

Toward a Geometric Theory of Passivity

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CONNECT LAB
Cooperative Networks and Controls



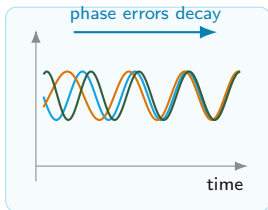
Diffusive Networks: Why We Need Passivity and What Is Missing



Why diffusive networks appear everywhere

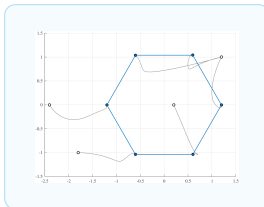
A diffusive law uses only relative information: each unit reacts to disagreement with its neighbors.

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} \psi(y_j - y_i)$$



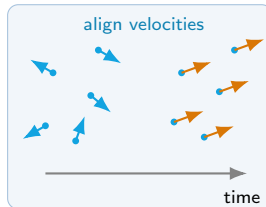
Synchronization

clocks, oscillators, generators, and rhythmic systems use relative phase errors to fall into step.



Formation control

Robots regulate relative states (distances, bearings, etc.) to maintain a desired formation.

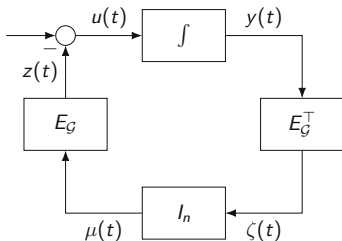


Flocking

Vehicles use only relative velocity measurements to achieve collective motion.



The Linear Consensus Protocol



Undirected Networks

$$\dot{x}(t) = -E_G (E_G^T y(t))$$

$$y(t) = x(t)$$

The linear consensus protocol may be analyzed using standard linear systems theory

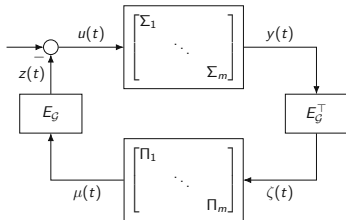
- the closed-loop is an autonomous linear system

$$x(t) = e^{-L_G t} x(0)$$

- the **graph Laplacian**, $L_G = E_G E_G^T$, creates a beautiful bridge between systems theory and graph theory



Nonlinear Diffusive Networks



nonlinear maps:

$$\Sigma_i : u_i \mapsto y_i, \quad \Pi_j : \zeta_j \mapsto \mu_j$$

Nonlinear Undirected Networks

$$\dot{x}(t) = -E_G \Pi (E_G^T y(t))$$

$$y(t) = \Sigma(u(t))$$

$$u(t) = -E_G \Pi(\zeta)$$

Nonlinear networks can be significantly more challenging

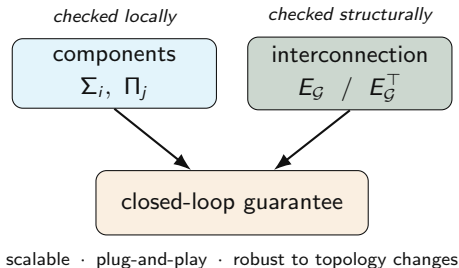
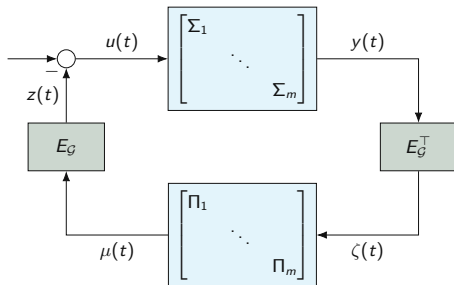
- how to characterize equilibria?
- how to assess convergence and stability properties?
- interplay between network structure and agent/controller dynamics more delicate
 - which system properties do we need to ensure convergence?
 - which graph properties do we need to ensure convergence?



Certify locally, conclude globally

Question

Can we certify the components separately and let the interconnection do the rest?

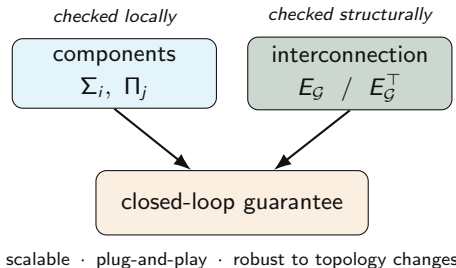
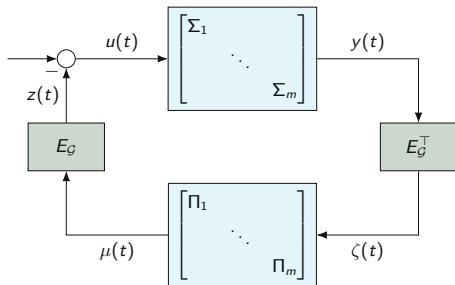




Certify locally, conclude globally

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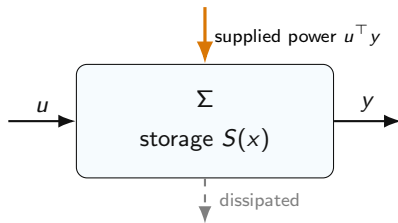
Can we certify the components separately and let the interconnection do the rest?



Passivity provides such a decomposition.

Passivity: storage and supplied power

$$\Sigma : \begin{cases} \dot{x} &= f(x, u), \\ y &= h(x, u) \end{cases}$$



e.g. mass-spring-damper:
 $S = \text{kinetic} + \text{potential energy}$
 $u^\top y = \text{force} \times \text{velocity}$

Storage function

A differentiable, positive semi-definite $S : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$
— the energy stored inside Σ .

Energy balance

Rate of storage cannot exceed supplied power:

$$\dot{S} = \nabla S(x)^\top f(x, u) \leq u^\top y \quad \forall (x, u)$$

equivalently, over any interval $[0, T]$,

$$S(x(T)) - S(x(0)) \leq \int_0^T u(t)^\top y(t) dt.$$

A passive system cannot generate energy — and this is a property of the component alone, checked without knowing how it may be interconnected with other systems.



Passivity: dissipation inequality

Definition (passivity)

The system Σ is passive if there exist a positive semi-definite storage function $S(x)$ and non-negative constants ε, γ such that

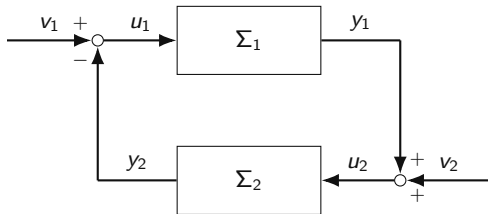
$$\dot{S}(x) \leq u^\top y - \varepsilon \|y\|^2 - \gamma \|u\|^2 \quad \forall (x, u).$$

passive	output-strict	input-strict	input-output strict
$\varepsilon = \gamma = 0$	$\varepsilon > 0, \gamma = 0$	$\varepsilon = 0, \gamma > 0$	$\varepsilon, \gamma > 0, \varepsilon\gamma < \frac{1}{4}$

Strict passivity means that some supplied energy is dissipated rather than stored.



Special interconnection: passivity is preserved by feedback



$$u_1 = v_1 - y_2, \quad u_2 = v_2 + y_1$$

$$u_1^\top y_1 + u_2^\top y_2 = v_1^\top y_1 + v_2^\top y_2 - y_2^\top y_1 + y_1^\top y_2 = v^\top y$$

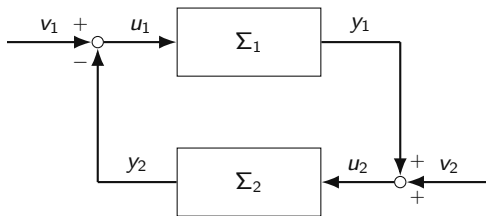
Composite storage

If Σ_1 and Σ_2 are passive with storages S_1, S_2 , then the feedback interconnection is passive with

$$S(x_1, x_2) = S_1(x_1) + S_2(x_2).$$



Special interconnection: passivity is preserved by feedback



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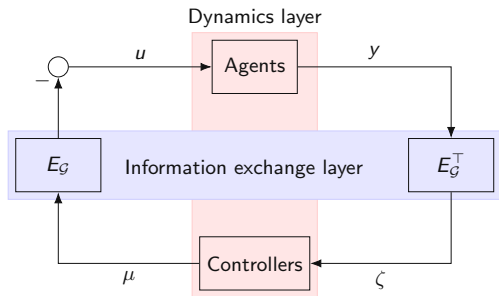
Passivity theorem

The negative-feedback interconnection of passive systems is passive.

Strictness gives dissipation, and dissipation gives stability or convergence.



Passivity and multi-agent networks



Diffusive networks are feedback interconnections+symmetric port interconnections.

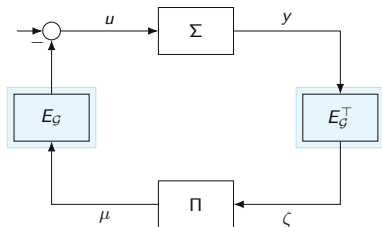
- agent/controller dynamics can be certified locally
- graph operators route relative information
- passivity separates dynamics from topology

Undirected networks

Symmetry of the incidence interconnection gives clean passivity cancellation.



Where the classical picture breaks

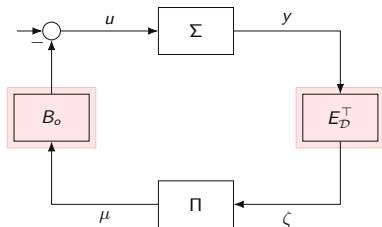


Undirected network $(\Sigma, \Pi, \mathcal{G})$

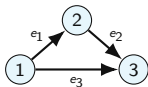
- **Undirected:** $u = -E_{\mathcal{G}}\mu$, $\zeta = E_{\mathcal{G}}^{\top}y$ — an adjoint pair, so the port powers cancel: passivity analysis ✓



Where the classical picture breaks



Directed network $(\Sigma, \Pi, \mathcal{D})$



$$E_{\mathcal{D}} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad B_o = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

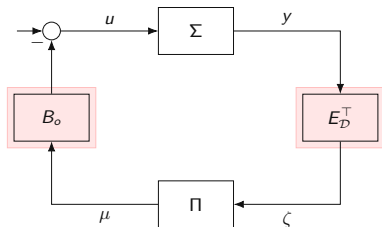
B_o keeps the +1 entries (where each edge starts); $B_i = E_{\mathcal{D}} - B_o$ keeps the -1 entries.

- **Undirected:** $u = -E_{\mathcal{G}}\mu$, $\zeta = E_{\mathcal{G}}^{\top}y$ — an adjoint pair, so the port powers cancel: passivity analysis ✓
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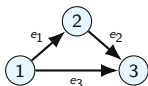
Directed feedback path can lose passivity even for the linear protocol: Yue & Zelazo, *Eur. J. Control*, 2025.



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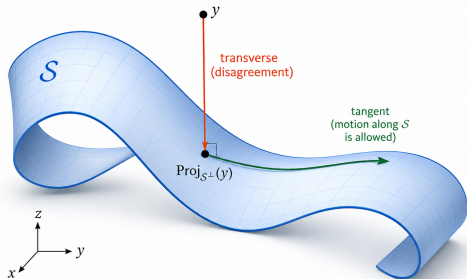
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- **Directed:** $u = -B_o \mu$, $\zeta = E_{\mathcal{D}}^T y$ with $B_o \neq E_{\mathcal{D}}$ — the cancellation is **lost**: passivity analysis ?

In the classical analysis the agreement set never appears in the certificates — it is hidden in the symmetric interconnection. Without symmetry, the target must enter the passivity notion itself.

Directed feedback path can lose passivity even for the linear protocol: Yue & Zelazo, *Eur. J. Control*, 2025.



Consensus is manifold stabilization



Agreement manifold $S = \text{span}\{\mathbb{1}\}$, with

$$\text{Proj}_S(y) = \frac{1}{n} \mathbb{1} \mathbb{1}^\top y.$$

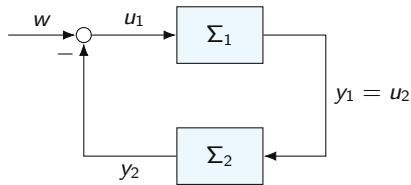
Output consensus:

$$\lim_{t \rightarrow \infty} y(t) \in S \iff \lim_{t \rightarrow \infty} \underbrace{y(t) - \text{Proj}_S(y(t))}_{\text{transverse deviation}} = 0.$$

- only the *transverse* deviation matters;
- motion *along* S is free.



A geometric input–output view



standard negative feedback

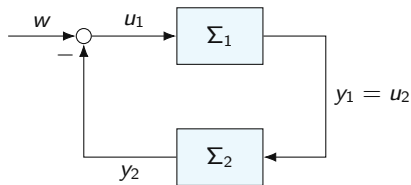
Montenbruck, Arcak & Allgöwer (2017):

- measure signals by their *distance* to a target submanifold: $\int_0^\infty \text{dist}(y(t), \mathcal{M})^2 dt < \infty$;
- passivity of input–output *relations* in these distances;
- a feedback theorem: enough excess passivity $\Rightarrow$ the loop's signals stay close to the submanifold.

Montenbruck, Arcak & Allgöwer, "An input–output framework for submanifold stabilization," *IEEE TAC*, 2017.



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- passivity of input–output *relations* in these distances;
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This talk keeps the transverse-error viewpoint, and adds (i) a state-space storage — a dissipation inequality relative to manifolds, and (ii) general symmetric channels $u_2 = My_1$, $u_1 = -M^\top y_2$, as in networks.

Montenbruck, Arcak & Allgöwer, "An input–output framework for submanifold stabilization," *IEEE TAC*, 2017.



The plan: three questions

1. Components

What should passivity mean when the target is a *manifold*?



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2. Interconnection

Does feedback *preserve* that geometry?



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Does feedback *preserve* that geometry?

3. Convergence

When does dissipation actually imply convergence *to the target*?

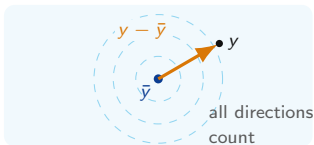
**Q1: What Should Passivity Mean
Relative to a Manifold?**



The geometry of passivity

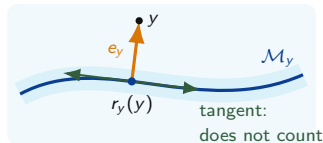
Classical viewpoint: point targets

Supply relative to a point: $(u - \bar{u})^\top (y - \bar{y})$.
Every displacement from the target enters the supply.



Geometric viewpoint: manifold targets

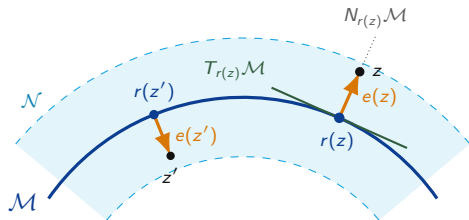
Supply relative to manifolds: $\langle e_u(u), e_y(y) \rangle$.
Only the *normal* errors enter; motion along the manifold is free.



Next: *turn this supply into a state-space property — a storage V with $\dot{V} \leq \langle e_u(u), e_y(y) \rangle$. Targets sit at both ports, because in a loop one system's output is the other's input.*



Tubular neighborhoods and nearest-point retractions



$\mathcal{M} \subset \mathbb{R}^n$ smooth embedded submanifold. On a tubular neighborhood $\mathcal{N}$ every z has a *unique* nearest point:

$$r(z) = \arg \min_{p \in \mathcal{M}} \|z - p\|^2, \quad e(z) := z - r(z) \in N_{r(z)}\mathcal{M},$$

so that $z = r(z) + e(z)$ and $\|e(z)\| = \text{dist}(z, \mathcal{M})$.

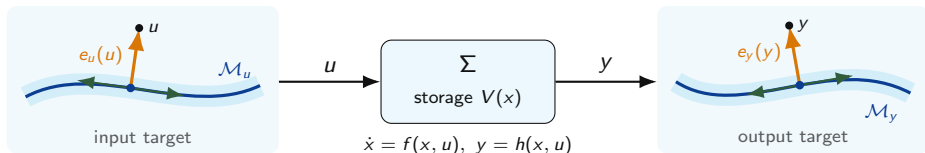
Lemma (projection calculus)

- on $\mathcal{M}$: $e = 0$, $d_z r = P_{T_z \mathcal{M}}$, $d_z e = P_{N_z \mathcal{M}}$
- on $\mathcal{N}$: $d_z \left(\frac{1}{2} \|e(z)\|^2 \right) [v] = \langle e(z), v \rangle$

*The gradient of the squared distance is the transverse error:
the engine behind every storage in this talk.*



Definition: transversal passivity



Definition (transversal passivity)

Σ is $(\mathcal{M}_u, \mathcal{M}_y)$ -passive if there exist a C^1 storage $V \geq 0$ and constants $\varepsilon, \gamma \geq 0$ such that

$$d_x V[f(x, u)] \leq \langle e_u(u), e_y(y) \rangle - \varepsilon \|e_y(y)\|^2 - \gamma \|e_u(u)\|^2$$

on the tubular domain where both errors are defined, $\mathcal{D} := \{(x, u) : u \in \mathcal{N}_u, h(x, u) \in \mathcal{N}_y\}$.

passive	output-strict	input-strict	input-output strict
$\varepsilon = \gamma = 0$	$\varepsilon > 0, \gamma = 0$	$\varepsilon = 0, \gamma > 0$	$\varepsilon, \gamma > 0, \varepsilon\gamma < \frac{1}{4}$

Normal errors enter the supply;
tangential motion does not.

Local: certifies behavior only while
 u and y stay in their tubes.



Example: one integrator, many targets

Integrator agents $\dot{x} = u$, $y = x$: any output target works

Take $\mathcal{M}_u = \{0\}$, any $\mathcal{M}_y$, and $V(x) = \frac{1}{2} \|e_y(x)\|^2$. Projection calculus gives

$$\dot{V} = \langle e_y(y), u \rangle = \langle e_u(u), e_y(y) \rangle.$$

Output target $\mathcal{M}_y$	Error $e_y(y)$	Storage $V(x)$	Task
$\{0\}$	y	$\frac{1}{2} \ x\ ^2$	stabilization (classical passivity)
$\{y^*\}$	$y - y^*$	$\frac{1}{2} \ x - y^*\ ^2$	set-point (shifted passivity)
$S = \text{span}\{\mathbb{1}\}$	$P_{S^\perp} y$	$\frac{1}{2} \ P_{S^\perp} x\ ^2$	consensus (disagreement energy)
$\mathbb{S}^{n-1}$ (unit sphere)	$y - \frac{y}{\ y\ }$	$\frac{1}{2} (\ x\ - 1)^2$	normalization; tube is $x \neq 0$

- The input target is not unique either: the integrator is also (S, S) -passive, since a common input $u = c\mathbb{1}$ moves all states together along S and enters no supply.
- Never output-strict ($\varepsilon = 0$): closed-loop dissipation must come from the controllers or the graph.



A constructive certificate: output-distance storage

Idea: use the squared distance of the output to its target as the storage. Then transversal passivity reduces to a single inequality: how fast the output moves away from $\mathcal{M}_y$ must be bounded by the supply.

Step 1: build the storage

For $y = h(x)$:

$$V(x) := \frac{1}{2} \text{dist}(h(x), \mathcal{M}_y)^2$$

Step 2: check one inequality

With $\dot{y} = d_x h[f(x, u)]$:

$$\langle e_y(y), \dot{y} \rangle \leq \langle e_u(u), e_y(y) \rangle - \varepsilon \|e_y(y)\|^2$$

$\Rightarrow$ transversally passive with $(\varepsilon, \gamma) = (\varepsilon, 0)$.

- **Why:** projection calculus gives $\dot{V} = \langle e_y(y), \dot{y} \rangle$.
- $\gamma = 0$ is unavoidable: with $y = h(x)$ there is no feedthrough.

For $y = h(x, u)$ the storage depends on u ; this needs extra assumptions or an augmented state.

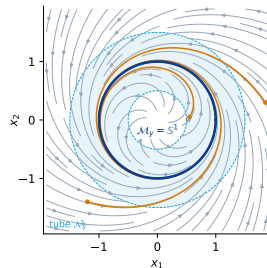


Example: an oscillator — transversally passive, but not passive

A limit-cycle oscillator

$$\dot{x} = \mathcal{J}x - \left(1 - \frac{1}{\|x\|}\right)x + u, \quad y = x, \quad \mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Unforced: $\dot{r} = 1 - r$, $\dot{\theta} = 1$ — converges to the unit circle, keeps rotating.



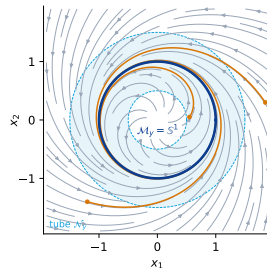


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Classical passivity fails

Hold the state still at $\bar{r} < 1$ with a constant $\bar{u}$: then $\bar{u}^\top \bar{y} = \bar{r}(\bar{r} - 1) < 0$ forever, so $\int_0^T \bar{u}^\top \bar{y} dt \rightarrow -\infty$ — no storage $S \geq 0$ allows this.

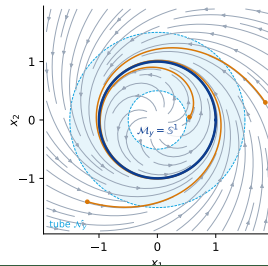


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Transversally passive w.r.t. $(\{0\}, \mathbb{S}^1)$

Distance-to-cycle storage $V = \frac{1}{2}(\|x\| - 1)^2$:
 $\dot{V} \leq \langle e_u(u), e_y(y) \rangle - \varepsilon \|e_y(y)\|^2 - \gamma \|e_u(u)\|^2$
holds (with equality) for $\varepsilon = 1$, $\gamma = 0$.

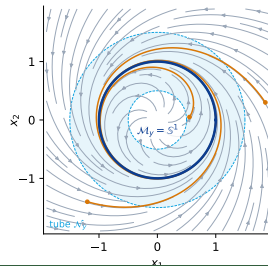


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 holds (with equality) for $\varepsilon = 1$, $\gamma = 0$.

Why care? *Relative to its cycle it is passive — so it can enter passivity-based networks.*

Oscillators via dissipativity with non-passive supply rates: Stan & Sepulchre, *IEEE TAC*, 2007.



Classical notions as special cases

Target manifolds	Transverse errors	Recovered notion	Classic ref.
$\mathcal{M}_u = \{0\}, \mathcal{M}_y = \{0\}$	u, y	standard passivity	[1]
$\{u^*\}, \{y^*\}$	$u - u^*, y - y^*$	shifted passivity; EIP/MEIP (one per equilibrium)	[2], [3]
affine $u^* + S_u, y^* + S_y$	$P_{S_u^\perp}(u - u^*), P_{S_y^\perp}(y - y^*)$	shifted dissipativity in normal coordinates	cf. [2]
diagonals $\Delta_u \subset \mathcal{U}^2$, $\Delta_y \subset \mathcal{Y}^2$	$\frac{1}{2}(u_1 - u_2, u_2 - u_1), \dots$	incremental passivity	[4]
infinitesimal, at an equilibrium	$P_u \delta u, P_y \delta y$	<i>related:</i> differential passivity	[5]

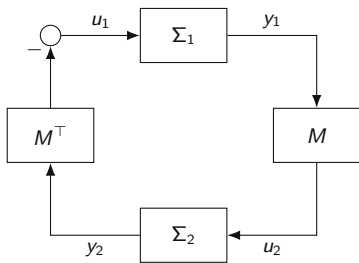
New behavior begins when the manifolds are curved, or when feedback reshapes the feasible target set.

- [1] Willems, *Arch. Rational Mech. Anal.*, 1972. [2] Monshizadeh, Monshizadeh, Ortega & van der Schaft, *SCL*, 2019. [3] Hines, Arcak & Packard, *Automatica*, 2011; Bürger, Zelazo & Allgöwer, *Automatica*, 2014. [4] Pavlov & Marconi, *SCL*, 2008. [5] van der Schaft, *IFAC NOLCOS*, 2013.

**Q2: Does Feedback Preserve
the Geometry?**



Feedback interconnection: what changes?



$$u_2 = My_1, \quad u_1 = -M^T y_2$$

Classical passivity. Adding $\dot{V}_i \leq u_i^T y_i$,

$$\dot{V}_1 + \dot{V}_2 \leq u_1^T y_1 + u_2^T y_2.$$

The interconnection is skew-symmetric — it is *power conserving*:

$$u_1^T y_1 + u_2^T y_2 = (-M^T y_2)^T y_1 + (My_1)^T y_2 = 0.$$

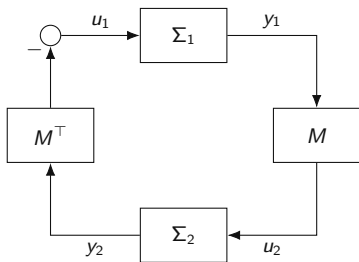
Closed-loop certificate for free

$$\dot{V}_1 + \dot{V}_2 \leq 0,$$

so $V_1 + V_2$ is a Lyapunov function for the interconnection.



Feedback interconnection: what changes?



$$u_2 = My_1, \quad u_1 = -M^T y_2$$

Transversal passivity. Now each Σ_i is $(\mathcal{M}_{u_i}, \mathcal{M}_{y_i})$ -passive with storage V_i . Adding,

$$\dot{V}_1 + \dot{V}_2 \leq \langle e_{u_1}(u_1), e_{y_1}(y_1) \rangle + \langle e_{u_2}(u_2), e_{y_2}(y_2) \rangle.$$

Power conservation constrains the ports, not their projections

With $u_2 = My_1$, $u_1 = -M^T y_2$ these transverse supplies need not cancel. They do when the channels *transport* transverse errors:

$$e_{u_2}(My_1) = Me_{y_1}(y_1), \quad e_{u_1}(-M^T y_2) = -M^T e_{y_2}(y_2).$$

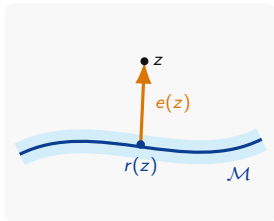
Question

A linear channel has no reason to map the transverse error of y_1 to the transverse error of $u_2 = My_1$. What replaces the classical cancellation?



Setup: measure, transport, close the loop

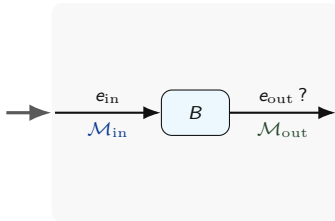
1. measure the error



How far is a port from its target?

nearest point r , transverse error
 $e = z - r(z)$ (seen)

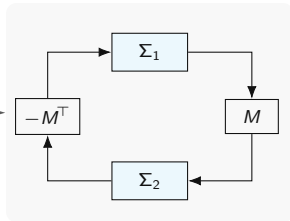
2. transport the error



Does a channel carry the error at its input to the error at its output?

compatibility, defect δ_B (next)

3. close the loop

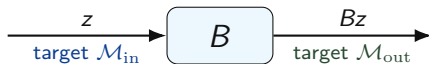


Two channels, $B = M$ and
 $B = -M^T$.

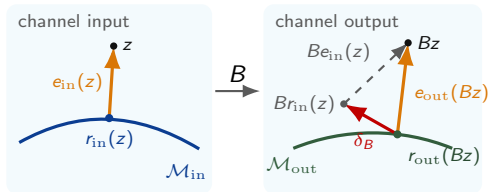
closed-loop dissipation



Channel compatibility: defects and feasible targets



1. **Defect:** the error is not carried along.

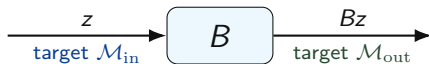


$$\delta_B(z) := \underbrace{e_{\text{out}}(Bz)}_{Bz - r_{\text{out}}(Bz)} - \underbrace{B e_{\text{in}}(z)}_{z - r_{\text{in}}(z)} = B r_{\text{in}}(z) - r_{\text{out}}(Bz)$$

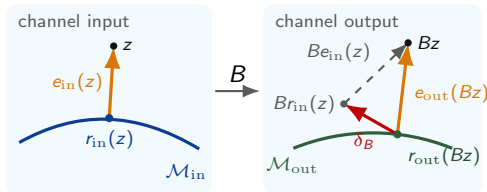
“map, then measure” $\neq$ “measure, then map”: the nearest points disagree.



Channel compatibility: defects and feasible targets



1. **Defect:** the error is not carried along.



2. **Feasible targets:** on target at the input need not mean on target at the output.

$$\mathcal{F}_B = \mathcal{M}_{\text{in}} \cap B^{-1}(\mathcal{M}_{\text{out}})$$

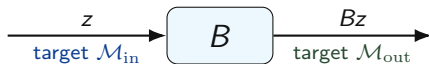
Domain: $\mathcal{D}_B = \{z \in \mathcal{N}_{\text{in}} : Bz \in \mathcal{N}_{\text{out}}\}$, where both errors are defined.

$$\delta_B(z) := \underbrace{e_{\text{out}}(Bz)}_{Bz - r_{\text{out}}(Bz)} - \underbrace{B e_{\text{in}}(z)}_{B r_{\text{in}}(z) - r_{\text{out}}(Bz)} = B r_{\text{in}}(z) - r_{\text{out}}(Bz)$$

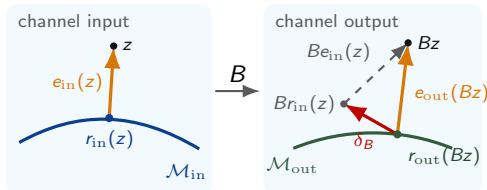
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Domain: $\mathcal{D}_B = \{z \in \mathcal{N}_{\text{in}} : Bz \in \mathcal{N}_{\text{out}}\}$, where both errors are defined.

In the loop: forward channel $B = M$ ($z = y_1 \mapsto u_2$, defect δ_{fc}), reverse channel $B = -M^\top$ ($z = y_2 \mapsto u_1$, defect δ_{rc}). The loop can only settle on $\mathcal{F}_M := \mathcal{F}_{B=M} \times \mathcal{F}_{B=-M^\top}$ — feedback reshapes the target.



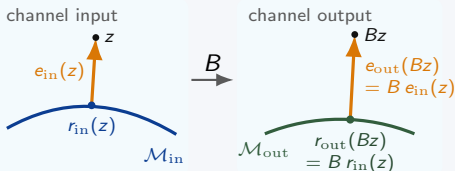
Exact compatibility: commutation and target preservation

Definition. B is *exactly compatible* on $\Omega \subseteq \mathcal{D}_B$ if $\delta_B \equiv 0$ on Ω (loop: for both M and $-M^\top$).

Proposition (commutation)

B is exactly compatible on $\Omega \iff$

$$r_{\text{out}}(Bz) = B r_{\text{in}}(z) \quad \forall z \in \Omega.$$



nearest point maps to nearest point: $\delta_B = 0$

Proposition (target preservation)

If B is exactly compatible on Ω , then for $z \in \Omega$

$$\text{dist}(z, \mathcal{F}_B) = \|e_{\text{in}}(z)\|, \quad \Omega \cap \mathcal{F}_B = \Omega \cap \mathcal{M}_{\text{in}}.$$

In words: no *extra* feasibility constraint — every target point is feasible.

The catch

Needs a nonlinear projection to commute with a linear B on an *open set*: restrictive for curved manifolds $\rightarrow$ **relax to first order**.



First-order compatibility

Definition (first-order compatibility)

Let $\delta_B \in C^1(\mathcal{D}_B)$. B is *first-order compatible along* $\mathcal{S} \subseteq \mathcal{F}_B$ if

$$d_z \delta_B = 0 \quad \forall z \in \mathcal{S}.$$

Order of vanishing. On $\mathcal{S} \subseteq \mathcal{F}_B$:

- feasibility kills the zeroth order: $\delta_B(z) = 0$;
- first-order compatibility kills the linear term:

$$\delta_B(p) = \frac{1}{2} d_z^2 \delta_B[p - z, p - z] + o(\|p - z\|^2).$$

Proposition (commuting differentials)

If $r_{\text{in}}, r_{\text{out}} \in C^1$, then B is first-order compatible along $\mathcal{S}$ iff

$$d_{Bz} r_{\text{out}} \circ B = B \circ d_z r_{\text{in}} \quad \forall z \in \mathcal{S}.$$

On $\mathcal{S}$ the differentials are tangent projections:

$$P_{T_{Bz} \mathcal{M}_{\text{out}}} B = B P_{T_z \mathcal{M}_{\text{in}}},$$

so along $\mathcal{S}$ the channel maps tangent directions to tangent directions.

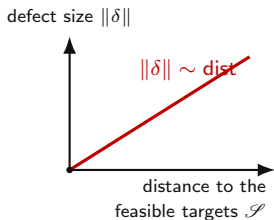
Exact compatibility on an open set implies first-order compatibility along the feasible targets inside it — but not conversely.



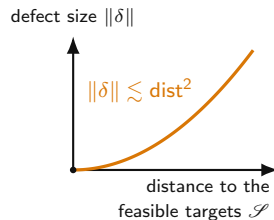
First-order compatibility: a uniform quadratic defect

Question

How fast does the defect grow as the ports move away from the feasible targets?



generic channel: linear growth



first-order compatible: quadratic growth

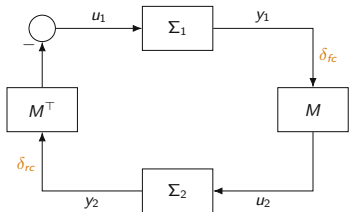
Lemma (uniform quadratic estimate)

If $\mathcal{S} = \mathcal{S}_{fc} \times \mathcal{S}_{rc} \subseteq \mathcal{F}_M$ is **compact**, both channels are first-order compatible along it, and $\delta_{fc}, \delta_{rc} \in C^2$, then near $\mathcal{S}$, for some $K > 0$,

$$\|\delta_{fc}(y_1)\| + \|\delta_{rc}(y_2)\| \leq \frac{K}{2} \text{dist}((y_1, y_2), \mathcal{S})^2.$$



Errors around the loop: transported, up to a defect



Without targets: the classical cancellation

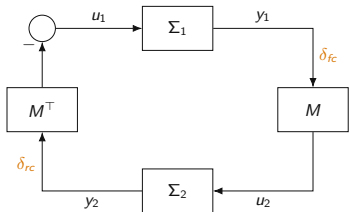
$$u_2 = My_1, \quad u_1 = -M^\top y_2 \Rightarrow$$

$$\langle u_1, y_1 \rangle + \langle u_2, y_2 \rangle = -\langle M^\top y_2, y_1 \rangle + \langle My_1, y_2 \rangle = 0.$$

The interconnection neither creates nor destroys power.



Errors around the loop: transported, up to a defect



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With targets: errors are transported, up to a defect

Defect of a channel: $\delta_B(z) = e_{\text{out}}(Bz) - B e_{\text{in}}(z)$. Hence

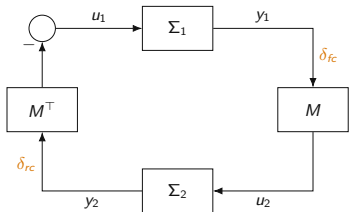
$$e_{u_2} = M e_{y_1} + \delta_{fc}, \quad \delta_{fc} := \delta_M(y_1),$$

$$e_{u_1} = -M^\top e_{y_2} + \delta_{rc}, \quad \delta_{rc} := \delta_{-M^\top}(y_2).$$

transported error + defect, wherever all port errors are defined.



Errors around the loop: transported, up to a defect



Without targets: the classical cancellation

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transported error + defect, wherever all port errors are defined.

Exactly compatible ($\delta \equiv 0$): in error coordinates, the loop is the classical interconnection.



Closed-loop dissipation with compatibility defects

Theorem (closed-loop dissipation)

Σ_1, Σ_2 transversally passive with storages $V_1, V_2 \Rightarrow V_{\text{cl}} = V_1 + V_2$ satisfies, wherever all port errors are defined,

$$\dot{V}_{\text{cl}} \leq s_{\delta} - Q, \quad s_{\delta} := \langle \delta_{rc}, e_{y_1} \rangle + \langle \delta_{fc}, e_{y_2} \rangle, \quad Q := \sum_i \varepsilon_i \|e_{y_i}\|^2 + \gamma_i \|e_{u_i}\|^2.$$



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Proof. Add the two dissipation inequalities and substitute the errors around the loop:

$$\begin{aligned} \dot{V}_{cl} &\leq \langle e_{u_1}, e_{y_1} \rangle + \langle e_{u_2}, e_{y_2} \rangle - Q = \langle -M^\top e_{y_2} + \delta_{rc}, e_{y_1} \rangle + \langle M e_{y_1} + \delta_{fc}, e_{y_2} \rangle - Q \\ &= \underbrace{-\langle M^\top e_{y_2}, e_{y_1} \rangle + \langle M e_{y_1}, e_{y_2} \rangle}_{= 0 \text{ (adjointness)}} + \textcolor{brown}{s}_\delta - Q. \quad \square \end{aligned}$$



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Exact compatibility

$$s_\delta \equiv 0 \Rightarrow \dot{V}_{cl} \leq -Q \leq 0$$

classical cancellation, in full

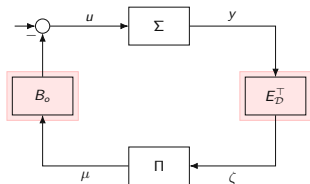
First-order compatibility (near $\mathcal{S}$, $d := \text{dist}(y, \mathcal{S})$)

$$|s_\delta| \leq \underbrace{\|(\delta_{fc}, \delta_{rc})\|}_{\lesssim d^2} \underbrace{\|(e_{y_1}, e_{y_2})\|}_{\leq d} \leq c_\delta d^3$$

harmless near $\mathcal{S}$ — if Q controls d^2 (Q3)



Back to directed consensus: compatible, yet not cancelling



1. Each channel is exactly compatible

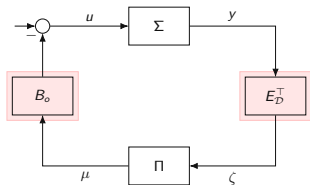
$$E_D^\top \mathbb{1} = 0: \quad e_\zeta(E_D^\top y) = E_D^\top e_y(y);$$

$$B_o(\ker B_o) = 0: \quad e_u(-B_o \mu) = -B_o e_\mu(\mu).$$

- agents Σ : $(\{0\}, S)$ -passive;
- controllers Π : $(\{0\}, \ker B_o)$ -passive;
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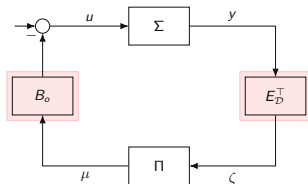
2. ... but cancellation needs more

Cross term $s = \langle e_y, (E_D - B_o) e_\mu \rangle = \langle e_y, B_i e_\mu \rangle$; it vanishes if the channels are *transversally adjoint*:

$$P_{S^\perp} B_i P_{(\ker B_o)^\perp} = 0.$$



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When it does not vanish: compensate it

$w = B_i \mu$ is absorbed by *compensation*: S -passive [= $(\{0\}, S)$ -transversally passive] agents + controller excess + graph $(\lambda_2, \text{out-degrees})$ — balanced^[1] or general^[2] digraphs.

[1] Yue & Zelazo, *Eur. J. Control*, 2025. [2] Yue & Zelazo, *IEEE CDC*, 2025.

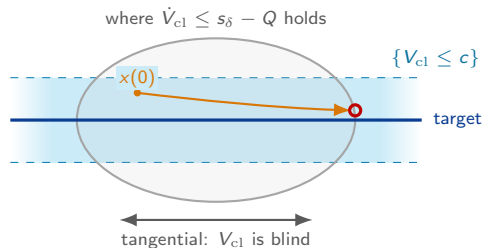
**Q3: When Does Dissipation
Imply Convergence?**



Why dissipation alone does not give convergence

The storage is blind along the target

V_{cl} only sees *transverse* errors: its sublevel sets are *unbounded* along the target, and $\dot{V}_{cl} \leq s_\delta - Q$ holds only *near* it (ports in their tubes). $\dot{V}_{cl} \leq 0$ does not stop a trajectory from sliding away.

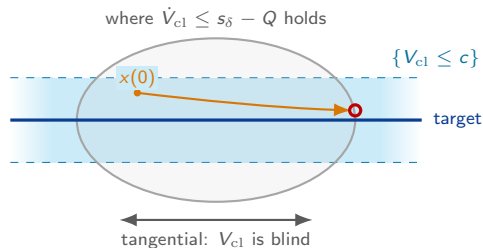




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Theorem (port convergence, informal)

(i) **Confinement:** the trajectory stays in a *compact* set where the dissipation inequality holds (e.g. via a conserved quantity);

(ii) **Margin:** $\dot{V}_{cl} \leq -\alpha \text{dist}(y, \mathcal{F}_M)^2$, $\alpha > 0$.

$\implies \text{dist}(y(t), \mathcal{F}_M) \rightarrow 0$: every port converges to its target (Barbalat).



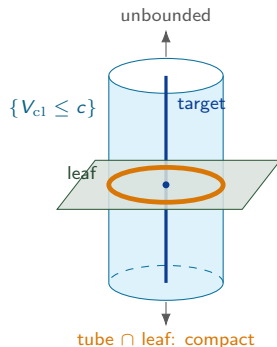
Confinement: where does compactness come from?

The problem. $\{V_{cl} \leq c\}$ is a *tube* around the target:
bounded across, unbounded along.

Route 1: a conserved quantity

If the loop conserves $\chi(x)$, the trajectory stays on its *leaf* $\{\chi = \chi(x_0)\}$: tube $\cap$ leaf can be **compact**, although neither is.

Consensus: the average $\mathbb{1}^\top x$ (directed: $w^\top x$) is conserved — the leaf fixes *which* agreement point; tube $\cap$ leaf is a ball around it.



Confinement picks where on the target we can end up; the margin makes us get there.



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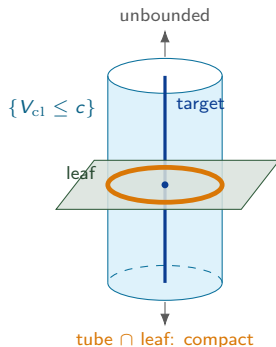
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Route 2: a compact target

Compact target (e.g. a periodic orbit): a thin tube is already compact.



Confinement picks where on the target we can end up; the margin makes us get there.



Where does the dissipation margin come from?

Convergence needs a margin: $\dot{V}_{cl} \leq -\alpha_p \text{dist}(y, \mathcal{F}_M)^2$. Where does $\alpha_p > 0$ come from?

Exact compatibility: from the subsystems

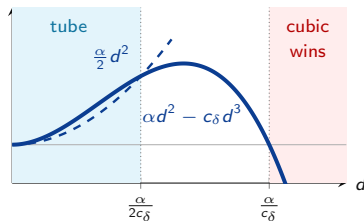
No residual supply ($s_\delta \equiv 0$), so the margin is the subsystems' own strictness: if both are output-strict, $\alpha_p = \min\{\varepsilon_1, \varepsilon_2\}$.

First-order: quadratic beats cubic

With $d = \text{dist}(y, \mathcal{S})$, a quadratic margin αd^2 beats the cubic residual:

$$\alpha d^2 - c_\delta d^3 \geq \frac{\alpha}{2} d^2$$

whenever $d < \alpha/(2c_\delta)$: $\alpha_p = \alpha/2$ on that tube.



solid: net dissipation; dashed: guaranteed margin



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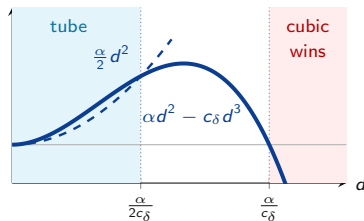
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Caveat: Q only charges normal displacement, so α is a hypothesis — supplied by the graph, a product structure, or state dissipation.



Three questions, three answers

1. Components

What should passivity mean when the target is a *manifold*?



Transversal passivity: dissipation on the nearest-point errors e_u, e_y at both ports.

check: a storage certificate for each subsystem



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Does feedback *preserve* that geometry?

Up to the channel defect: $\dot{V}_{cl} \leq s_\delta - Q$. Exact
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3. Convergence

When does dissipation actually imply convergence *to the target*?

Confinement + margin: a compact region (conserved
 $\Rightarrow$ quantity or compact target) and $\dot{V}_{cl} \leq -\alpha_p \text{dist}^2$.
check: confinement and a quadratic margin



Three questions, three answers

1. Components

What should passivity mean when the target is a *manifold*?

Transversal passivity: dissipation on the nearest-point
 $\Rightarrow$ errors e_u, e_y at both ports.
check: a storage certificate for each subsystem

2. Interconnection

Does feedback *preserve* that geometry?

Up to the channel defect: $\dot{V}_{cl} \leq s_\delta - Q$. Exact
 $\Rightarrow$ compatibility: classical cancellation; first order: cubic residual.
check: compatibility of M and $-M^\top$ along the feasible targets

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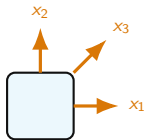
The recipe: *targets* $\rightarrow$ *certificates* $\rightarrow$ *compatibility* $\rightarrow$ *confinement* $\rightarrow$ *margin*. Next: two case studies, checked line by line.

**Case Studies:
the Recipe at Work**



Case study I: thrust allocation with a curved demand

Problem. A planar vehicle with three fixed thrusters $x \in \mathbb{R}^3$ produces the net force $F = Bx$.



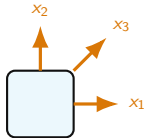
$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- **Redundancy:** $\ker B = \text{span}(1, 1, -1)$ — thrusters push against each other: *internal force*, no net force.
- **Demand:** $|F| = f$, direction free (e.g. hold a tether at constant tension)



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The target in thruster space

Split $x = J\eta + \xi$, $\eta = J^\top x \in \mathbb{R}^2$, $\xi \in \ker B$:

$$J = B^\top (BB^\top)^{-1/2} \quad (J^\top J = I).$$

Then $F = (BB^\top)^{1/2}\eta$: the force depends on η only, and

$$\mathcal{M}_x = \{x : |Bx| = f\} = J\mathcal{C} + \ker B,$$

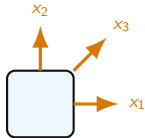
$$\mathcal{C} = \{\eta \in \mathbb{R}^2 : |(BB^\top)^{1/2}\eta| = f\},$$

an ellipse (semi-axes f , $f/\sqrt{3}$).



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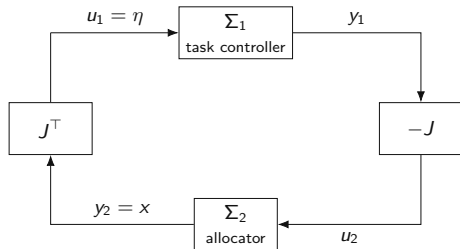
an ellipse (semi-axes f , $f/\sqrt{3}$).

$\mathcal{M}_x$ is an elliptic **cylinder**: curved, and unbounded along the internal force.

$$\text{dist}(x, \mathcal{M}_x) = \text{dist}(J^\top x, \mathcal{C}).$$



Case study I: the loop



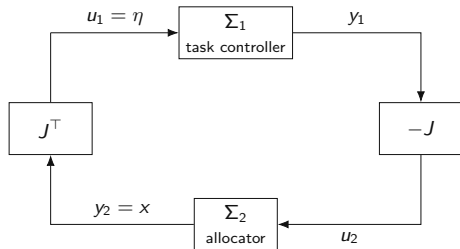
Σ_1 (static): $y_1 = \kappa e_C(u_1)$ — force correction

Σ_2 : $\dot{x} = u_2, y_2 = x$ — rate-commanded thrusters

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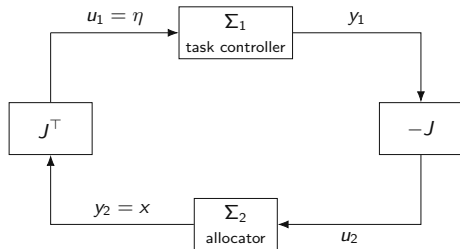
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Closed loop

$$\begin{aligned}\dot{x} &= -\kappa J e_C(J^\top x) = -\kappa e_x(x) \\ &\text{using } J e_C(J^\top x) = e_x(x)\end{aligned}$$



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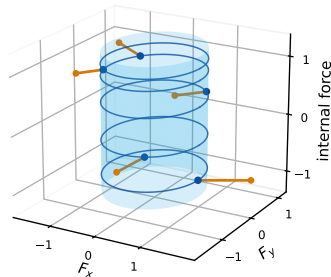
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Closed loop

$$\dot{x} = -\kappa J e_C(J^\top x) = -\kappa e_x(x)$$

using $J e_C(J^\top x) = e_x(x)$



Simulated: the net force reaches $|F| = f$; the internal force never changes.



Case study I: the checklist

Targets	task controller: $(\mathcal{C}, \{0\})$; allocator: $(\{0\}, \mathcal{M}_x)$
Certificates	task controller output-strict, $\varepsilon_1 = 1/\kappa$ (static); allocator <i>lossless</i> : $V_2 = \frac{1}{2}\ e_x\ ^2$, $\dot{V}_2 = \langle u_2, e_x \rangle$
Compatibility	exact , everywhere — the orthogonal splitting makes $J^\top e_x(x) = e_{\mathcal{C}}(J^\top x)$
Confinement	the internal force $(I - JJ^\top)x$ is <i>conserved</i> : its leaf cuts the noncompact cylinder down to a compact set
Margin	from the <i>structure</i> , not the allocator: $y_1 = \kappa J^\top e_x(y_2)$ with J an isometry, so $Q = \kappa\ e_x\ ^2 = \frac{\kappa}{1+\kappa^2} \text{dist}(y, \mathcal{F}_M)^2$
Result	$ Bx(t) \rightarrow f$: the demanded force magnitude is reached; its direction and the internal force are never constrained



Case study II: stabilizing a limit cycle

Plant. State $x = (q, z)$, $q \in \mathbb{R}^2$, $z \in \mathbb{R}$:

$$\dot{q} = \mathcal{J}q + u_1, \quad \dot{z} = -z, \quad y_1 = (q, z),$$

with $\mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: q rotates, z decays on its own.

Goal. Converge to the circle

$$\mathcal{O} = \{\|q\| = 1, z = 0\},$$

and keep rotating on it: a **limit cycle**. The *phase is free* — only the orbit matters.

(oscillators, rotating machinery, periodic motion: the goal is a closed orbit, not a point)



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The plant is transversally passive

Output target: the *cylinder* $\{\|q\| = 1\}$ (z free). With $r = \|q\|$ and

$$V_1 = \frac{1}{2}(r - 1)^2 + \frac{1}{2}z^2:$$

$$\dot{V}_1 = \langle u_1, e_{y_1} \rangle - z^2.$$

Rotation is invisible: $\mathcal{J}$ is skew, so spinning does not change the distance to the cylinder.



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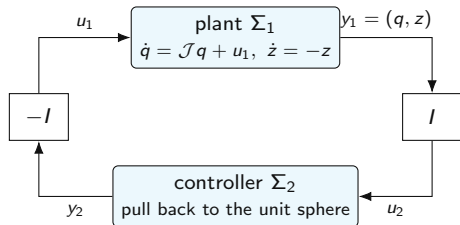
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Remember the $-z^2$: the plant's own dissipation will supply the margin.



Case study II: the loop



Controller (static), with $\rho = \|u_2\|$:

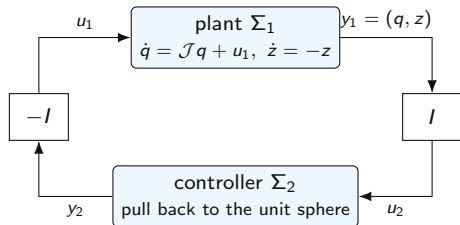
$$y_2 = u_2 - \frac{u_2}{\rho} = e_{\mathbb{S}^2}(u_2) :$$

the vector from the unit sphere out to u_2 , along its ray.

$(\mathbb{S}^2, \{0\})$ -passive, output-strict ($\varepsilon_2 = 1$).



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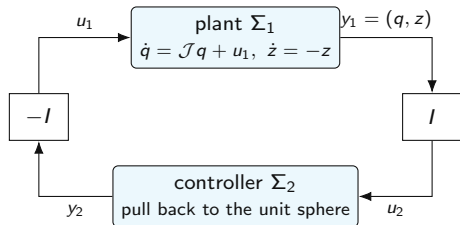
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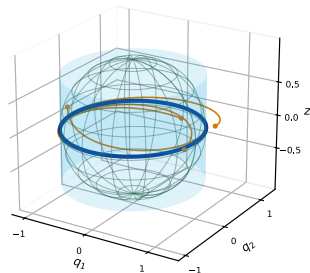
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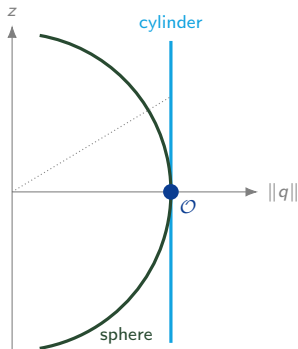
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plant target: cylinder (blue); controller target: sphere (green); orbit $\mathcal{O}$ (navy); a trajectory (orange).



Case study II: why this example — targets that only touch



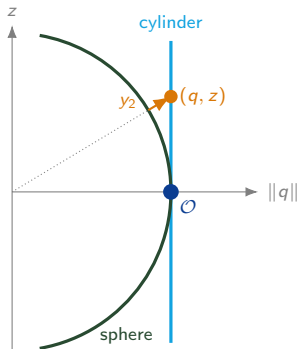
cross-section: one ray of q

The two targets differ

Channel I , but plant output target = **cylinder**,
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and are *tangent* there.



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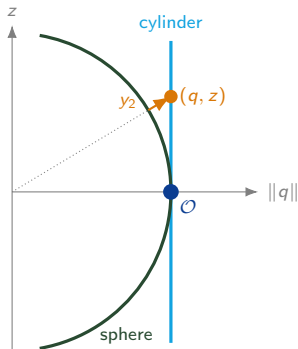
Off the orbit: a quadratic defect

At $\|q\| = 1$, height z : the plant sees *no* error; the
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$\Rightarrow$ **first-order** compatible, not exact (cubic residual,
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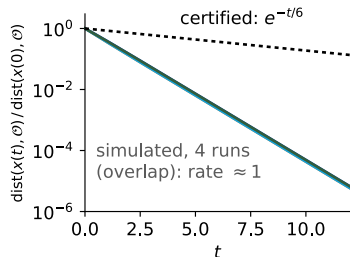
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*The ports are blind to z : their penalty $\|y_2\|^2 \approx \frac{1}{4}z^4$,
but $\text{dist}^2 \approx z^2$. Only the plant's $-z^2$ gives a
quadratic margin.*



Case study II: the checklist

Targets	plant: $(\{0\}, \text{cylinder})$; controller: $(\mathbb{S}^2, \{0\})$
Certificates	$V_1 = \frac{1}{2}(\ q\ - 1)^2 + \frac{1}{2}z^2$; $\varepsilon_2 = 1$
Compatibility	first-order only : cylinder and sphere are tangent along $\mathcal{O}$
Confinement	a compact solid torus around $\mathcal{O}$ — no conserved phase needed
Margin	from <i>state</i> dissipation: $\ y_2\ ^2 + z^2 \geq \frac{1}{3} \text{dist}(y, \mathcal{S})^2$
Result	$\text{dist}(x(t), \mathcal{O}) \leq e^{-t/6} \text{dist}(x(0), \mathcal{O})$



distance to $\mathcal{O}$, normalized: every run stays *below* the certified bound $e^{-t/6}$ (dotted); the bound is conservative.

Easy vs. hard: Case I was exact with a margin from structure; here compatibility is only first-order, and the margin comes from the plant's own $-z^2$.



Case study II: where the rate $e^{-t/6}$ comes from

Notation: $r = \|q\|$, $\rho = \|(q, z)\|$, $d^2 = \text{dist}(y, \mathcal{S})^2 = \underbrace{(r-1)^2 + z^2}_{\text{dist}(x, \mathcal{O})^2} + \underbrace{(\rho-1)^2}_{\|y_2\|^2}$.

1. Dissipation

$V_{\text{cl}} = V_1 = \frac{1}{2} \text{dist}(x, \mathcal{O})^2$ ($V_2 = 0$):

$$\dot{V}_{\text{cl}} = \langle \delta, y_2 \rangle - Q, \quad Q = \underbrace{\|y_2\|^2}_{\varepsilon_2=1} + \underbrace{z^2}_{\text{plant}}.$$



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2. Margin: $Q \geq \frac{1}{3} d^2$

$$\begin{aligned} \|y_2\| &= |\rho - 1| \quad \text{and} \quad |\rho - r| \leq |z|, \text{ so} \\ (r-1)^2 &\leq (|\rho-1| + |z|)^2 \leq 2(\rho-1)^2 + 2z^2, \\ \text{hence } d^2 &\leq 3((\rho-1)^2 + z^2) = 3Q: \quad \alpha = \frac{1}{3}. \end{aligned}$$



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$\|y_2\| = |\rho - 1|$ and $|\rho - r| \leq |z|$, so

$$(r-1)^2 \leq (|\rho-1| + |z|)^2 \leq 2(\rho-1)^2 + 2z^2,$$

hence $d^2 \leq 3((\rho-1)^2 + z^2) = 3Q$: $\alpha = \frac{1}{3}$.

3. Residual: cubic

Defect $\delta = e_{\mathbb{S}^2}(y_1) - e_{\text{cyl}}(y_1)$; the targets are tangent along $\mathcal{O}$: $\delta = O(d^2)$, $y_2 = O(d)$, so $|\langle \delta, y_2 \rangle| \leq c_\delta d^3$.

4. Quadratic beats cubic, then Grönwall

In the tube $c_\delta d < \frac{1}{6}$, with $d^2 \geq 2V_{\text{cl}}$:

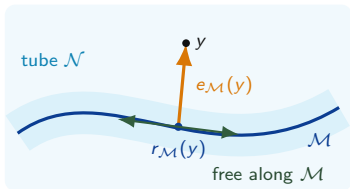
$$\dot{V}_{\text{cl}} \leq -\frac{1}{3}d^2 + c_\delta d^3 \leq -\frac{1}{6}d^2 \leq -\frac{1}{3}V_{\text{cl}}$$

$$\Rightarrow \text{dist}(x(t), \mathcal{O}) \leq e^{-t/6} \text{dist}(x(0), \mathcal{O}).$$

Outlook



Potential applications



- Desired behavior: $y(t) \rightarrow \mathcal{M}$
- Penalize transverse deviations $e_{\mathcal{M}}(y) = y - r_{\mathcal{M}}(y)$
- Allow motion along the manifold

Task	Target manifold
Setpoint regulation	$\mathcal{M}$ is a singleton
Consensus	$\mathcal{M}$ is the agreement manifold
Formation control	$\mathcal{M} = \{e \in \text{Im}(E \otimes I_d) : \ e_k\ ^2 = d_{ij}^2, \forall e_k = (i, j)\}$
Synchronization on n-sphere	$\mathcal{M} = (S^n)^N$
Path following	$\mathcal{M}$ is a curve



Outlook: learning & optimization — a dictionary

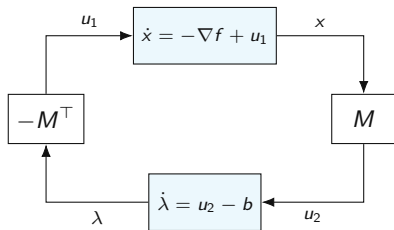
Optimization and learning problems routinely have *solution manifolds*, not points. Several pieces of the theory map onto known objects:

Transversal passivity	Optimization / learning
target manifold $\mathcal{M}$	set of minimizers (overparameterization, symmetries)
output-strict, $\varepsilon = \mu$	restricted secant inequality relative to $\arg \min f$
quadratic dissipation margin	PL / error-bound / quadratic-growth conditions
conserved quantity $\Rightarrow$ compact leaf	balancedness invariants in factorization / deep linear nets
negative-adjoint channels $M, -M^\top$	primal–dual and saddle-point dynamics
directed channels	push-sum, gradient tracking over digraphs

Further directions: learning dynamics that are transversally passive *by construction*; data-driven estimation of (ε, γ) relative to targets; Nash seeking with non-unique equilibria.



Outlook: primal–dual dynamics are our interconnection



$M = A$ for the constraint $Ax = b$

$$\dot{x} = -\nabla f(x) - A^\top \lambda, \quad \dot{\lambda} = Ax - b$$

is the feedback loop of the talk with $u_2 = Mx$,
 $u_1 = -M^\top \lambda$, $M = A$.

- non-unique optimizers or multipliers $\Rightarrow$ *manifold* targets at the ports;
- curved constraint sets $\Rightarrow$ *channel compatibility* becomes a property of the algorithm;
- distributed optimization: $M = E \otimes I$ (consensus constraint); digraphs break the adjoint pair.

Saddle-point dynamics: Arrow, Hurwicz & Uzawa, 1958; Cherukuri, Gharesifard & Cortés, *SICON*, 2017.



Summary

Q1. Transversal passivity

Dissipation on nearest-point errors e_u, e_y at both ports.

- output-distance certificate
- oscillators: transversally passive, not passive
- standard, shifted, incremental passivity as special cases

Q2. Channel compatibility

Do $M, -M^\top$ carry errors to errors?

- exact: no residual
- first-order: cubic residual
- directed channels: compatible, yet not cancelling

Q3. Port convergence

Dissipation is blind along the target; add

- confinement (e.g. conserved quantities)
- a quadratic margin
- case studies: thrust allocation, limit cycle

**Classical passivity is built around equilibria.
Transversal passivity is built around target sets.**

target geometry + component dissipation + channel compatibility $\implies$ compositional convergence



Future directions

Theory

- margins from graph structure: consensus and *directed* agreement
- compatibility for nonlinear channels and curved feasible sets
- global results beyond tubular neighborhoods
- incremental and differential transversal passivity

Control design

- transversal-passivity-based synthesis: shaping W_0 to supply the margin
- compatibility-aware choice of port targets and channels

Applications

- synchronization on spheres and $SO(3)$
- formation control with distance constraints
- control allocation and path following

F.-Y. Yue and D. Zelazo, Transversal passivity: a geometric extension of passivity theory, in preparation.

Directed consensus: F.-Y. Yue and D. Zelazo, "A passivity analysis for nonlinear consensus on balanced digraphs," *Eur. J. Control* 86, 2025; "A passivity analysis for nonlinear consensus on digraphs," *IEEE CDC*, 2025.