t)$$

- structure idea
  - representative vertices in $\mathcal{V}_0$ take care of distances
  - other vertices just maintain symmetry constraints

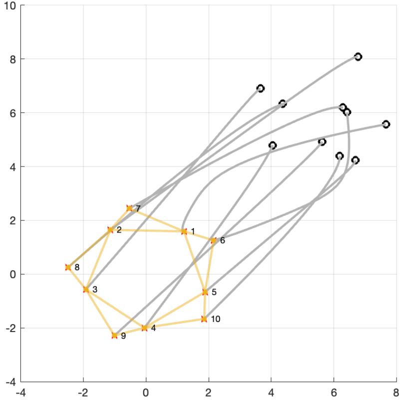
EXAMPLE

- $\mathcal{V}_0 = \{1, 6\}$
- $\mathcal{E}_0 = \{16, 17, 12\}$

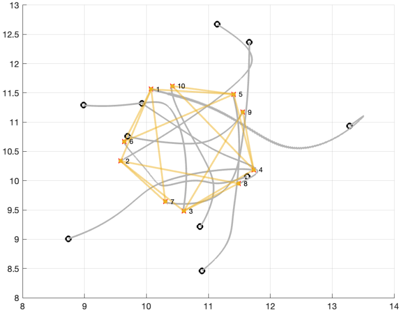




- strategy requires only 3 distance constraints and 8 symmetry constraints
- compared to 17 distance constraint for MIR classic approach



- symmetry relies on a fixed inertial frame



- can add consensus term to agree on arbitrary centroid

Symmetry preserving motion coordination aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body and preserving symmetry of configuration.

BACK TO MOTION COORDINATION

Symmetry preserving motion coordination aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body and preserving symmetry of configuration.

- can we maneuver a symmetric formation in space?
- if we relax rigidity requirement, can you introduce symmetry-preserving motions?

Theorem - Distance Constrained Formation Control

Consider the potential function

$$V(p) = \frac{1}{4}\sum_{i\sim j}\left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2\right)^2$$

and assume the desired distances $d_{ij}^\star$ correspond to a feasible formation. Then the gradient dynamical system

$$\dot{p} = -\nabla_p V(p) = -R^T(p)R(p)p + R^T(p)(d^\star)^2$$

asymptotically converges to the critical points of the potential function, i.e., $\frac{\partial V(p)}{\partial p} = 0$.

- $R(p)$ is the *rigidity matrix* for the framework $(\mathcal{G}, p)$
- rigidity theory used here to understand more about the equilibrium sets

PROOF SKETCH

(following De Queiroz ’18)

Define some notations...

- relative positions: $\tilde{p}_{ij} = p_i - p_j$
- distance error: $e_{ij} = \|\tilde{p}_{ij}\| - d^\star_{ij}$
- intermediate variable: $z_{ij} = \|\tilde{p}_{ij}\|^2 - (d^\star_{ij})^2 = e_{ij}(e_{ij} + 2d^\star_{ij})$

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introduce Lyapunov candidate:

$$V(e) = \frac{1}{4}\sum_{i\sim j} z_{ij}^2 = z^T z$$

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time-derivative of Lyapunov function along trajectories

$$\boxed{\dot{V} = z^T R(p) u}$$

IDEA: Design control u to ensure Lyapunov function is decreasing!

- Formation acquisition: $u = -R(p)^T z$
  ensures stable formation dynamics
  "classic" distance-constrained formation controller

Formation maneuvering aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body.

...recall our earlier Lyapunov function

$$\dot{V} = z^T R(p) u$$

choose $u = u_a + u_m$

- $u_a = -R(p)^T z$: used to attain desired formation
- $u_m = \mathbb{1} \otimes v_0 + \begin{bmatrix} \vdots \\ \omega_0 \times \tilde{q}_i \\ \vdots \end{bmatrix}$: rigid body translation (v_0) and rotation about a point ($\omega_0 \times \tilde{q}_i$)

Main Idea: rigid body rotations and translations are in the Kernel of the rigidity matrix!

BACK TO MOTION COORDINATION

...recall our earlier Lyapunov function

$$\dot{W} = z^T R(p) u$$

choose $u = u_a + u_m + u_s$

- $u_a = -R(p)^T z$: used to attain desired formation
- $u_m = \mathbb{1} \otimes v_0 + \begin{bmatrix} \vdots \\ \omega_0 \times \tilde{q}_i \\ \vdots \end{bmatrix}$: rigid body translation (v_0) and rotation about a point ($\omega_0 \times \tilde{q}_i$)
- u_s obtained from kernel of Orbit rigidity matrix

SYMMETRY PRESERVING MOTION COORDINATION

Summary

- exploit notions of symmetry in formation control
- $\tau(\Gamma)$-symmetric graphs captures symmetry of configurations and graphs
- symmetric formation potential used to design distributed control law with less edges compared to "traditional" formation control strategies
- opportunities for more sophisticated motion coordination

Future Work

- formation maneuvering requires time-varying point group symmetries
- is it possible to distributedly decide on certain symmetries?
- can we eliminate need for requiring self-state in protocol?
- more?

Questions?