Uncertain Consensus Networks: Robustness and its Connection to Effective Resistance

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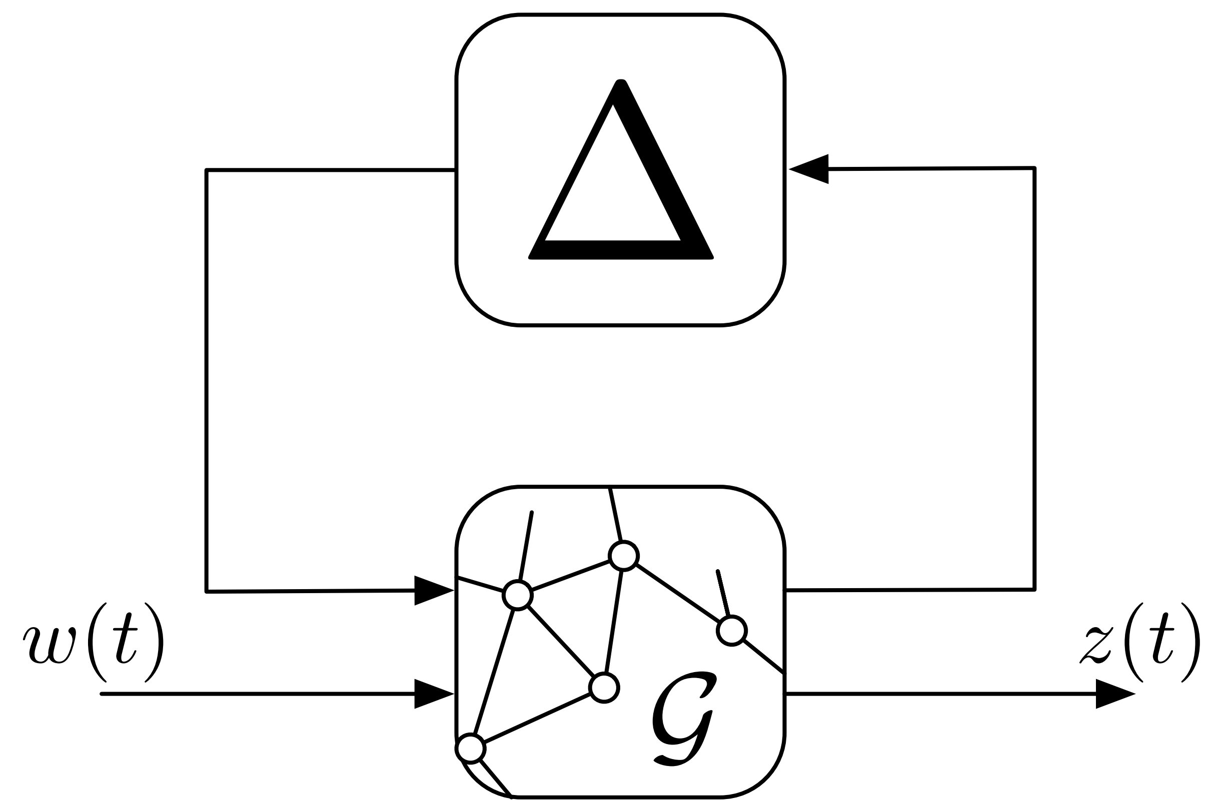

2nd Swedish-Israeli Control Conference
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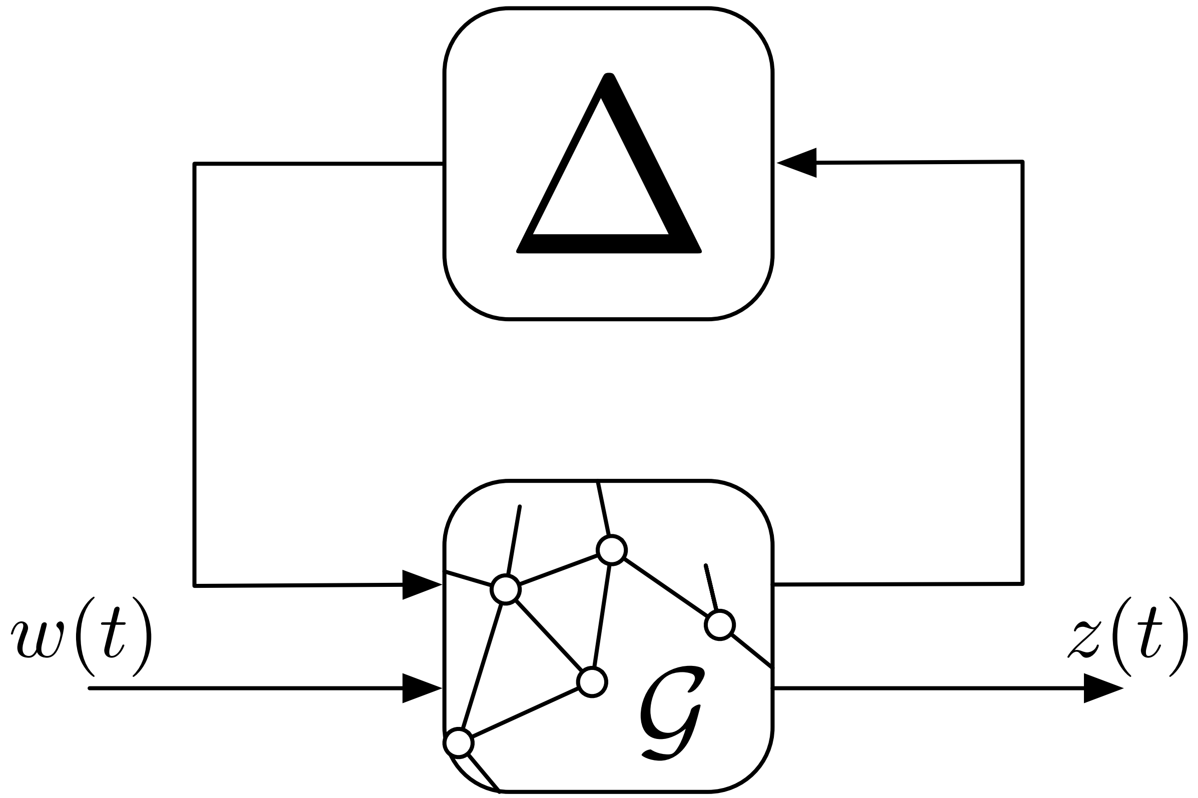


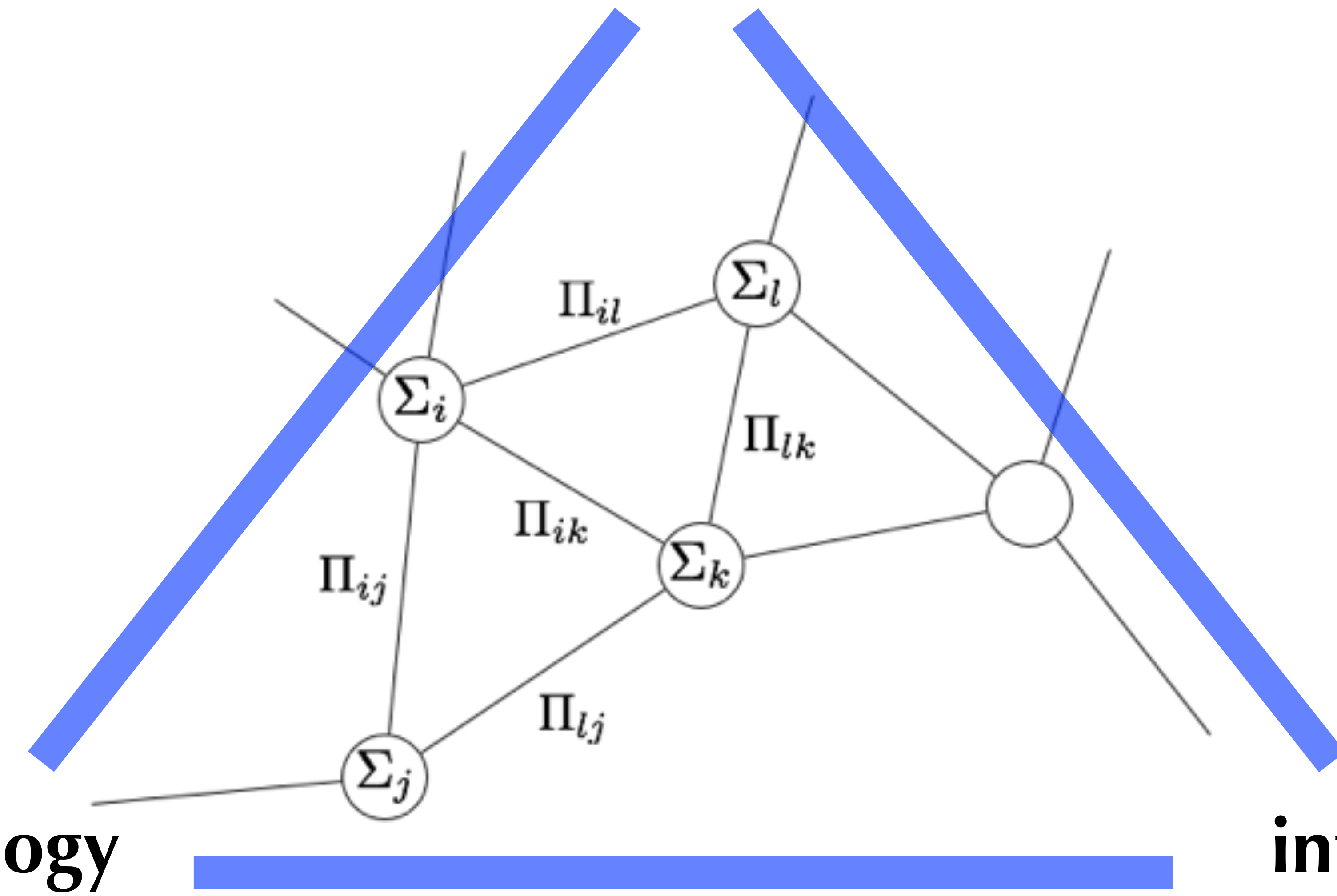
Networked Dynamic Systems

networks of dynamical systems are one of *the* enabling technologies of the future

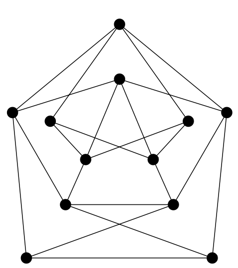
Networked Dynamic Systems

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t))$$

dynamics



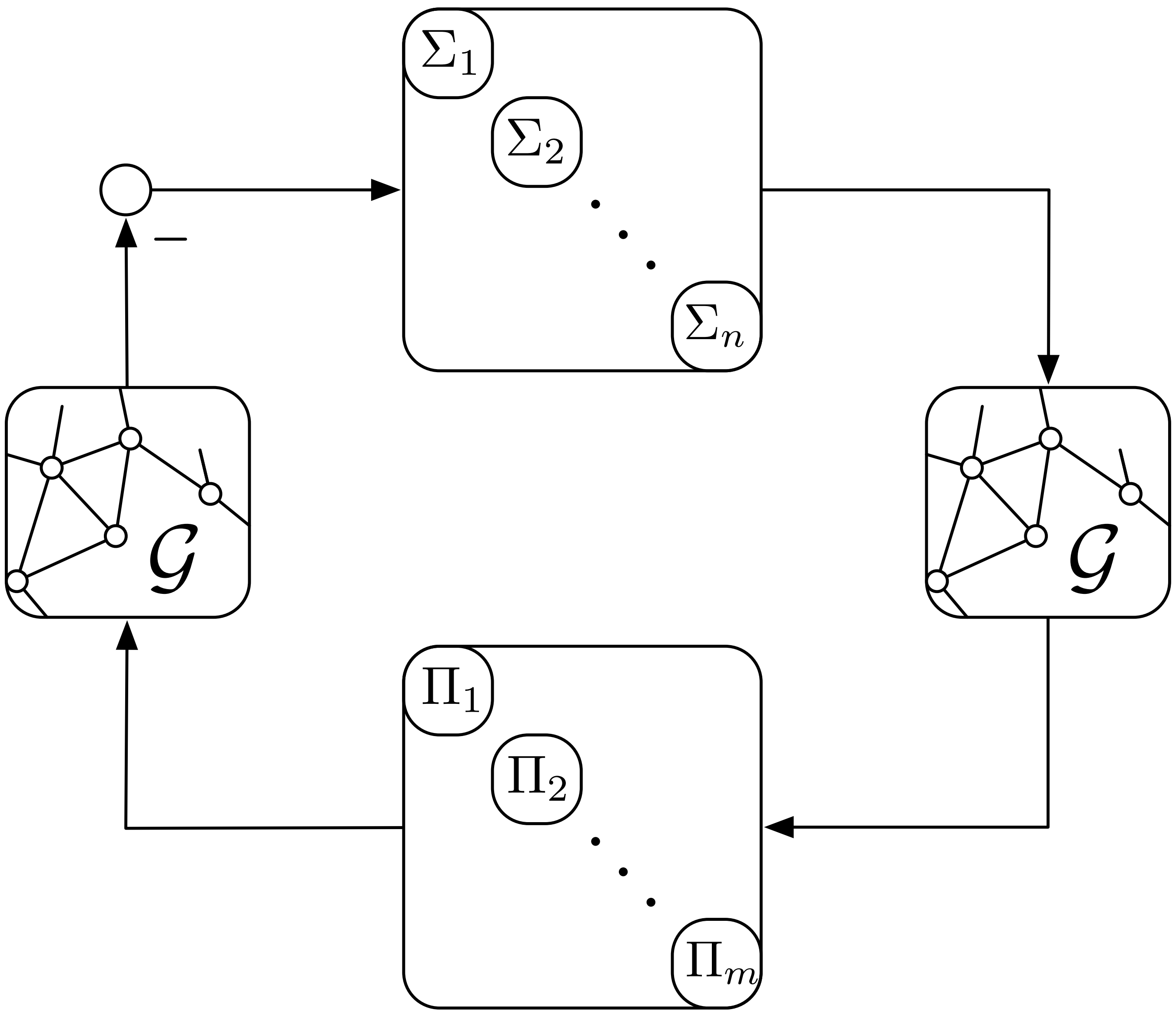
topology
(graph)



interaction
protocol

$$u_i(t) = \Pi_i(x(t), \mathcal{G})$$

Diffusively Coupled Networks



Kumamoto Model

$$\dot{\theta}_i = -k \sum_{i \sim j} \sin(\theta_i - \theta_j)$$

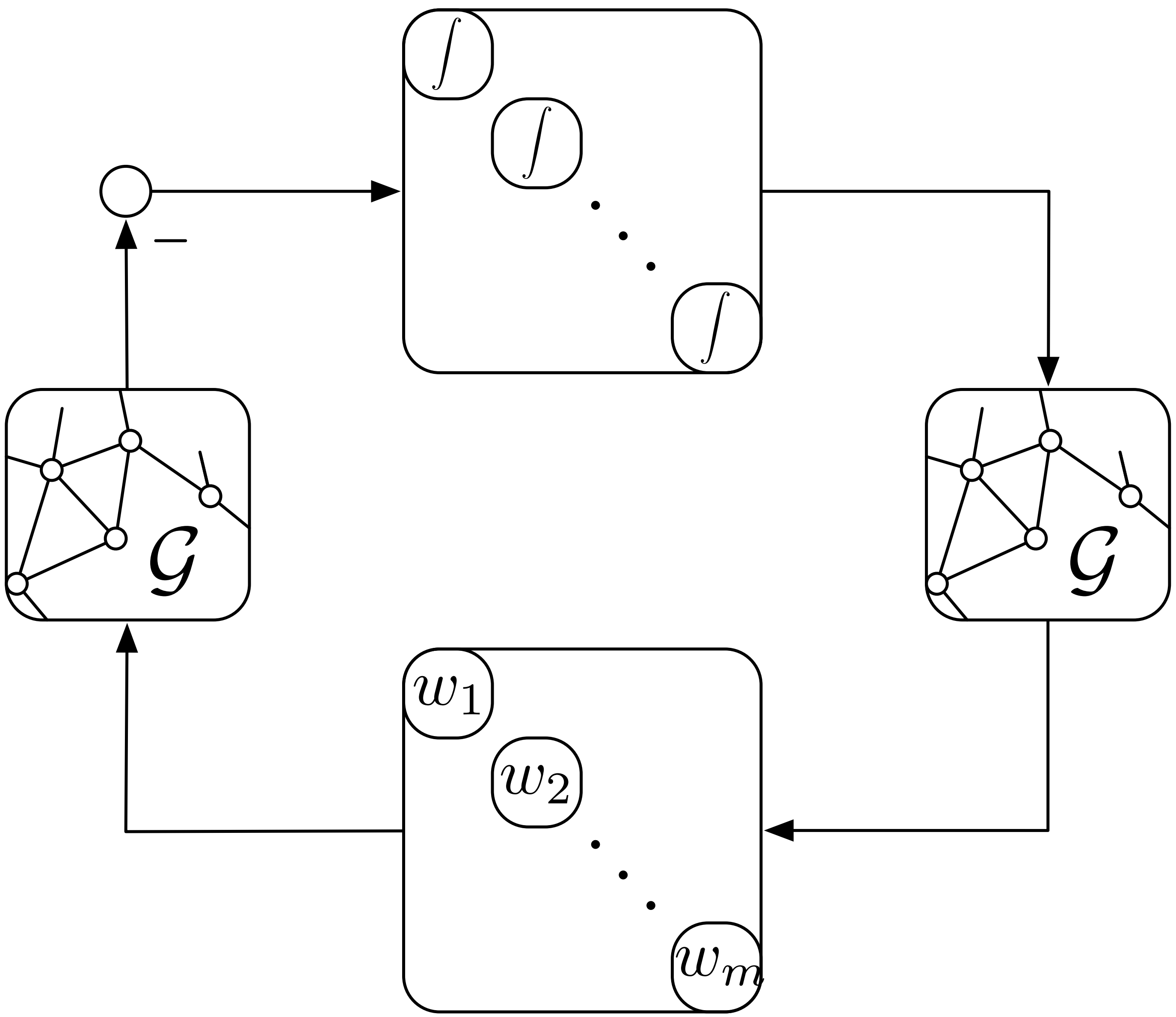
Traffic Dynamics Model

$$\dot{v}_i = \kappa_i \left(V_i^0 - v_i + V_i^1 \sum_{i \sim j} \tanh(p_j - p_i) \right)$$

Neural Network

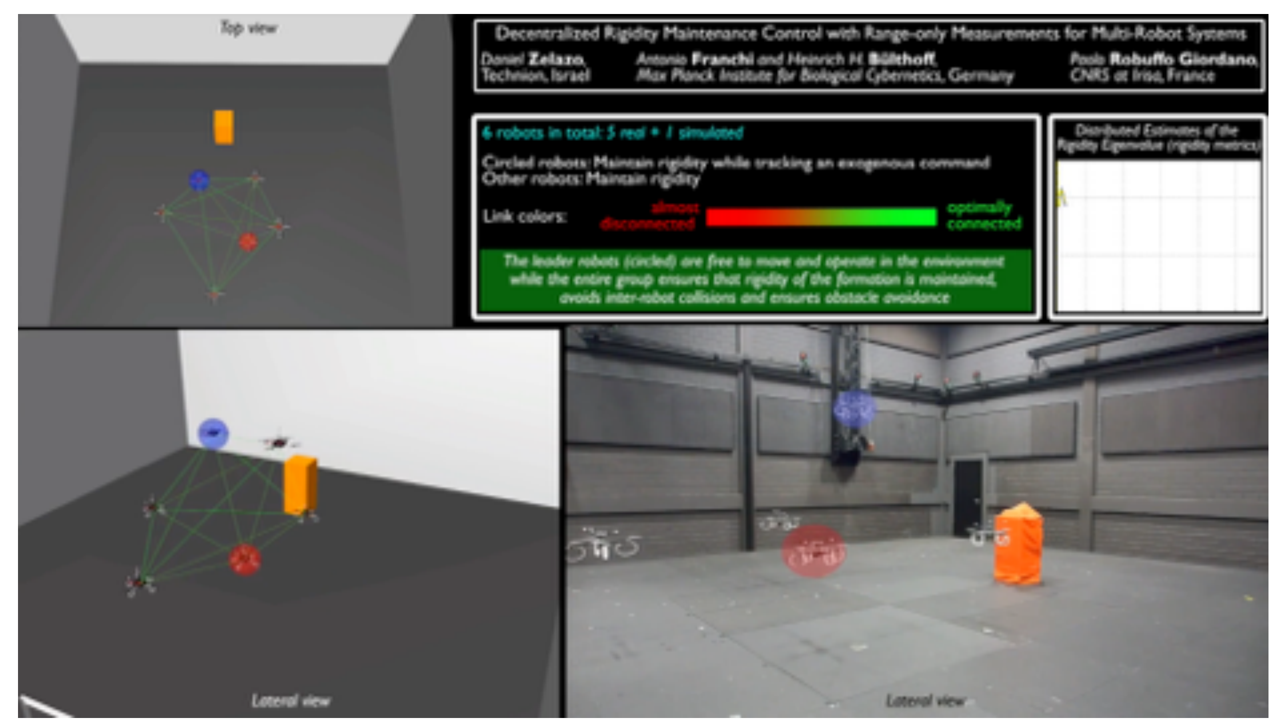
$$\begin{aligned} C\dot{V}_i &= f(V_i, h_i) + \textstyle\sum_{i \sim j} g_{ij}(V_j - V_i) \\ \dot{h}_i &= g(V_i, h_i) \end{aligned}$$

Diffusively Coupled Networks



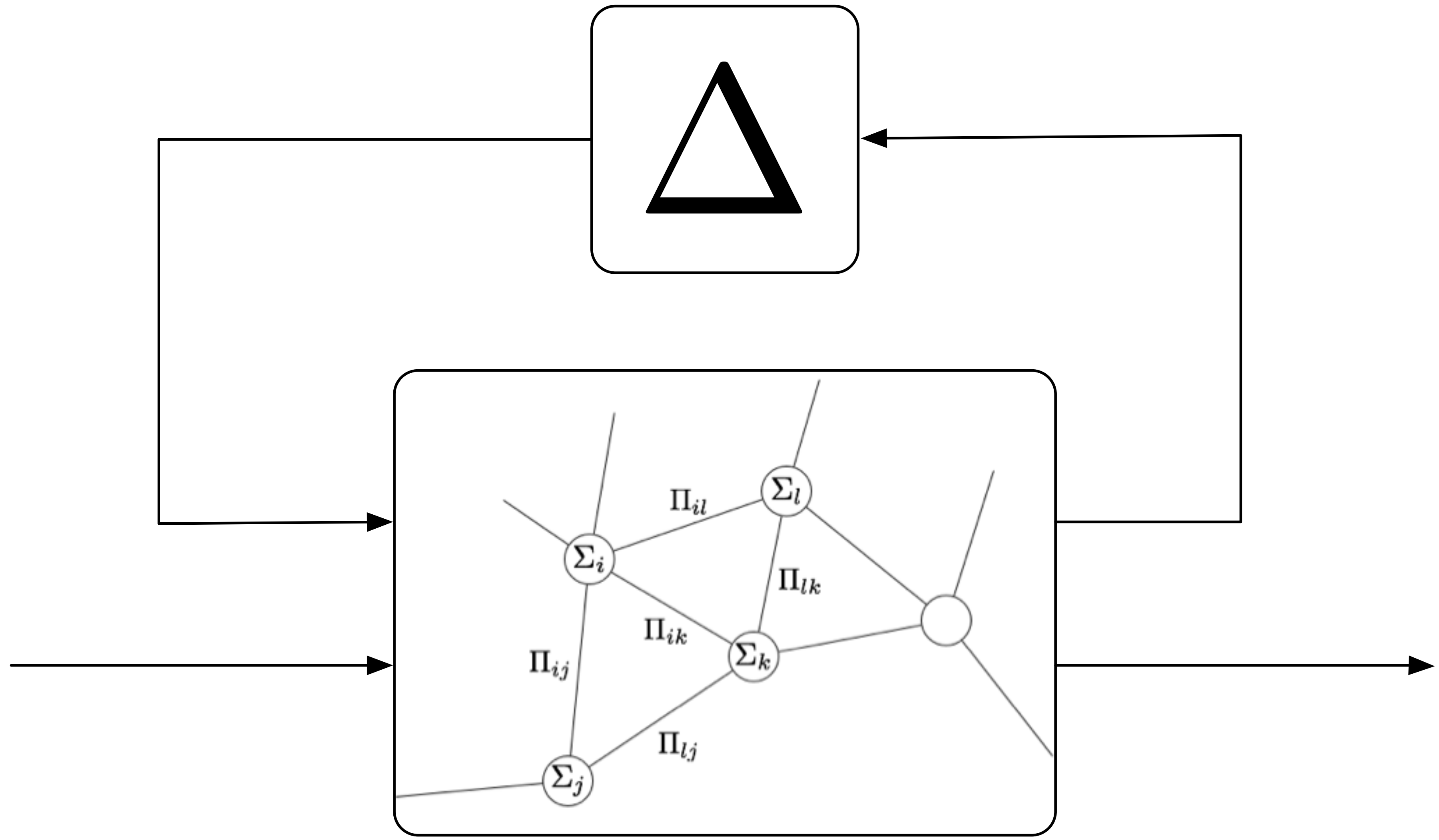
Consensus Protocol

$$\dot{x}_i = \sum_{j \sim i} w_{ij}(x_j - x_i)$$



Networked Dynamic Systems

What about robustness?



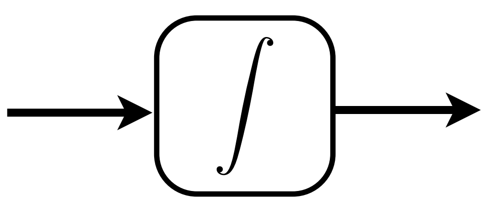
what is the right way to approach *robustness* of networked dynamic systems?

The Consensus Protocol

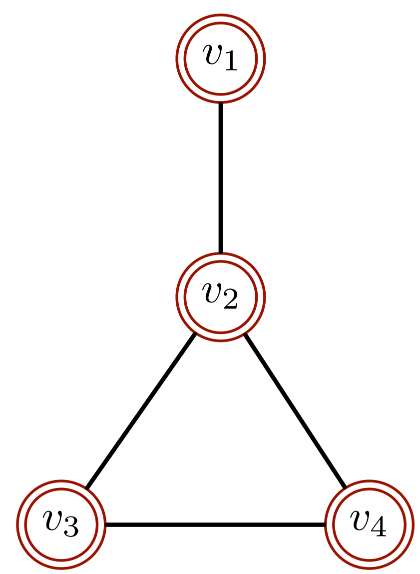
The consensus protocol is a *distributed and dynamic protocol* used for computing the average of a set of numbers.

<u>Agent Dynamics</u>

$$\dot{x}_i(t) = u_i(t)$$



<u>Information Exchange Network</u>



$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

$$\mathcal{W} : \mathcal{E} \rightarrow \mathbb{R}$$

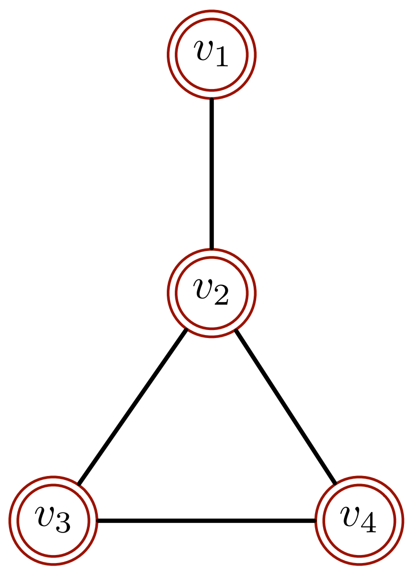
Incidence Matrix

$$E(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$$

$$E(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The Consensus Protocol

The consensus protocol is a *distributed and dynamic protocol* used for computing the average of a set of numbers.



<u>Consensus Protocol</u>

$$u_i(t) = \sum_{i\sim j} w_{ij}(x_j(t) - x_i(t))$$

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

Laplacian Matrix

- $L(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$
- $L(\mathcal{G}) = E(\mathcal{G})WE(\mathcal{G})^T$
- $L(\mathcal{G})\mathbb{1} = 0$

$$e = (v_i, v_j) \in \mathcal{E}$$

$$\mathcal{W}(e) = w_{ij} = [W]_{ee}$$

The Consensus Protocol

Consensus Protocol

$$u_i(t) = \sum_{i \sim j} w_{ij}(x_j(t) - x_i(t))$$

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

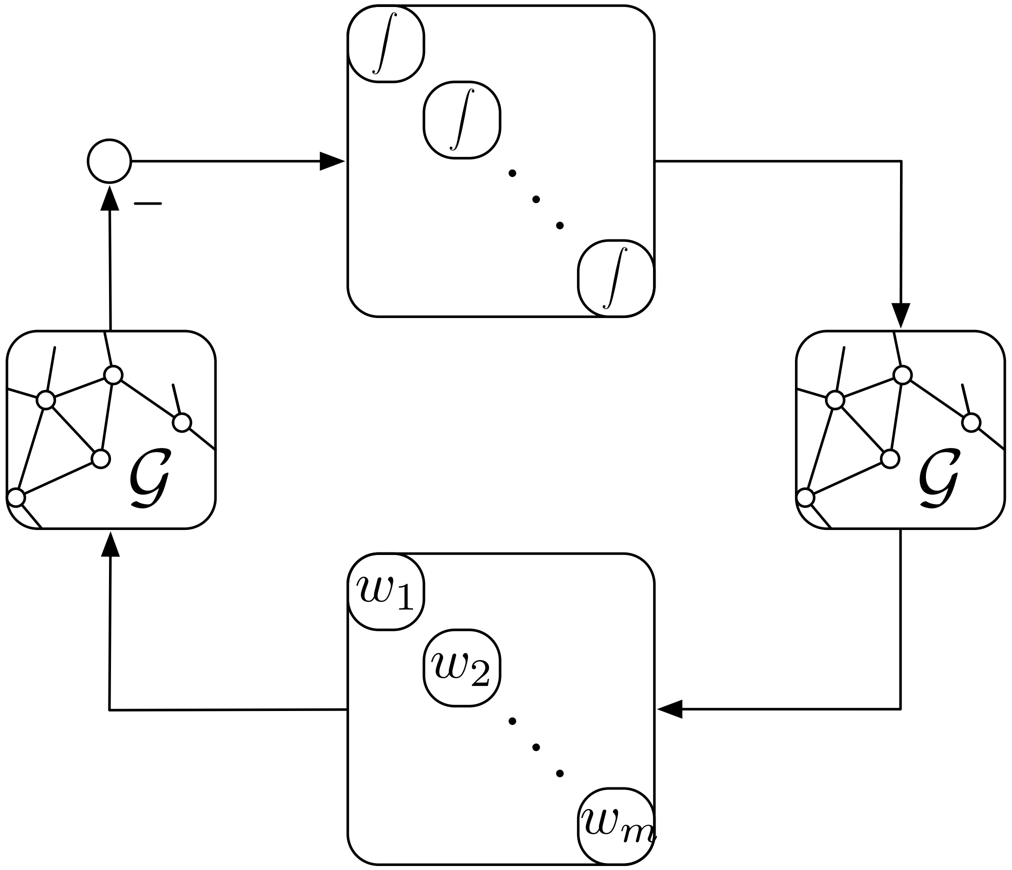
Theorem *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a weighted and connected graph with positive edge weights $\mathcal{W}(k) > 0$ for $k = 1, \ldots, |\mathcal{E}|$. Then the consensus dynamics synchronizes; i.e., $\lim_{t \to \infty} x_i(t) = \beta$ for $i = 1, \ldots, |\mathcal{V}|$.*

Mesbahi & Egerstedt, Olfati-Saber, Ren

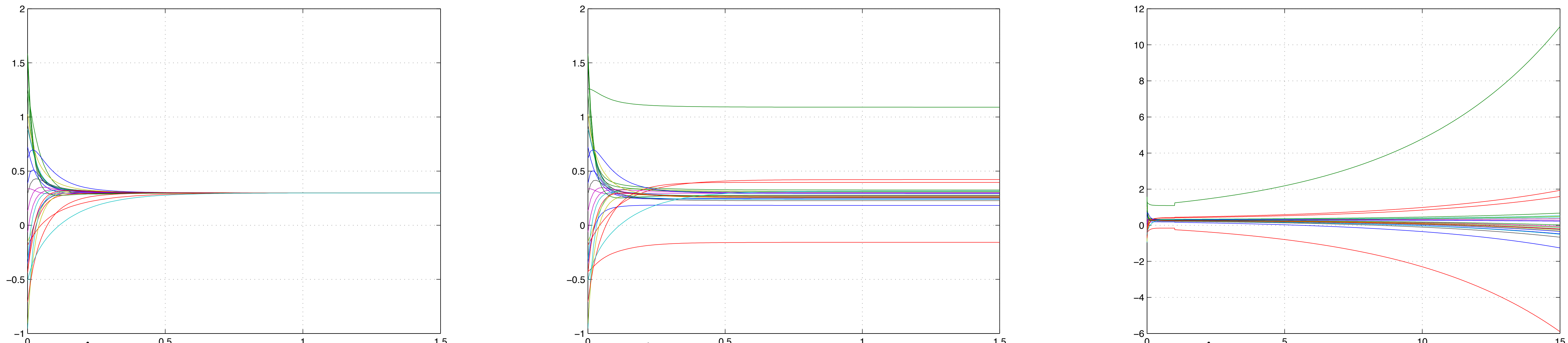
Robustness in Consensus Networks

The Linear Weighted Consensus Protocol

$$\dot{x}_i(t) = \sum_{i\sim j} w_{ij}(x_j(t) - x_i(t))$$



$\mathcal{G}$ 25 nodes
98 edges

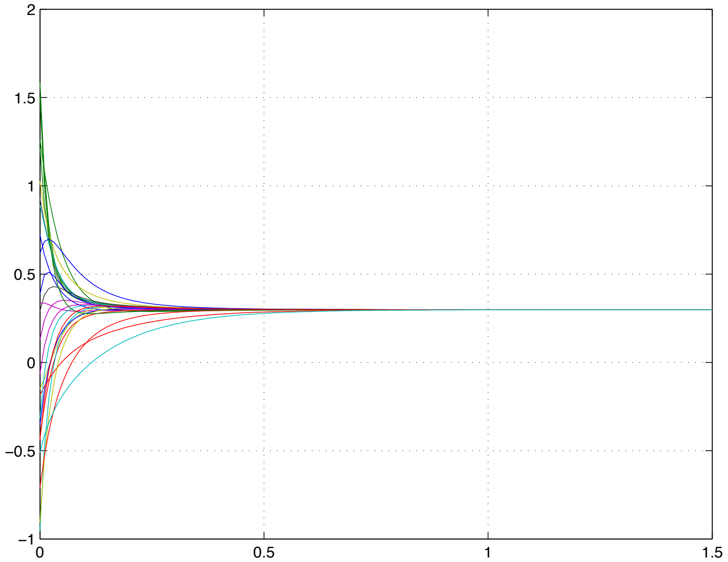


$w^* > -10.1911$ $w^* = -10.1911$ $w^* < -10.1911$

Synchronization and the Laplacian

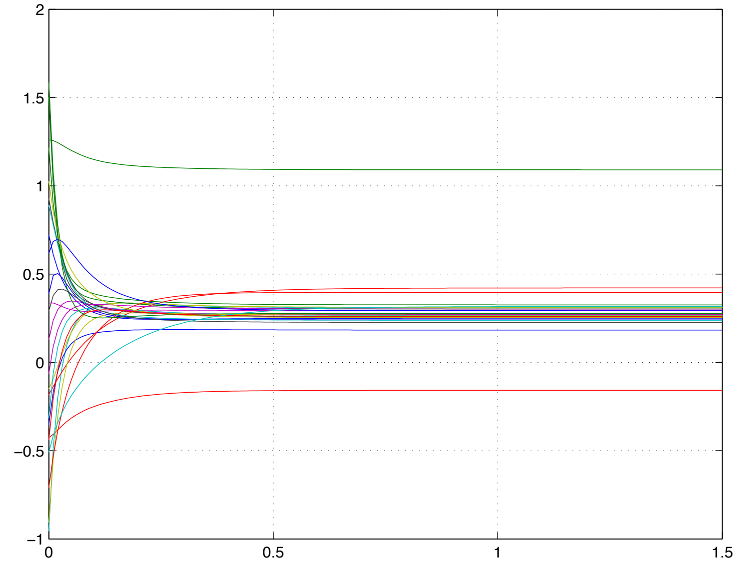
$$x(t) = e^{-L(\mathcal{G})t}x_0$$

$\lim_{t\to\infty} x(t) = \beta\mathbb{1} \Leftrightarrow L(\mathcal{G})$ has only **one** eigenvalue at the origin



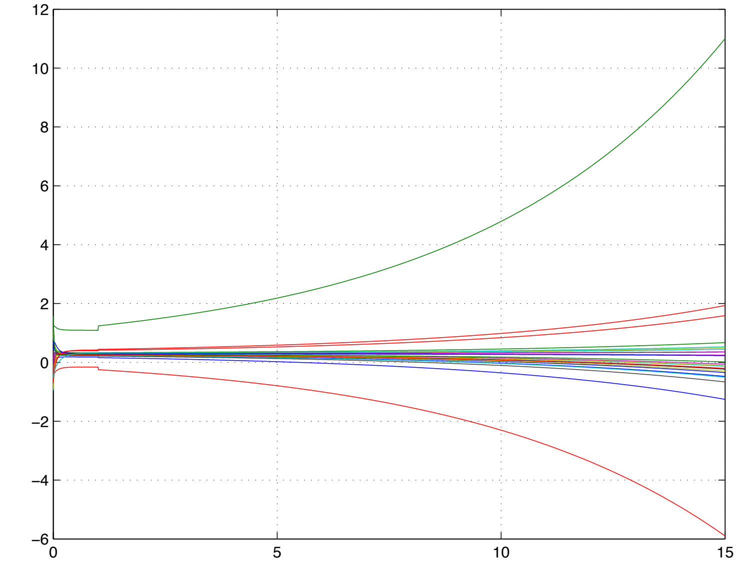
$L(\mathcal{G}) \geq 0$

has only **one** eigenvalue at the zero



$L(\mathcal{G}) \geq 0$

has **more than one** eigenvalue at the zero

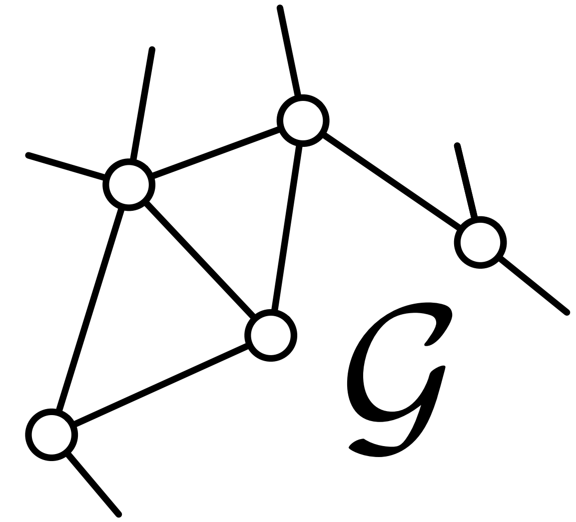


$L(\mathcal{G})$

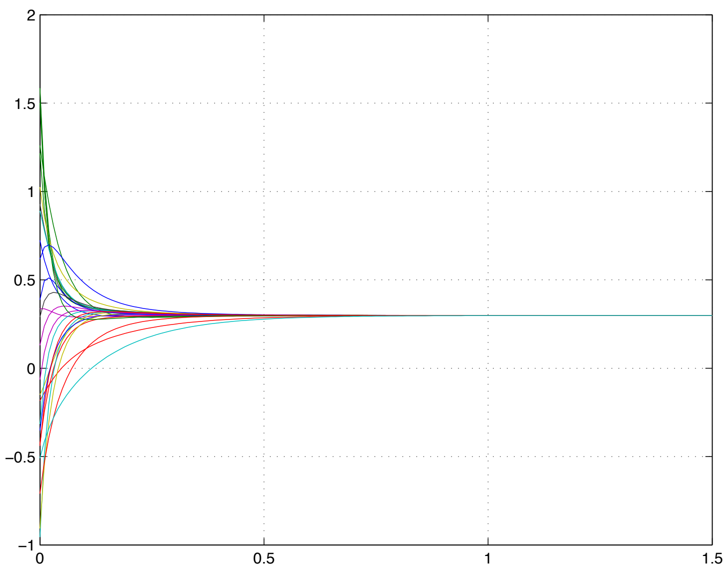
has **at least one** negative eigenvalue (indefinite)

Synchronization and the Laplacian

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

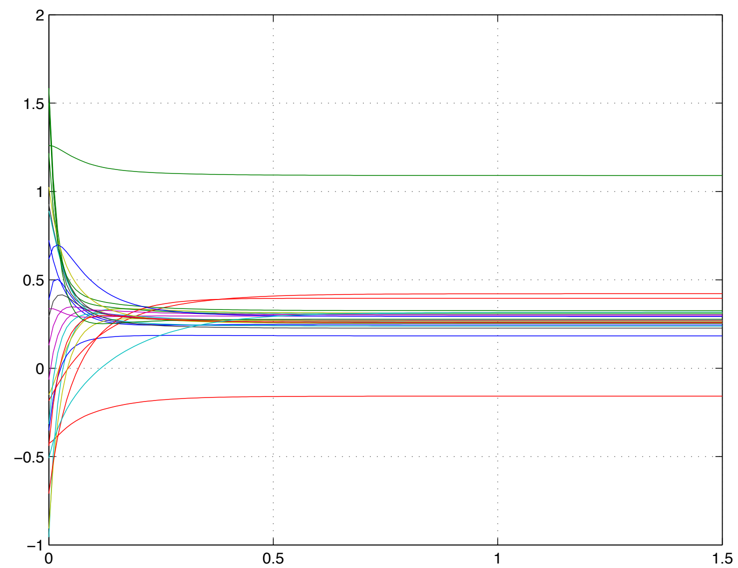


system behavior depends on the spectral properties of the graph Laplacian



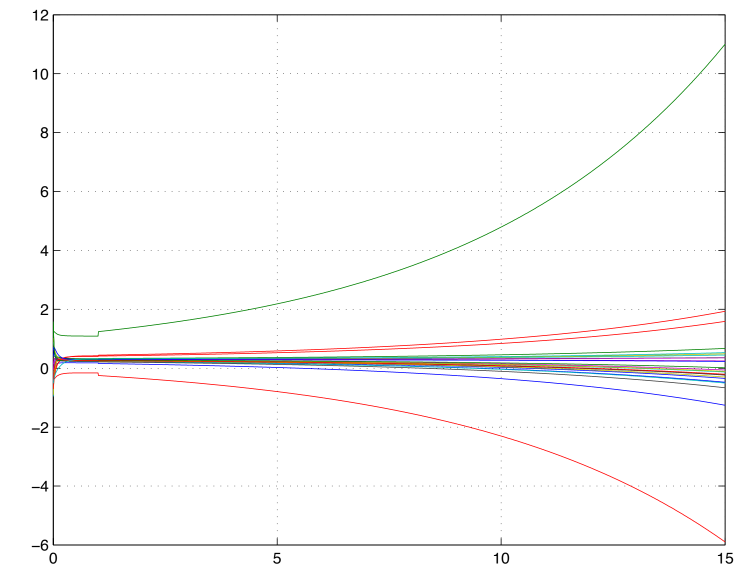
$L(\mathcal{G}) \geq 0$

has only **one** eigenvalue at the zero



$L(\mathcal{G}) \geq 0$

has **more than one** eigenvalue at the zero



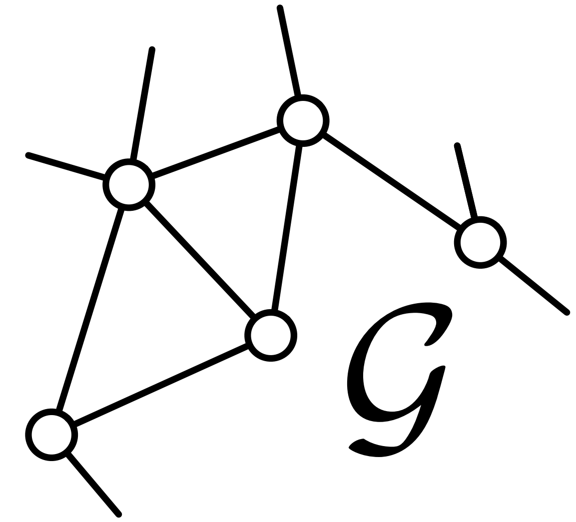
$L(\mathcal{G})$

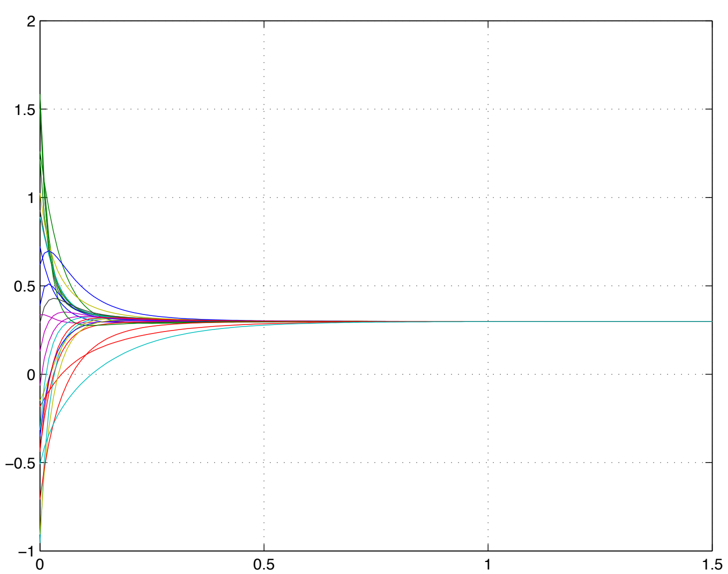
has **at least one** negative eigenvalue (indefinite)

Synchronization and the Laplacian

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

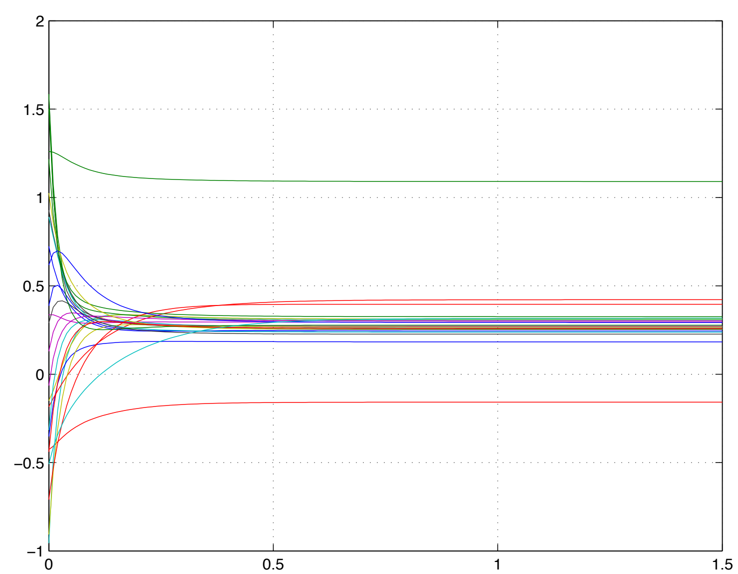
can we understand spectral properties of the Laplacian from the structure of the graph?





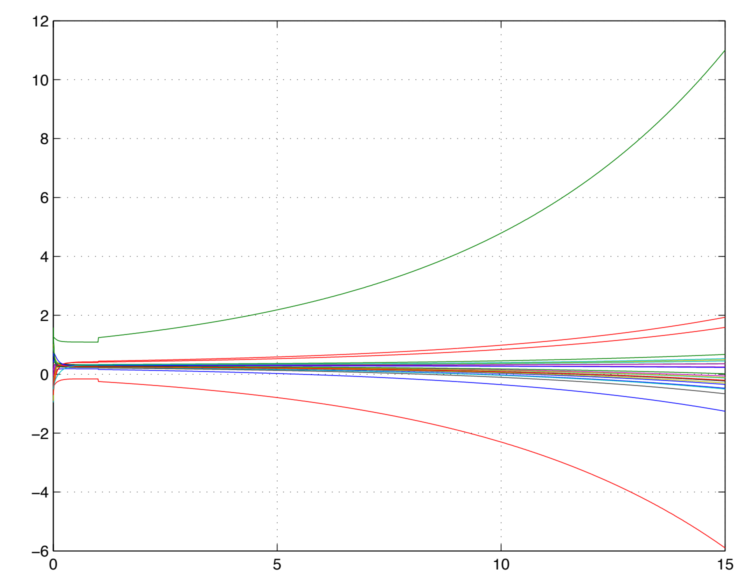
$L(\mathcal{G}) \geq 0$

has only **one** eigenvalue at the zero



$L(\mathcal{G}) \geq 0$

has **more than one** eigenvalue at the zero



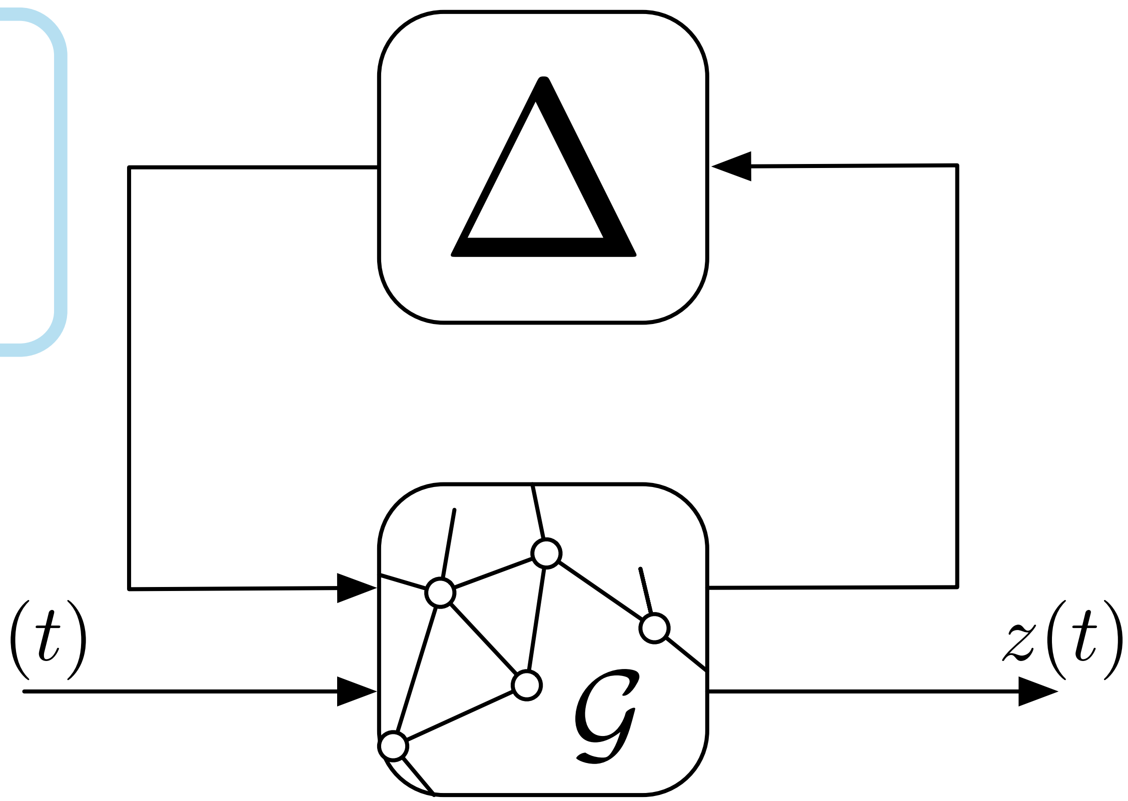
$L(\mathcal{G})$

has **at least one** negative eigenvalue (indefinite)

The Uncertain Consensus Protocol

the *nominal* consensus protocol

$$\Sigma(\mathcal{G}) \, : \, \left\{ \begin{array}{rcl} \dot{x}(t) & = & -L(\mathcal{G})x(t) + \mathrm{w}(t) \\ z(t) & = & E(\mathcal{G}_o)^T x(t) \end{array} \right.$$



- assume finite-energy disturbances $\mathrm{w}(t) \in \mathcal{L}_2^n[0, \infty)$
- controlled variable are relative states over *any* graph of interest

additive uncertainty in the edge weights

$$\boldsymbol{\Delta} \; = \; \{\Delta \, : \, \Delta = \mathbf{diag}\{\delta_1, \ldots, \delta_{|\mathcal{E}_\Delta|}\}, \; \|\Delta\| \leq \overline{\delta}\}$$

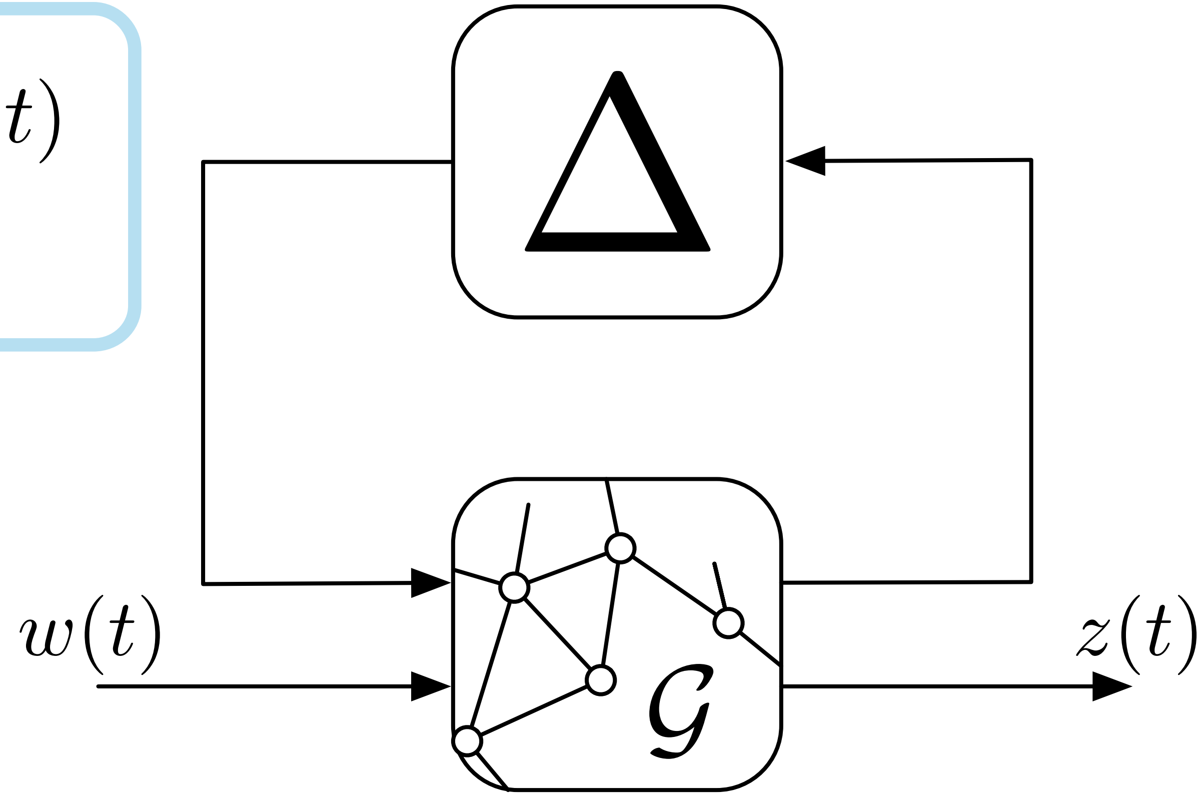
$$\Sigma(\mathcal{G}, \Delta) \, : \, \left\{ \begin{array}{rcl} \dot{x}(t) & = & -E(\mathcal{G})(W + \Delta)E(\mathcal{G})^T x(t) + \mathrm{w}(t) \\ z(t) & = & E(\mathcal{G}_o)^T x(t) \end{array} \right.$$

The Uncertain Consensus Protocol

the *nominal* consensus protocol

$$\Sigma(\mathcal{G}) : \begin{cases} \dot{x}(t) = -L(\mathcal{G})x(t) + \mathrm{w}(t) \\ z(t) = E(\mathcal{G}_o)^T x(t) \end{cases}$$

- assume finite-energy disturbances $\mathrm{w}(t) \in \mathcal{L}_2^n[0, \infty)$
- controlled variable are relative states over *any* graph of interest



sector-bounded non-linearities in the edge weights

$$\Phi(y) = [\phi_1(y_1) \cdots \phi_{|\mathcal{E}_\Delta|}(y_{|\mathcal{E}_\Delta|})] \quad \alpha_i u_i^2 \leq u_i \phi_i(y_i) \leq \beta_i u_i^2$$

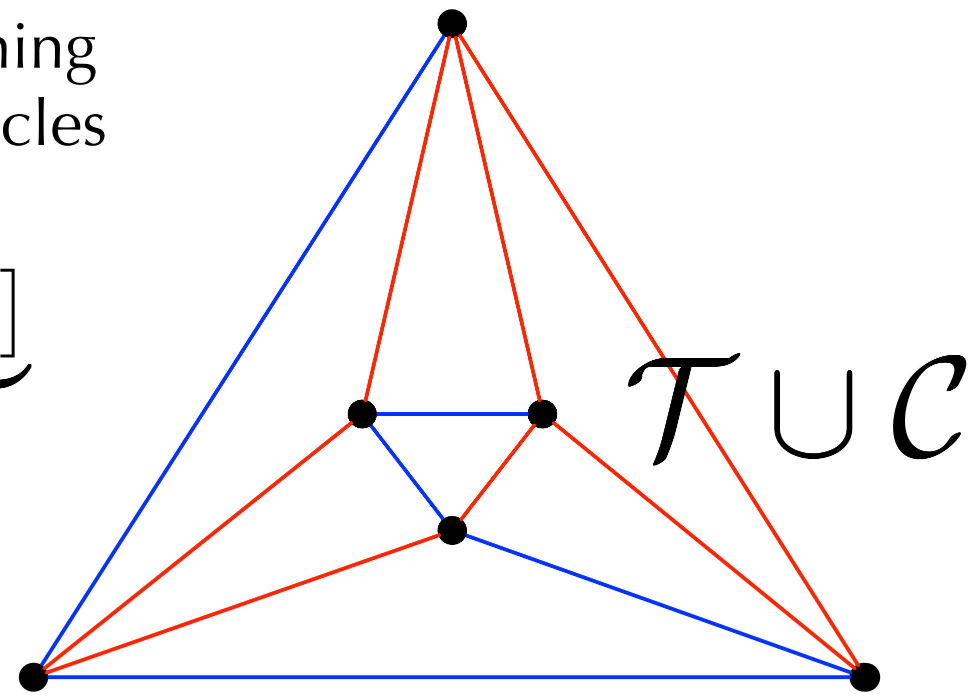
$$\Sigma(\mathcal{G}, \Phi) : \begin{cases} \dot{x}(t) = -L(\mathcal{G})x(t) - E(G_\Delta)\Phi\left(E(G_\Delta)^T x(t)\right) + \mathrm{w}(t) \\ z(t) = E(\mathcal{G}_o)^T x(t) \end{cases}$$

Spanning Trees and Cycles

A graph as the union of a spanning tree and edges that complete cycles

$$E(\mathcal{G}) = E(\mathcal{T}) \underbrace{\left[\begin{array}{cc} I & T_{(\mathcal{T},\mathcal{C})} \end{array} \right]}_{\mathcal{R}_{(\mathcal{T},\mathcal{C})}}$$

$$T_{(\mathcal{T},\mathcal{C})} = \underbrace{(E_{\mathcal{T}}^T E_{\mathcal{T}})^{-1} E_{\mathcal{T}}^T}_{E_{\mathcal{T}}^L} E(\mathcal{C})$$



a spanning tree

remaining edges "complete cycles"

Weighted Edge Laplacian

$$L_e(\mathcal{G}) = W^{\frac{1}{2}} E(\mathcal{G})^T E(\mathcal{G}) W^{\frac{1}{2}}$$

Essential Edge Laplacian

$$L_e(\mathcal{T}) R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T := L_{ess}(\mathcal{G})$$

$\mathcal{R}_{(\mathcal{T},\mathcal{C})}$ rows form a basis for the *cut space* of the graph

similarity between edge and graph Laplacians

$L(\mathcal{G}) \rightleftharpoons L_e(\mathcal{G})$

The Edge Agreement

the *uncertain linear edge agreement*

$$\Sigma_{\mathcal{F}}(\mathcal{G},\Delta)$$

$$\begin{cases} \dot{x}_{\mathcal{F}} &= -L_e(\mathcal{F})R_{(\mathcal{F},\mathcal{C})}(W+P\Delta P^T)R^T_{(\mathcal{F},\mathcal{C})}x_{\mathcal{F}} + E^T_{\mathcal{F}}\mathrm{w} \\ z &= E(\mathcal{G}_o)^T(E^L_{\mathcal{F}})^T x_{\mathcal{F}} \end{cases}$$

- a *minimal* realization of consensus network
- $z(t) \in \mathcal{L}_2^m[0,\infty)$.

the *uncertain non-linear edge agreement*

$$\Sigma_{\mathcal{F}}(\mathcal{G},\Phi)$$

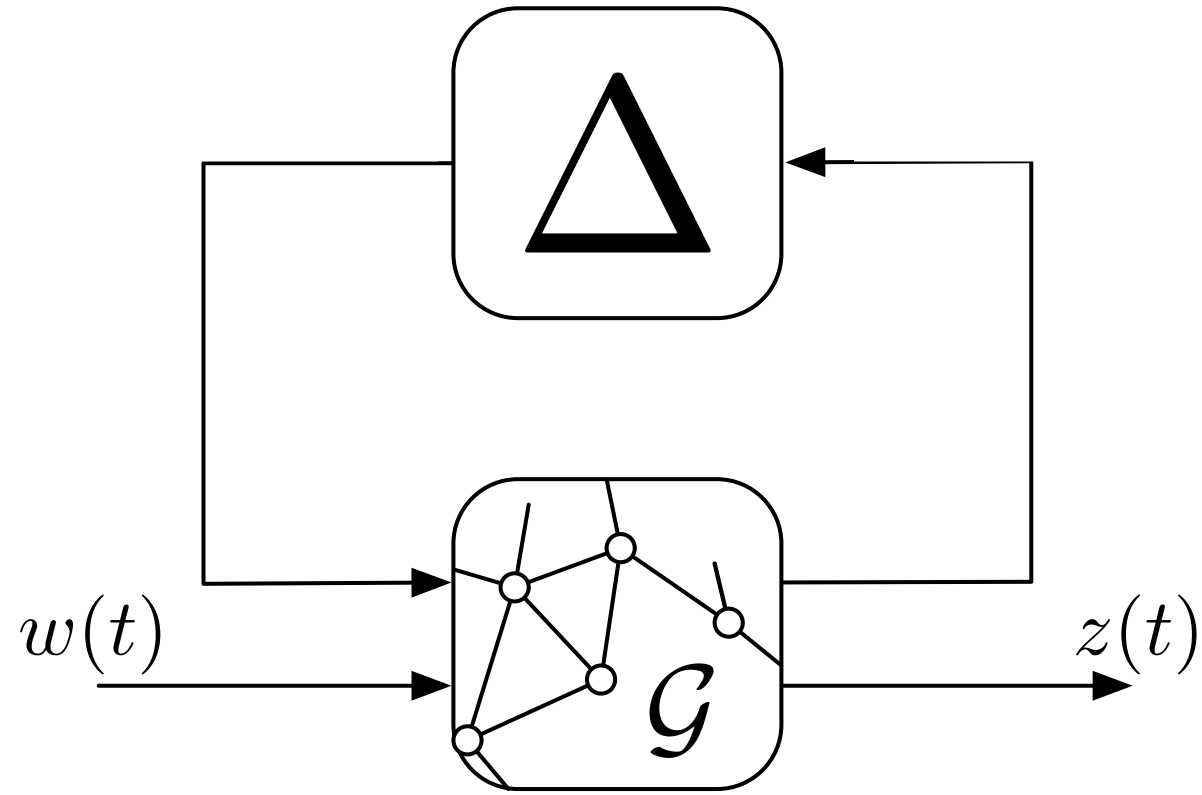
sector-bounded nonlinear couplings

$$\begin{cases} \dot{x}_{\mathcal{F}} &= -L_{ess}(\mathcal{F})x_{\mathcal{F}} - L_e(\mathcal{F})R_{(\mathcal{F},\mathcal{C})}P(\Phi(P^T R^T_{(\mathcal{F},\mathcal{C})}x_{\mathcal{F}})) + E^T_{\mathcal{F}}\mathrm{w} \\ z &= E(\mathcal{G}_o)^T(E^L_{\mathcal{F}})^T x_{\mathcal{F}} \end{cases}$$

The Edge Agreement

What are the *robustness margins* or a consensus network with bounded additive perturbations to the edge weights?

- robust stability
- robust performance
- robust synthesis



$$\left\{ \begin{array}{rcl} \dot{x}_{\mathcal{F}} & = & -L_e(\mathcal{F})R_{(\mathcal{F},\mathcal{C})}(W + P\Delta P^T)R^T_{(\mathcal{F},\mathcal{C})}x_{\mathcal{F}} + E^T_{\mathcal{F}}\mathrm{w} \\ z & = & E(\mathcal{G}_o)^T(E^L_{\mathcal{F}})^T x_{\mathcal{F}} \end{array} \right.$$

Some Properties of $L_e(\mathcal{G})$

Proposition *The matrix* $L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})}$ *has the same inertia as* $R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})}$*. Similarly, the matrix* $(L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})})^{-1}$ *has the same inertia as* $(R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})})^{-1}$.

Recall: The *inertia* of a matrix is the number of negative, 0, and positive eigenvalues

Proof:

$$L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})} \sim L_e(\mathcal{T})^{\frac{1}{2}}R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})}L_e(\mathcal{T})^{\frac{1}{2}}$$

$L_e(\mathcal{T})^{\frac{1}{2}}R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})}L_e(\mathcal{T})^{\frac{1}{2}}$ is congruent to $R_{(\mathcal{T},\mathcal{C})}WR^T_{(\mathcal{T},\mathcal{C})}$

congruent matrices have the same inertia

Some Properties of $L_e(\mathcal{G})$

Proposition

$$L(\mathcal{G}) \geq 0 \Leftrightarrow R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T \geq 0$$

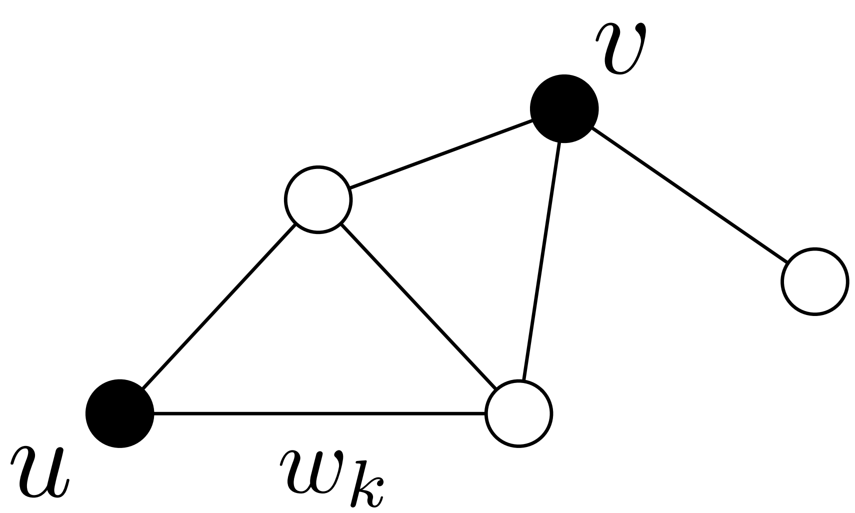
The definiteness of the graph Laplacian can be studied through another matrix!

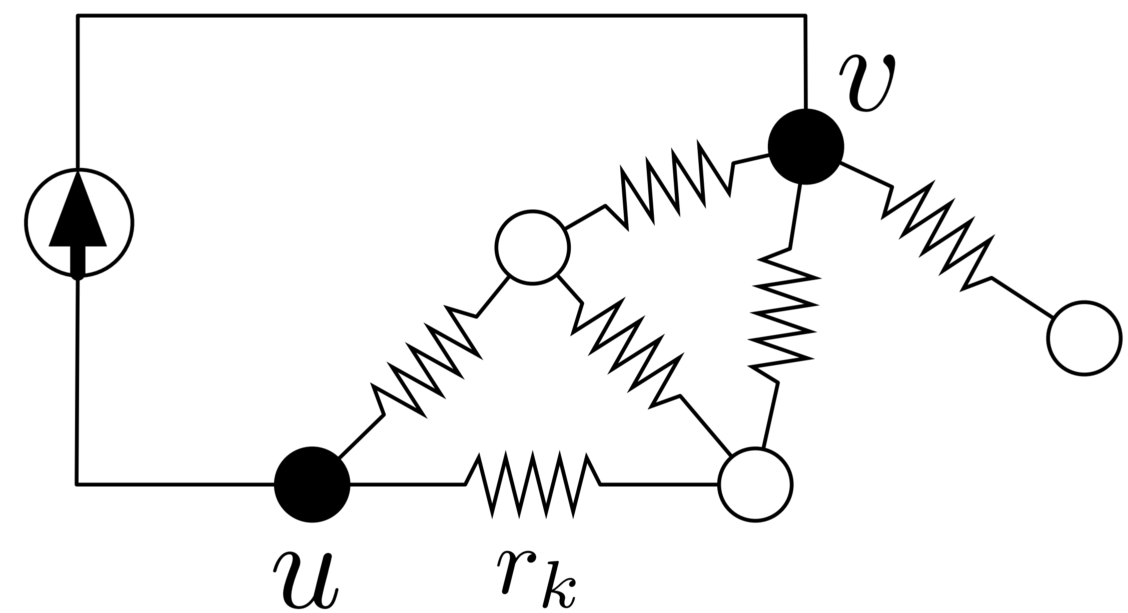
$$R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T$$

intimately related to the notion of **effective resistance** of a network

Effective Resistance of a Graph

The **effective resistance** between two nodes *u* and *v* is the electrical resistance measured across the nodes when the graph represents an electrical circuit with each edge a resistor





$$r_k = \frac{1}{w_k}$$ edge weights are the *conductance* of *each resistor*

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T L^\dagger(\mathcal{G})(\mathbf{e}_u - \mathbf{e}_v)$$
$$= \left[L^\dagger(\mathcal{G})\right]_{uu} - 2\left[L^\dagger(\mathcal{G})\right]_{uv} + \left[L^\dagger(\mathcal{G})\right]_{vv}$$

Klein and Randić 1993

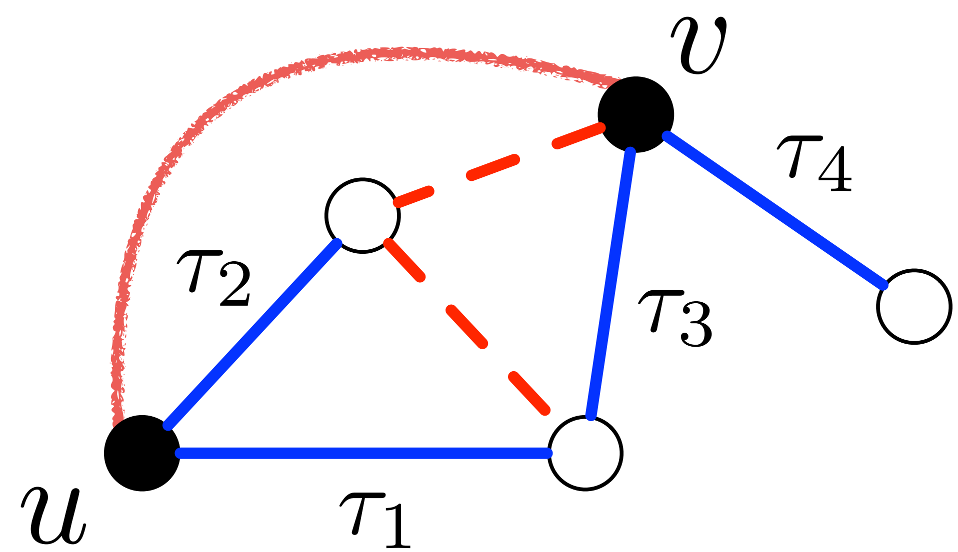
Effective Resistance of a Graph

Proposition

$$L^{\dagger}(\mathcal{G}) = (E_{\mathcal{T}}^{L})^{T}\left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^{T}\right)^{-1} E_{\mathcal{T}}^{L}$$
$$= \quad (E_{\mathcal{T}}^{L})^{T} L_{ess}(\mathcal{T})^{-1} E_{\mathcal{T}}^{T}$$

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T L^{\dagger}(\mathcal{G})(\mathbf{e}_u - \mathbf{e}_v)$$

$$E_{\mathcal{T}}^{L}(\mathbf{e}_u - \mathbf{e}_v) = \begin{bmatrix} \pm 1 \\ 0 \\ \pm 1 \\ 0 \end{bmatrix} \begin{matrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{matrix}$$



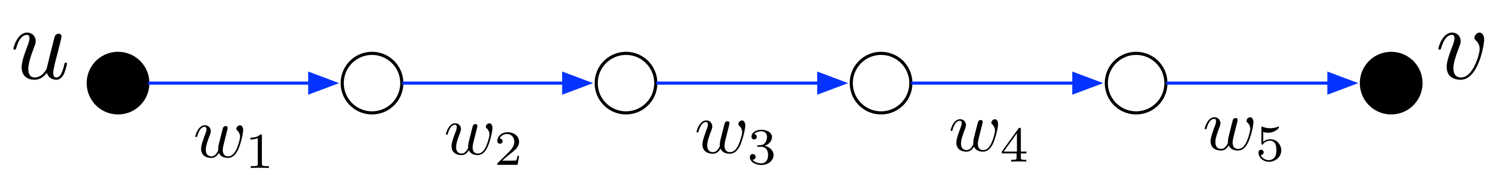
$$\mathcal{G} = \mathcal{T} \cup \mathcal{C}$$

indicates a path from node u to v using only edges in the spanning tree

$$T_{(\mathcal{T},\mathcal{C})} = \underbrace{(E_{\mathcal{T}}^{T} E_{\mathcal{T}})^{-1} E_{\mathcal{T}}^{T}}_{E_{\mathcal{T}}^{L}} E(\mathcal{C})$$

Effective Resistance of a Graph

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T (E_{\mathcal{T}}^L)^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v)$$



$$R_{(\mathcal{T},\mathcal{C})} = I$$

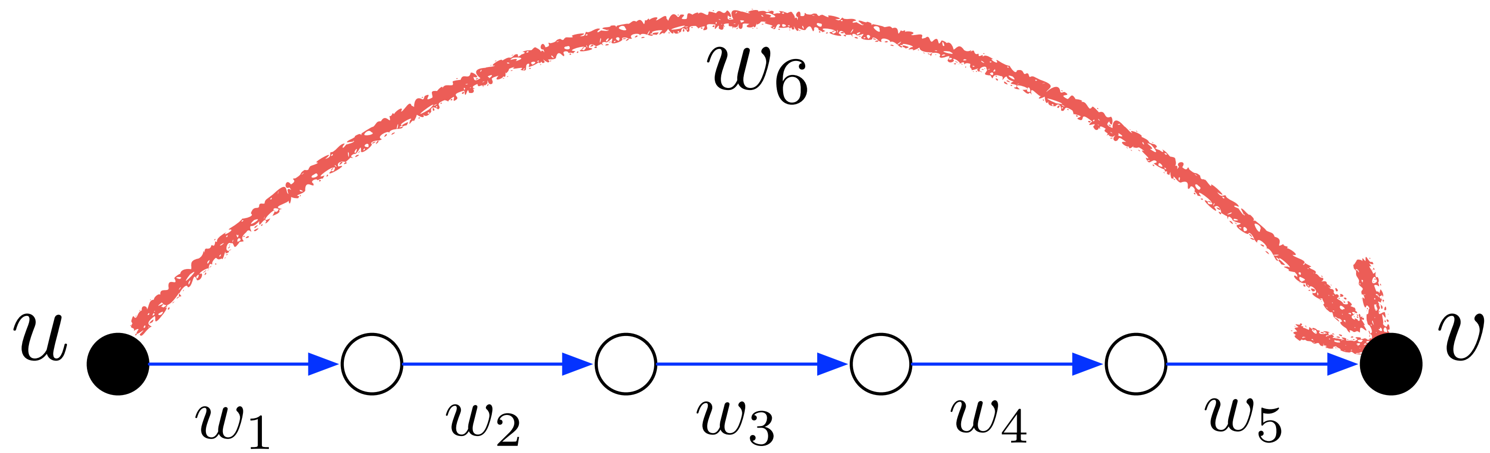
$$E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v) = \mathbb{1}$$

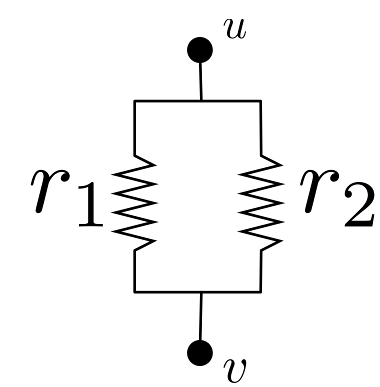
$$r_k = \frac{1}{w_k}$$

$$r_{uv} = \mathbb{1}^T W^{-1} \mathbb{1} = \sum_{i=1}^{5} \frac{1}{w_i}$$

Effective Resistance of a Graph

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T (E_{\mathcal{T}}^L)^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v)$$





$$r_{uv} = \frac{r_1 r_2}{r_1 + r_2}$$

$$R_{(\mathcal{T},\mathcal{C})} = \left[\begin{array}{cc} I & \mathbb{1} \end{array}\right]$$

$$E_{\mathcal{T}}^L(\mathbf{e}_u - \mathbf{e}_v) = \mathbb{1}$$

$$r_k = \frac{1}{w_k}$$

$$W_{\mathcal{T}} = \text{diag}\{w_1, \ldots, w_5\}$$

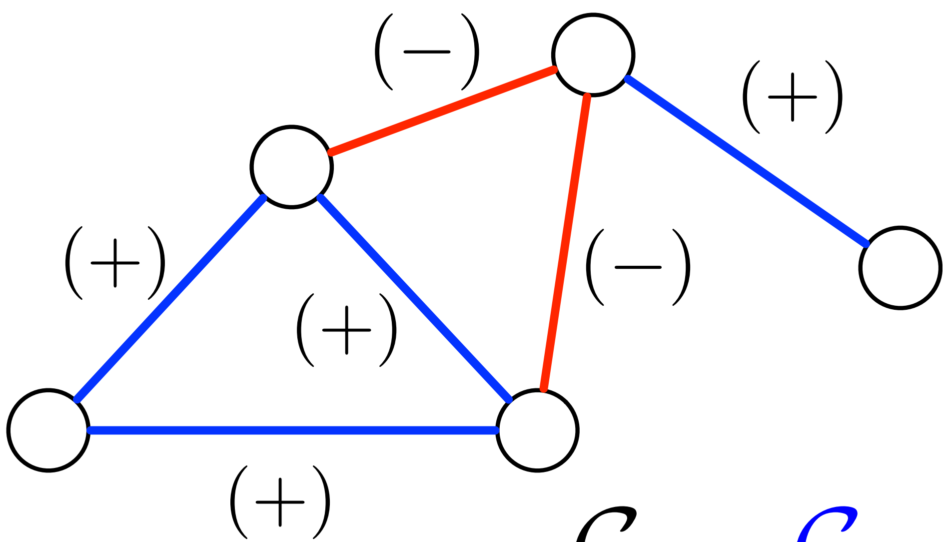
$$\begin{aligned} r_{uv} &= \mathbb{1}^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} \mathbb{1} \\ &= \mathbb{1}^T \left(W_{\mathcal{T}} + w_6 \mathbb{1}\mathbb{1}^T\right)^{-1} \mathbb{1} \\ &= \frac{\left(\mathbb{1}^T W_{\mathcal{T}}^{-1} \mathbb{1}\right) w_6^{-1}}{\mathbb{1}^T W_{\mathcal{T}}^{-1} \mathbb{1} + w_6^{-1}} \end{aligned}$$

Signed Graphs

a **signed graph** is a graph with positive and negative edge weights

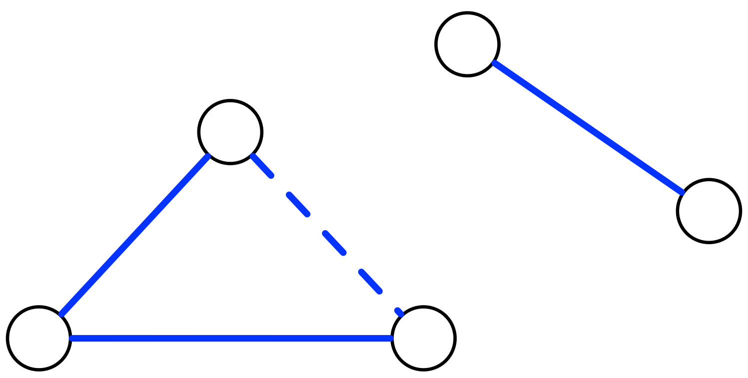
$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

$$\mathcal{W} : \mathcal{E} \rightarrow \mathbb{R}$$

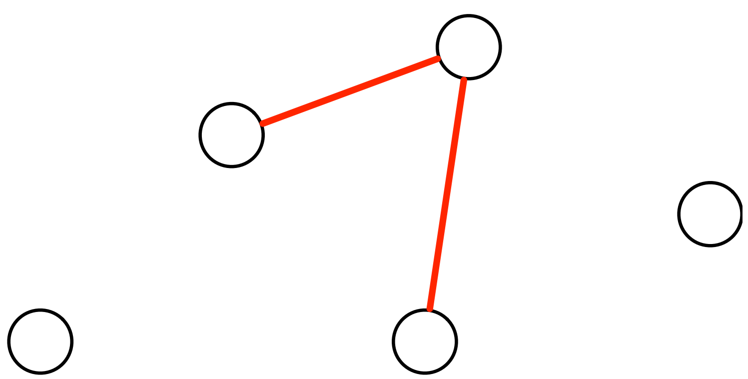


$$\mathcal{G} = \mathcal{G}_+ \cup \mathcal{G}_-$$

$$\mathcal{E}_+ = \{e \in \mathcal{E} \, : \, \mathcal{W}(e) > 0\} \qquad \mathcal{E}_- = \{e \in \mathcal{E} \, : \, \mathcal{W}(e) < 0\}$$



$$E(\mathcal{G}_+) = E_+ = E_{\mathcal{F}_+} R_{(\mathcal{F}_+, \mathcal{C}_+)}$$



$$E(\mathcal{G}_-) = E_-$$

$$L(\mathcal{G}) = E(\mathcal{G}_+) W_+ E(\mathcal{G}_+)^T - E(\mathcal{G}_-)|W_-|E(\mathcal{G}_-)^T$$

Spectral Properties of Signed Graphs

Proposition

$$L(\mathcal{G}) \geq 0 \Leftrightarrow \begin{bmatrix} |W_-|^{-1} & E_-^T \\ E_- & E_+ W_+ E_+^T \end{bmatrix} \geq 0$$

Proof:

Schur Complement

$$L(\mathcal{G}) = E(\mathcal{G}_+) W_+ E(\mathcal{G}_+)^T - E(\mathcal{G}_-)|W_-|E(\mathcal{G}_-)^T$$

Spectral Properties of Signed Graphs

Proposition

$$L(\mathcal{G}) \geq 0 \Leftrightarrow \begin{bmatrix} |W_-|^{-1} & E_-^T (E_{\mathcal{F}_+}^L)^T & E_-^T N_{\mathcal{F}_+} \\ E_{\mathcal{F}_+}^L E_- & R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T & 0 \\ N_{\mathcal{F}_+}^T E_- & 0 & \mathbf{0} \end{bmatrix} \geq 0$$

Proof:

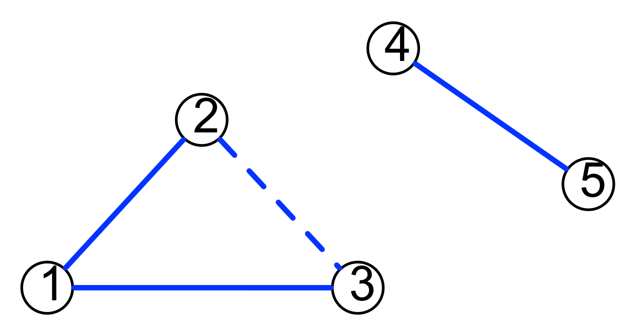
Congruent Transformation $S = \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} (E_{\mathcal{F}_+}^L)^T & N_{\mathcal{F}_+} \end{bmatrix} \end{bmatrix}$

applied to $\begin{bmatrix} |W|_- & E_-^T \\ E_- & E_+ W_+ E_+^T \end{bmatrix}$

$$E(\mathcal{G}_+) = E_+ = E_{\mathcal{F}_+} R_{(\mathcal{F}_+,\mathcal{C}_+)}$$

$$\text{IM}[N_{\mathcal{F}_+}] = \text{span}[\mathcal{N}(E_{\mathcal{F}_+}^T)]$$

Identifies how the positive weight graph is partitioned



$$N_{\mathcal{F}_+} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Spectral Properties of Signed Graphs

Proposition

$$L(\mathcal{G}) \geq 0 \Leftrightarrow \begin{bmatrix} |W_-|^{-1} & E_-^T (E_{\mathcal{F}_+}^L)^T & E_-^T N_{\mathcal{F}_+} \\ E_{\mathcal{F}_+}^L E_- & R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T & 0 \\ N_{\mathcal{F}_+}^T E_- & 0 & \mathbf{0} \end{bmatrix} \geq 0$$

Proof:

Congruent Transformation $S = \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} (E_{\mathcal{F}_+}^L)^T & N_{\mathcal{F}_+} \end{bmatrix} \end{bmatrix}$

applied to $\begin{bmatrix} |W|_- & E_-^T \\ E_- & E_+ W_+ E_+^T \end{bmatrix}$

If the positive portion weighted graph is connected…

$$N_{\mathcal{F}_+} = \mathbb{1}$$

$$L(\mathcal{G}) \geq 0 \Leftrightarrow \begin{bmatrix} |W_-|^{-1} & E_-^T (E_{\mathcal{F}_+}^L)^T \\ E_{\mathcal{F}_+}^L E_- & R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T \end{bmatrix} \geq 0$$

Spectral Properties of Signed Graphs

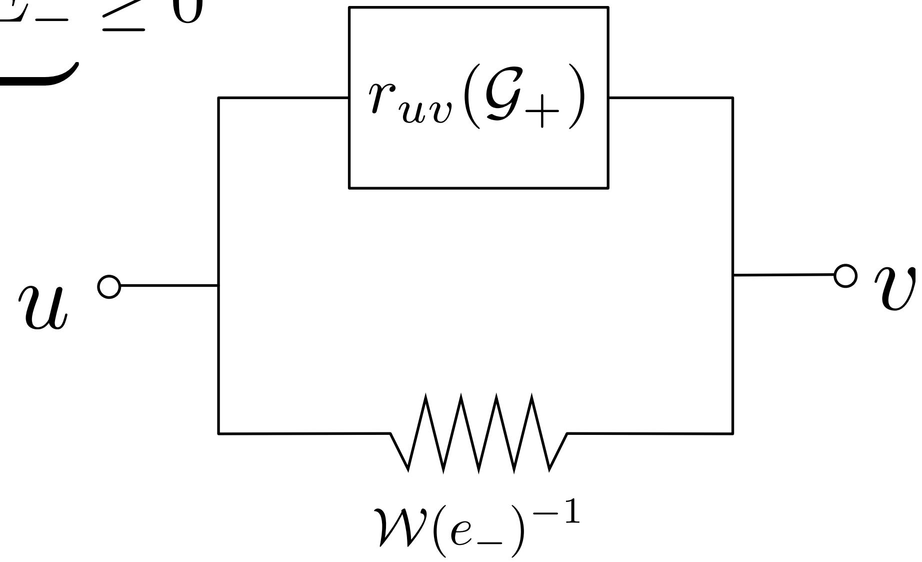
Theorem *Assume that $\mathcal{G}_+$ is connected and $|\mathcal{E}_-| = 1$ and let $\mathcal{E}_- = \{e_- = (u,v)\}$. Let r_{uv} denote the effective resistance between nodes $u, v \in \mathcal{V}$ over the graph $\mathcal{G}_+$. Then*

$$L(\mathcal{G}) \geq 0 \Leftrightarrow |\mathcal{W}(e_-)| \leq r_{uv}^{-1}$$

Proof:

$$|W_-|^{-1} - \underbrace{E_-^T (E_{\mathcal{F}_+}^L)^T (R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T)^{-1} E_{\mathcal{F}_+}^L E_-}_{r_{uv}(\mathcal{G}_+)} \geq 0$$

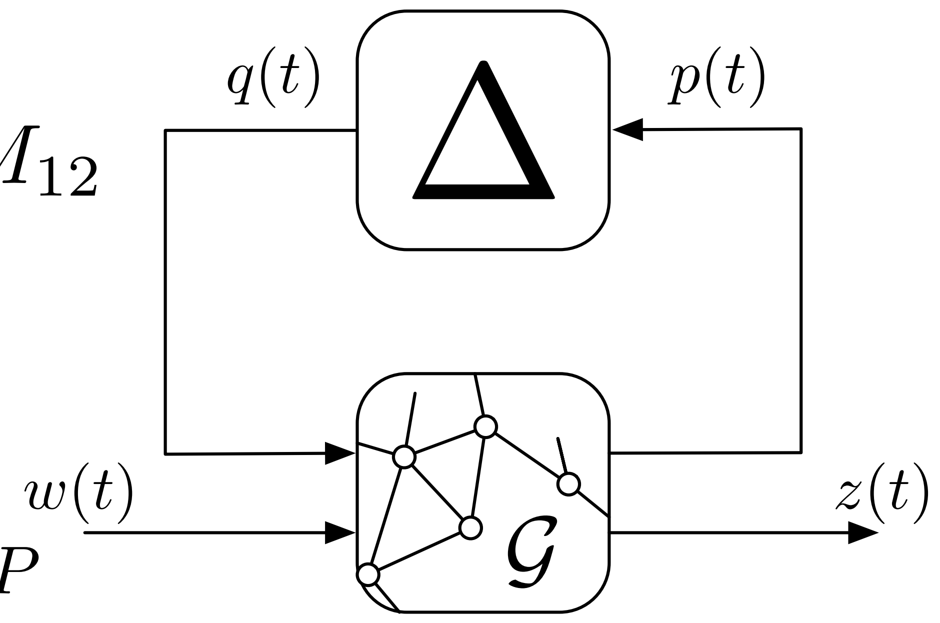
any single edge can destabilize a consensus network with a "negative enough" edge weight



A Small-Gain Interpretation

upper fractional transformation

$$\overline{S}(\Sigma_{\mathcal{F}}(\mathcal{G}),\Delta) = M_{22} + M_{21}\Delta\left(I - M_{11}\Delta\right)^{-1}M_{12}$$



$$\begin{aligned}
M_{11}(s) &= P^T R^T_{(\mathcal{F},\mathcal{C})}(sI + L_{ess}(\mathcal{F}))^{-1}L_e(\mathcal{F})R_{(\mathcal{F},\mathcal{C})}P\\
M_{12}(s) &= P^T R^T_{(\mathcal{F},\mathcal{C})}(sI + L_{ess}(\mathcal{F}))^{-1}E(\mathcal{F})^T\\
M_{21}(s) &= E(\mathcal{G}_o)^T(E^L_{\mathcal{F}})^T(sI + L_{ess}(\mathcal{F}))^{-1}L_e(\mathcal{F})R_{(\mathcal{F},\mathcal{C})}P\\
M_{22}(s) &= E(\mathcal{G}_o)^T(E^L_{\mathcal{F}})^T(sI + L_{ess}(\mathcal{F}))^{-1}E(\mathcal{F})^T.
\end{aligned}$$

Small-Gain Theorem

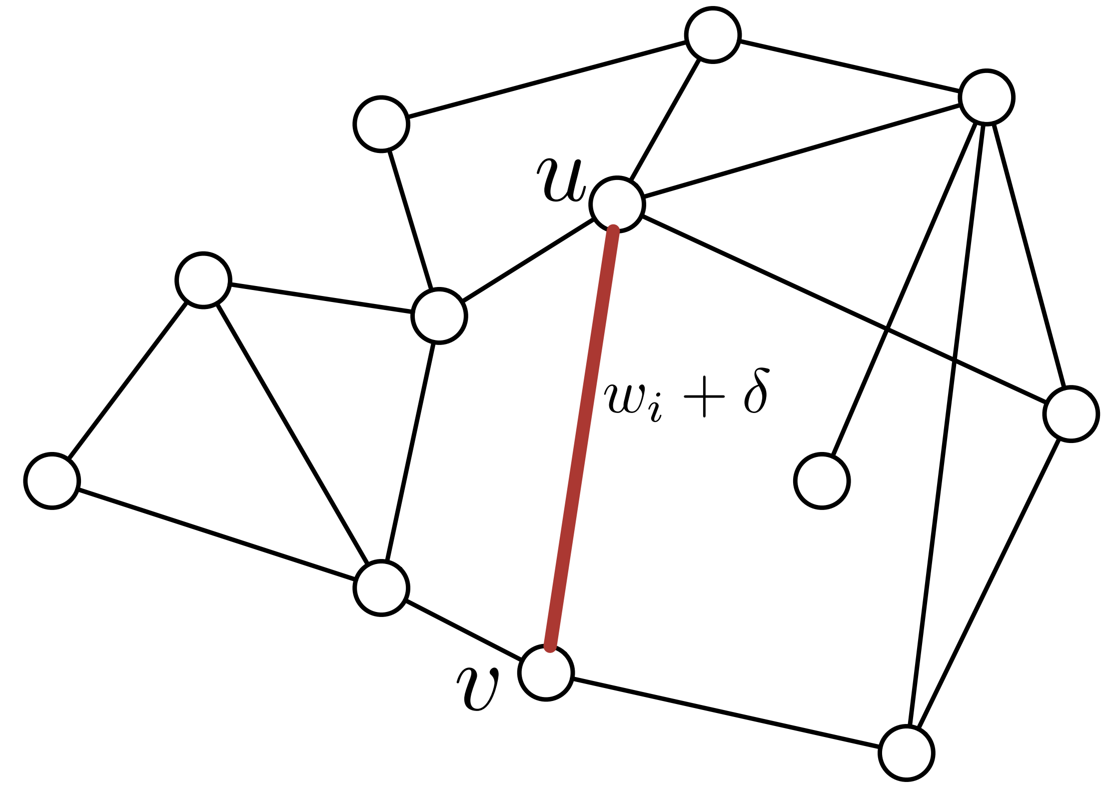
$$\|\Delta\| < \overline{\sigma}(M_{11}(0))^{-1}$$

A Small-Gain Interpretation

assume *nominal* network is stable

consider a network with only a *single* uncertain edge

$$\mathcal{E}_\Delta = \{\{u,v\}\}$$



Theorem

- $\|M_{11}(s)\|_\infty = \mathcal{R}_{uv}$
- The uncertain consensus network is stable for any $\|\Delta\|_\infty < \mathcal{R}_{uv}^{-1}$

$$M_{11}(s) \;=\; P^T R^T_{(\mathcal{F},\mathcal{C})}(sI + L_{ess}(\mathcal{F}))^{-1} L_e(\mathcal{F}) R_{(\mathcal{F},\mathcal{C})} P$$

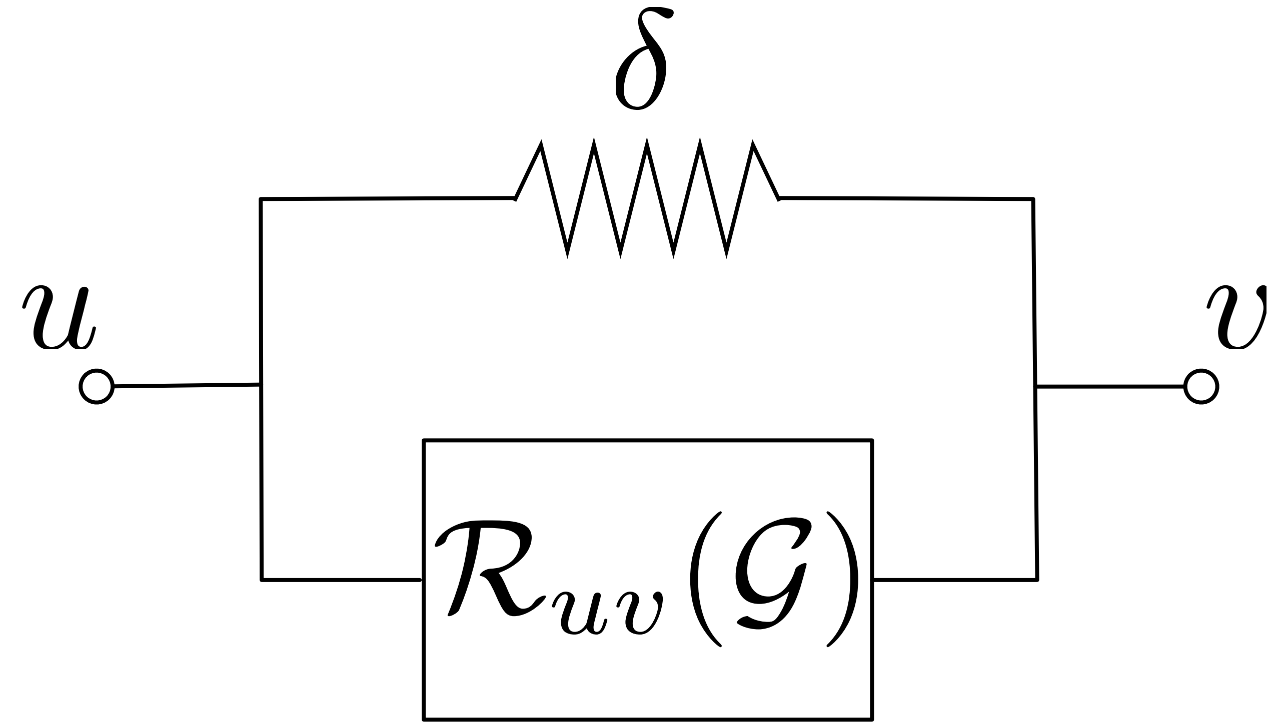
$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T (E^L_{\mathcal{T}})^T \left(R_{(\mathcal{T},\mathcal{C})} W R^T_{(\mathcal{T},\mathcal{C})}\right)^{-1} E^L_{\mathcal{T}} (\mathbf{e}_u - \mathbf{e}_v)$$

A Small-Gain Interpretation

assume *nominal* network is stable

consider a network with only a *single* uncertain edge

$$\mathcal{E}_\Delta = \{\{u,v\}\}$$



Theorem

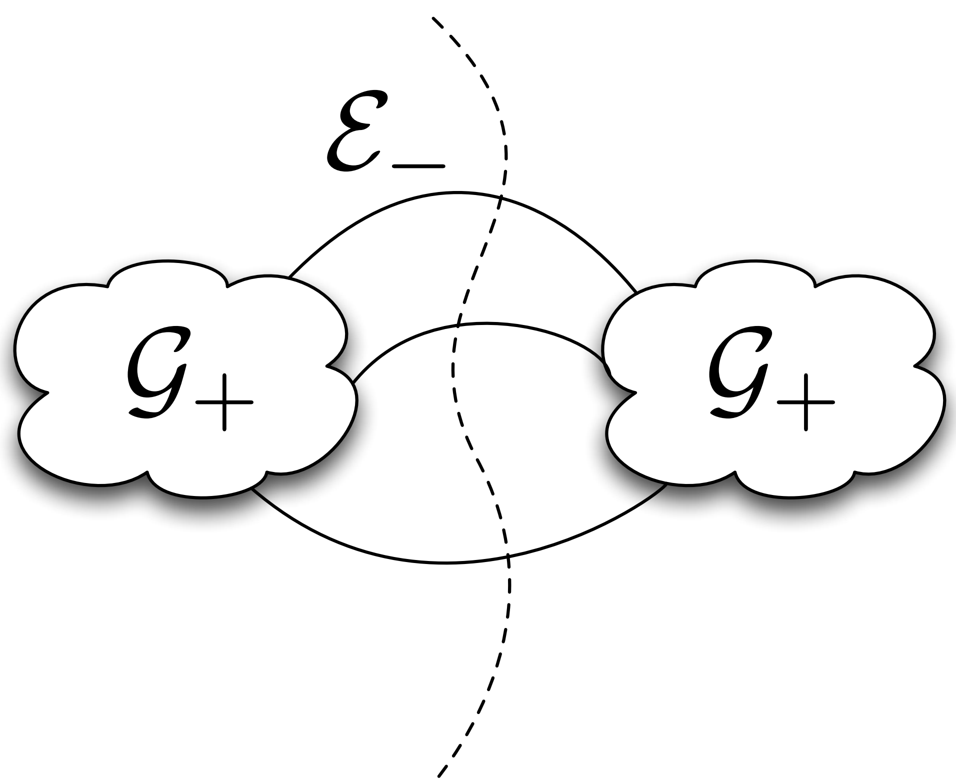
- $\|M_{11}(s)\|_\infty = \mathcal{R}_{uv}$
- The uncertain consensus network is stable for any $\|\Delta\|_\infty < \mathcal{R}_{uv}^{-1}$

for single edge uncertainty, small-gain condition is *exact* (i.e., no conservatism)

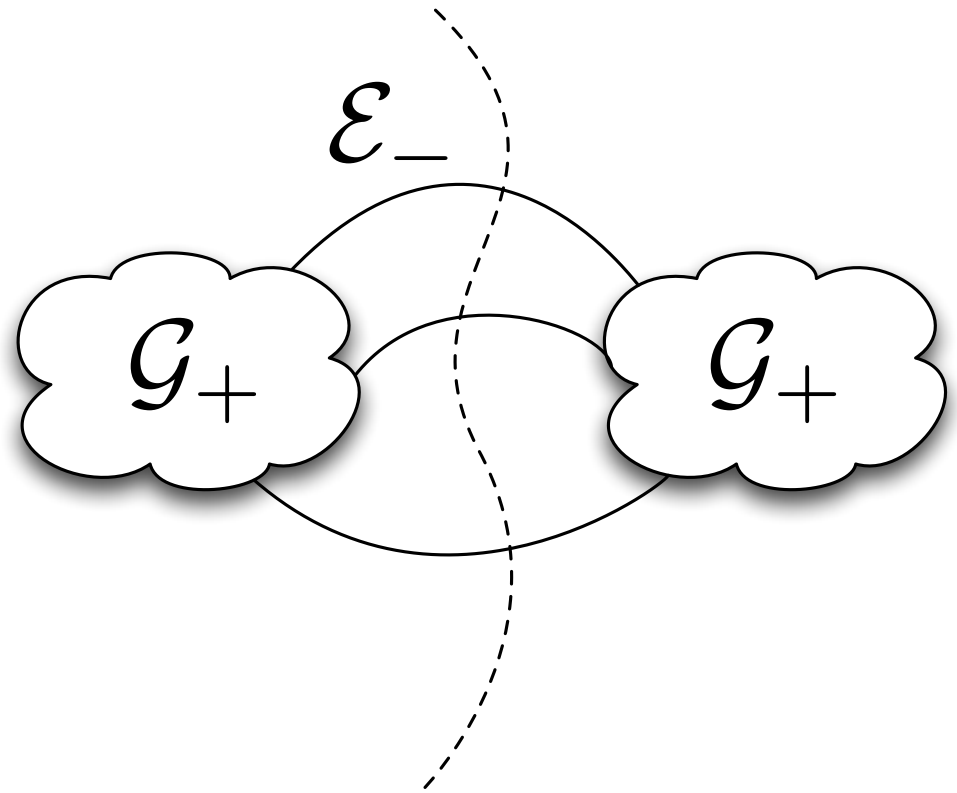
Spectral Properties of Signed Graphs

Corollary *Assume that both $\mathcal{E}_+$ and $\mathcal{E}_-$ are not empty. If $\mathcal{G}_+$ is not connected, then $L(\mathcal{G})$ is indefinite for any choice of negative weights.*

a *balanced* signed graph

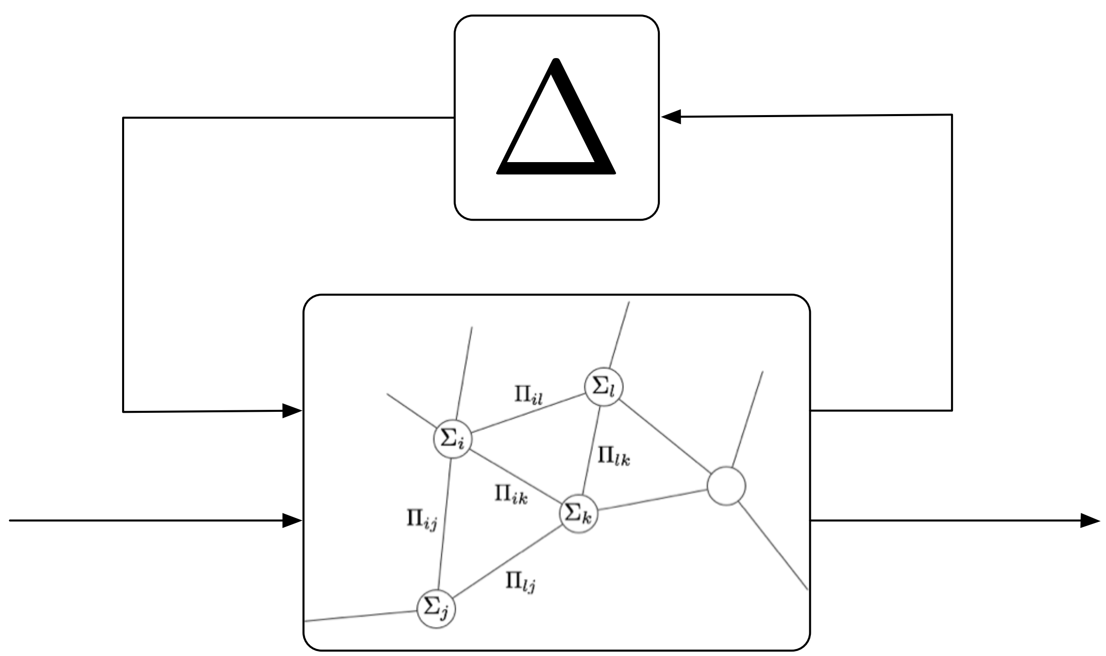


Graph Cuts and Robustness



The smallest cardinality cut of a graph can be thought of as a **combinatorial robustness measure** for linear consensus protocols
==> but *always* conservative

$$\left(\max_{e\in\mathcal{E}_\Delta} \mathcal{W}(e)\right)^{-1} \leq \max_{e\in\mathcal{E}_\Delta} \mathcal{R}_e(\mathcal{G}) \leq \overline{\sigma}(M_{11}(0))$$



As in the single negative weight edge example, graph cuts act to make an "open circuit"

- max-flow/min-cut algorithms
- minimum cardinality cut algorithms (Karger)

A Small-Gain Interpretation

$\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$ sector-bounded nonlinear couplings

$$\begin{cases} \dot{x}_{\mathcal{F}} & = & -L_{ess}(\mathcal{F})x_{\mathcal{F}} - L_e(\mathcal{F})R_{(\mathcal{F},\mathcal{C})}P(\Phi(P^T R^T_{(\mathcal{F},\mathcal{C})}x_{\mathcal{F}})) + E^T_{\mathcal{F}}\mathrm{w} \\ z & = & E(\mathcal{G}_o)^T(E^L_{\mathcal{F}})^T x_{\mathcal{F}} \end{cases}$$

Corollary *Consider the nonlinear edge agreement protocol* $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$ *with* $\mathcal{E}_\Delta = \{\{u, v\}\}$ *(i.e.,* $|\mathcal{E}_\Delta| = 1$*) and assume* $\Sigma(\mathcal{G})$ *is nominally stable. Then* $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$ *is asymptotically stable for all* $\Phi \in \mathbf{\Phi}$ *satisfying*

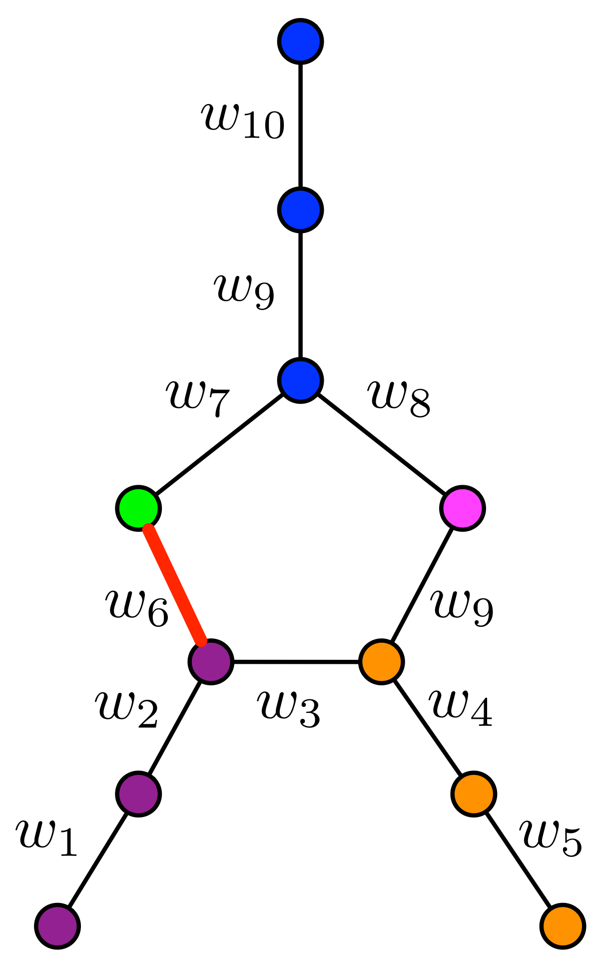
$$|\alpha| < \mathcal{R}^{-1}_{uv}(\mathcal{G}) \text{ and } ((\beta-\alpha)^2 - 2(\beta-\alpha) - 1) > -2w_{uv}.$$

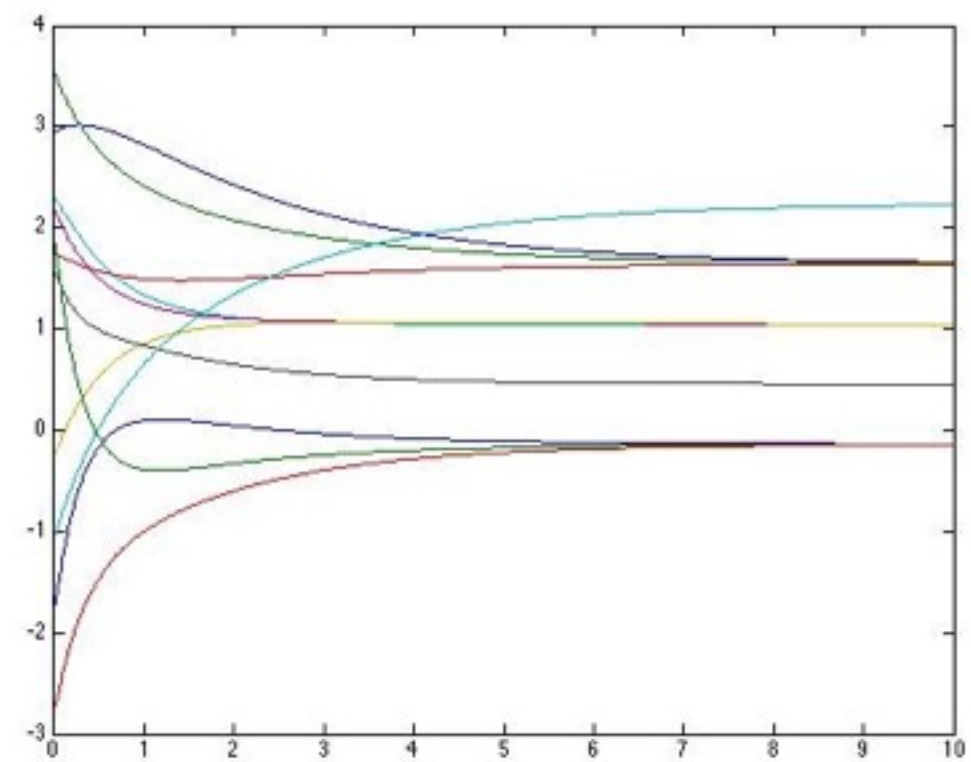
An Illustrative Example

any single edge in the cycle can make the Laplacian indefinite

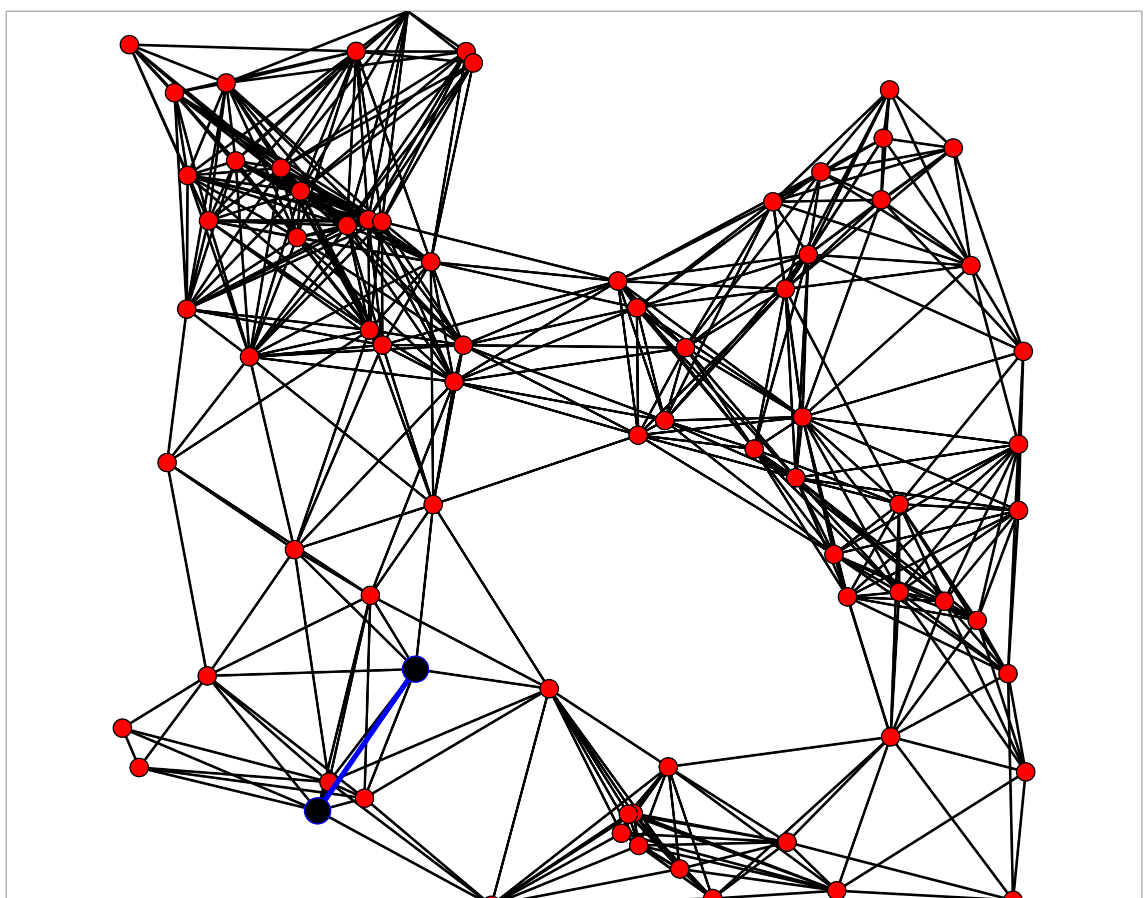
$$w_6 = -\frac{1}{r_6} = -\frac{1}{4}$$

$L(\mathcal{G})$ has two eigenvalues at the origin



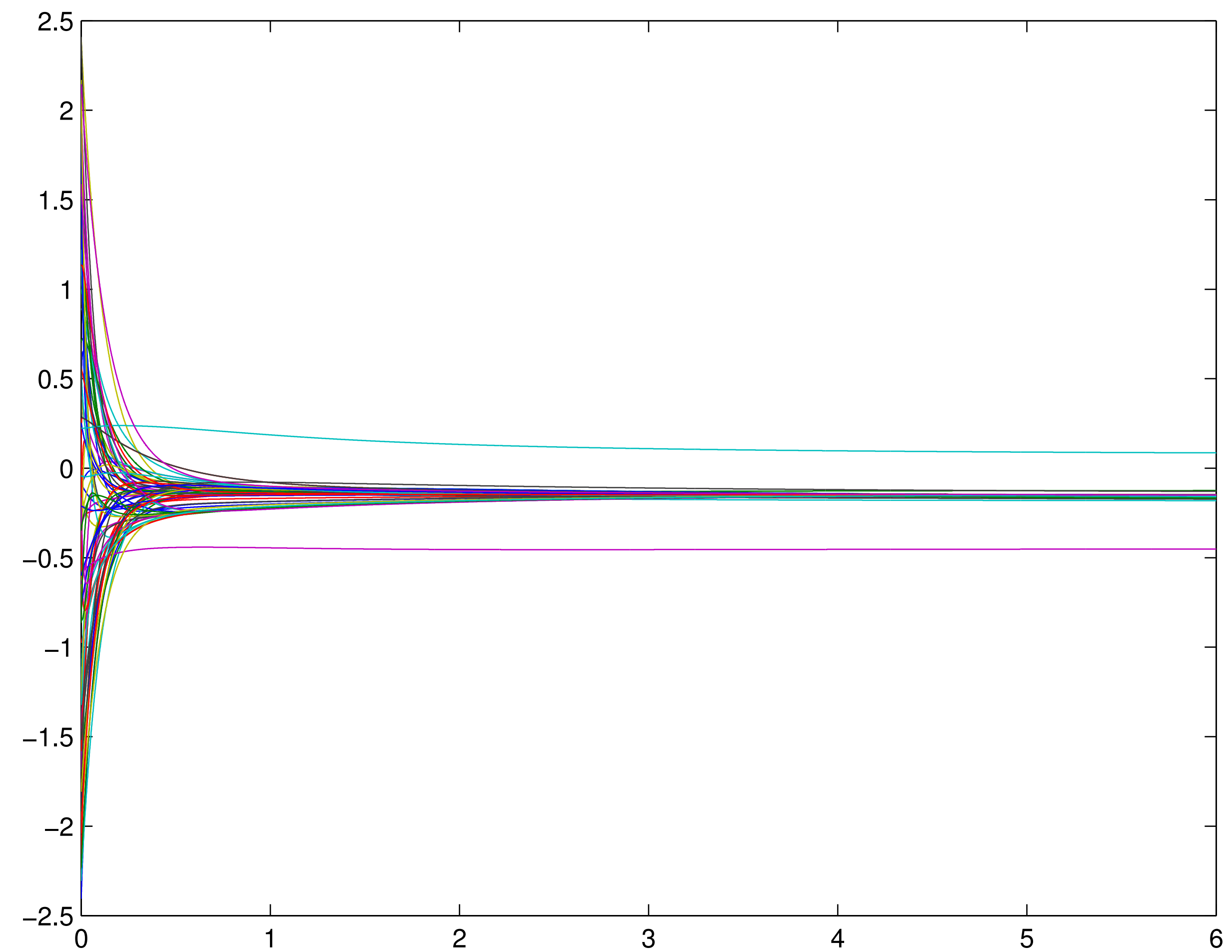


An Illustrative Example

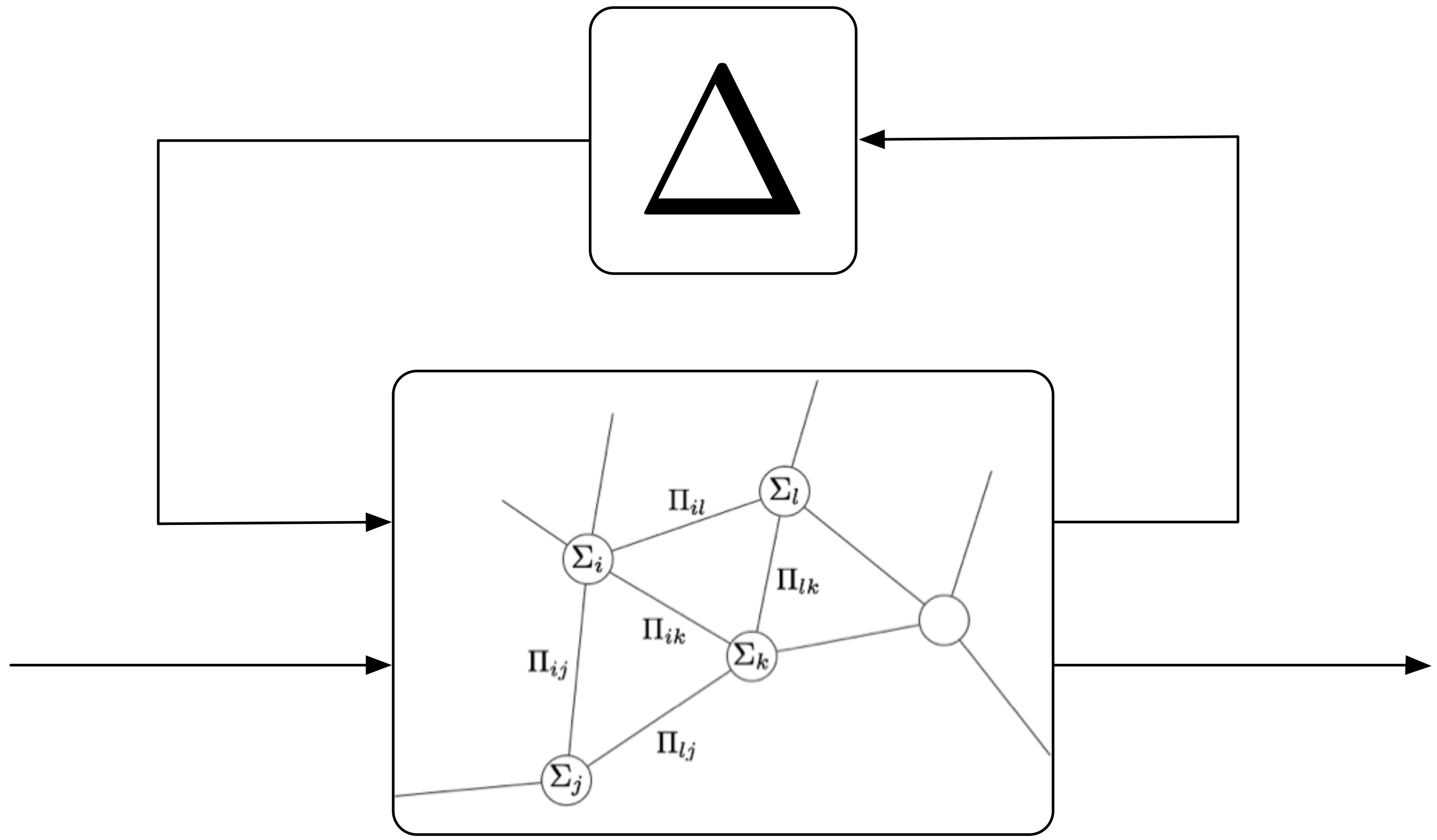


random geometric graph on 75 nodes

uncertain edge in blue



Concluding Remarks



- networked dynamic systems require new tools for robustness analysis
- graph properties have real system theoretic implications

Acknowledgements

Dr. Mathias Bürger

Cognitive Systems Group
at Robert Bosch GmbH

Thank-you!

Questions?

[1] D. Zelazo and M. Bürger, "On the Definiteness of the Weighted Laplacian and its Connection to Effective Resistance," IEEE CDC, Los Angeles, CA, 2014.
[2] D. Zelazo and M. Bürger, "On the Robustness of Uncertain Consensus Networks," submitted to IEEE Transactions on Control of Network Systems, 2014 (preprint on arXiv)