SYMMETRY-FORCED FORMATION CONTROL

Daniel Zelazo

DEPARTMENT OF AEROSPACE ENGINEERING
TECHNION Israel Institute of Technology

CoNeCt Cooperative Networks and Controls Lab

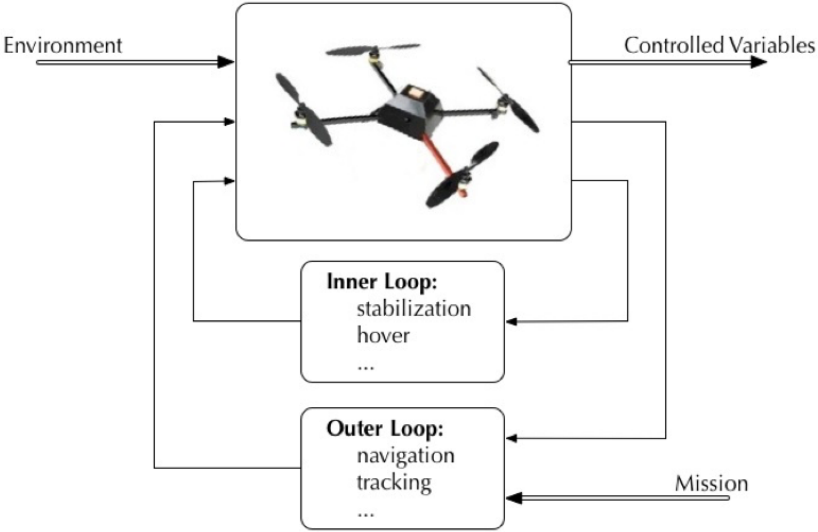
Philadelphia Flight Control Laboratory

University of Stuttgart

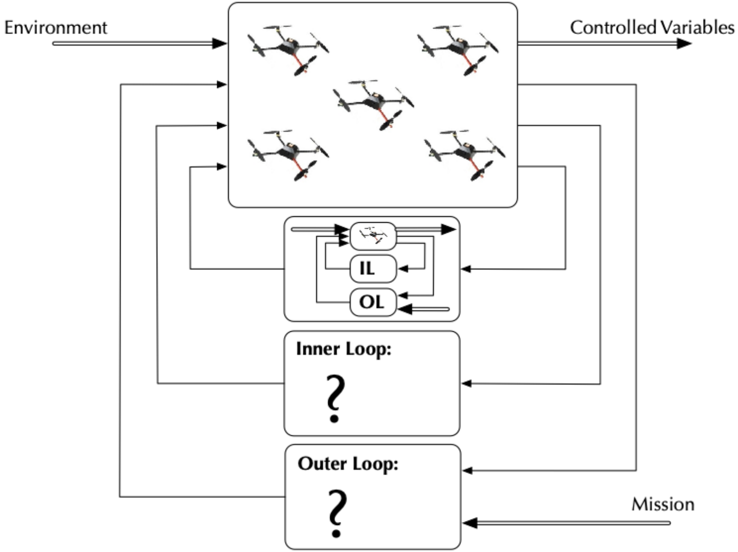
August 15, 2023

with Shin-Ichi Tanigawa (University of Tokyo) and Bernd Shulze (Lancaster University)

CONTROL ARCHITECTURES



CONTROL ARCHITECTURES

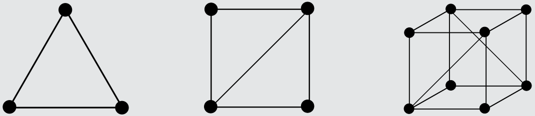


...this talk is about formation control

...this talk is about formation control

Formation Control Objective

Given a team of robots endowed with the ability to sense/ communicate with neighboring robots, design a control for each robot using only local information that moves the team into a desired spatial configuration.



FORMATION CONSTRAINTS

- The desired formation is characterized by a set of M constraints, encoded in the function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, and a configuration $\mathbf{p}^\star$ satisfying the constraints.
- The set of all feasible formations is

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \,|\, F(p) = F(\mathbf{p}^\star)\}$$

- The desired formation is characterized by a set of M constraints, encoded in the function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, and a configuration $\mathbf{p}^\star$ satisfying the constraints.
- The set of all feasible formations is

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \,|\, F(p) = F(\mathbf{p}^\star)\}$$

Formation Control Objective

For an ensemble of n agents with dynamics

$$\dot{p}_i = u_i,$$

with $p_i(t) \in \mathbb{R}^d$, an information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and formation constraint function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, design a distributed control law for each agent $i \in \{1, \dots, n\}$ such that the set

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \,|\, F(p) = F(\mathbf{p}^\star)\},$$

is asymptotically stable.

Theorem - Distance Constrained Formation Control [Krick 2009]

Consider the potential function

$$F_f(p) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d^\star_{ij})^2 \right)^2$$

and assume the desired distances $d^\star_{ij}$ correspond to a feasible formation. Then the gradient dynamical system

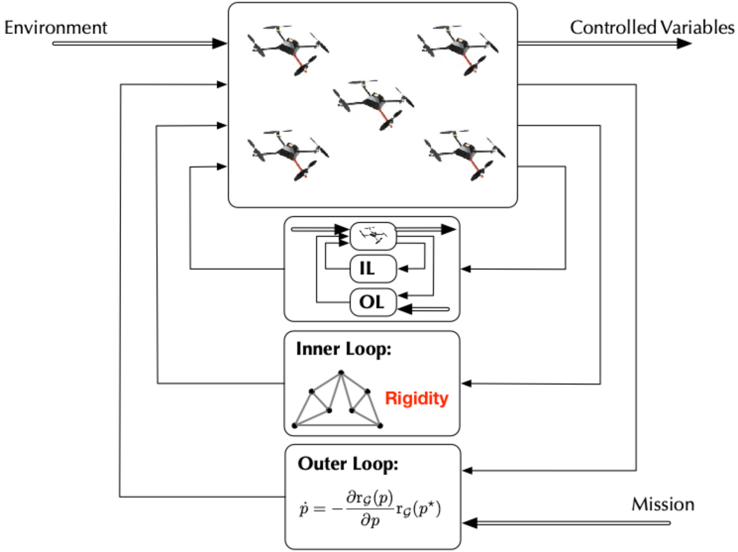
$$u_i = -\nabla_{p_i} F_f(p) = \sum_{ij \in \mathcal{E}} \left(\|p_i - p_j\|^2 - (d^\star_{ij})^2 \right) (p_j - p_i)$$

$$\dot{p} = -\nabla_p F_f(p) = -R^T(p)R(p)p + R^T(p)(d^\star)^2$$

asymptotically converges to the critical points of the potential function, i.e., $\frac{\partial F_f(p)}{\partial p} = 0$.

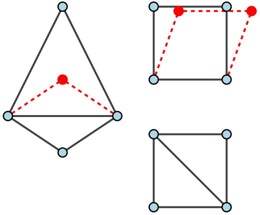
- $R(p)$ is the *rigidity matrix* for the framework $(\mathcal{G}, p)$

CONTROL ARCHITECTURES



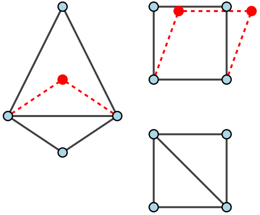
Rigidity theory helps us understand

- how many constraints are required to ensure uniqueness of formation shape (modulo translations, rotations, and flip ambiguities)
- how the constraints should be distributed in the network



Rigidity theory helps us understand

- how many constraints are required to ensure uniqueness of formation shape (modulo translations, rotations, and flip ambiguities)
- how the constraints should be distributed in the network



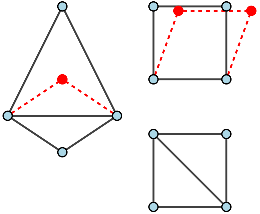
A widely accepted architectural requirement for distance constrained formation control is that minimally infinitesimally rigid frameworks are required. Equivalent to:

$$\operatorname{rk} R(p) = 2|\mathcal{V}| - 3 \text{ and } |\mathcal{E}| = 2|\mathcal{V}| - 3 \quad (\text{in } \mathbb{R}^2)$$

Q: is this a necessary condition? (can we solve the problem with fewer edges?)

Rigidity theory helps us understand

- how many constraints are required to ensure uniqueness of formation shape (modulo translations, rotations, and flip ambiguities)
- how the constraints should be distributed in the network



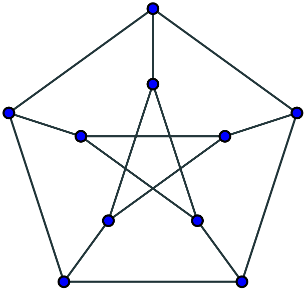
A widely accepted architectural requirement for distance constrained formation control is that minimally infinitesimally rigid frameworks are required. Equivalent to:

$$\mathrm{rk}\, R(p) = 2|\mathcal{V}| - 3 \text{ and } |\mathcal{E}| = 2|\mathcal{V}| - 3 \quad (\text{in } \mathbb{R}^2)$$

Q: is this a necessary condition? (can we solve the problem with fewer edges?)

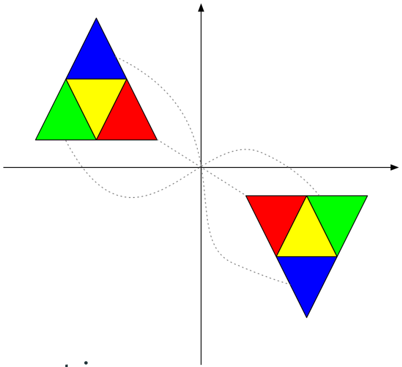
A: Impose additional symmetry constraints without requiring more information exchange (in fact, less!)

Graph Symmetries



- graph automorphisms

Point Groups

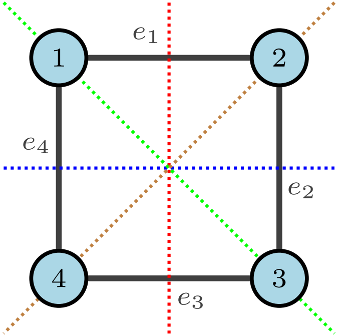


- isometries

Graph Automorphism

An automorphism of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of of its vertex set such that

$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$



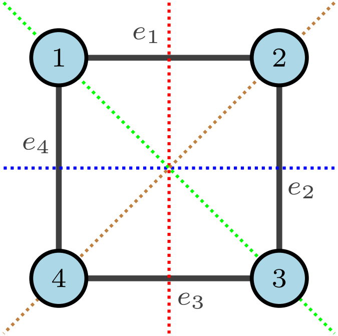
Automorphisms encode graph symmetries

- identity: $\mathrm{Id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$
- $90°$ rotation: $\psi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
- $180°$ rotation: $\psi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
- $270°$ rotation: $\psi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

Graph Automorphism

An automorphism of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of of its vertex set such that

$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$



Automorphisms encode graph symmetries

- reflection: $\psi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
- reflection: $\psi_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
- reflection: $\psi_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$
- reflection: $\psi_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

Definition

Let X be a set, and let Γ be a collection of invertible functions $X \to X$. Then Γ is called a group if the identity map, Id, belongs to Γ, and for any $\Gamma \ni f, g : X \to X$, both the composite function $f \circ g$ and the inverse function f^{-1} belong to Γ.

Automorphisms of a graph form a *group* - $\mathrm{Aut}(\mathcal{G})$

- $\mathrm{Aut}(\mathcal{G}) = \{\mathrm{Id}, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\}$

A subgroup is a subset of a group, and also satisfies all properties of a group

- $\{\mathrm{Id}, \psi_1, \psi_2, \psi_3\}$
- $\{\mathrm{Id}, \psi_2, \psi_4, \psi_5\}$
- $\{\mathrm{Id}, \psi_2\}$
- $\{\mathrm{Id}, \psi_6\}$
- $\{\mathrm{Id}, \psi_7\}$

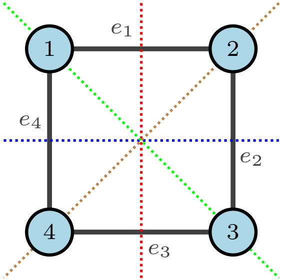
Γ-SYMMETRIC GRAPHS

- Subgroups of $\mathrm{Aut}(\mathcal{G})$ define specific symmetries in $\mathcal{G}$
- for any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$, we say that $\mathcal{G}$ is Γ-symmetric

- Subgroups of $\mathrm{Aut}(\mathcal{G})$ define specific symmetries in $\mathcal{G}$
- for any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$, we say that $\mathcal{G}$ is Γ-symmetric

Definition

For a Γ-symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and vertex $i \in \mathcal{V}$, the set $\Gamma_i = \{\gamma(i) \mid \gamma \in \Gamma\}$ is called the vertex orbit of i. Similarly, for an edge $e = ij \in \mathcal{E}$, the set $\Gamma_e = \{\gamma(i)\gamma(j) \mid \gamma \in \Gamma\}$ is termed the edge orbit of e.



Consider $\Gamma = \{\mathrm{Id}, \psi_2\}$ (ψ_2 is the $180°$ rotation)

- **Vertex Orbit:**
  $\Gamma_1 = \Gamma_3 = \{1, 3\},\ \Gamma_2 = \Gamma_4 = \{2, 4\}$

  vertices inside a vertex orbit are equivalent
  representative vertex set: $\mathcal{V}_0 = \{1, 2\}$
- **Edge Orbit:**
  $\Gamma_{e_1} = \Gamma_{e_3} = \{e_1, e_3\}$,
  $\Gamma_{e_2} = \Gamma_{e_4} = \{e_2, e_4\}$
  representative edge set: $\mathcal{E}_0 = \{e_1, e_2\}$

$\tau(\Gamma)$-SYMMETRIC FRAMEWORK

combine notions of graph symmetries with point groups

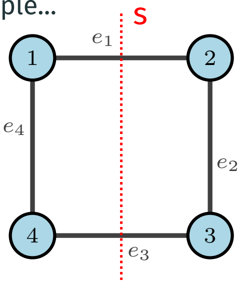
- let $\mathcal{G}$ be a Γ-symmetric graph
- Γ also represented as a *point group*
  - a set of isometries that preserve symmetries
  - homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$
  - τ assigns an orthogonal matrix (describing an isometry of $\mathbb{R}^d$ such as a rotation or reflection) to each element of Γ

Definition

A framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called $\tau(\Gamma)$-symmetric if

$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}.$$

example...



- consider $\Gamma = \{\mathrm{Id}, \psi_4\} \subseteq \mathrm{Aut}(\mathcal{G})$
- $\gamma = \psi_4 \in \Gamma$ (reflection about mirror S)
- isometry $\tau(\gamma) : (a, b) \mapsto (-a, b)$

satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$.

- note: for a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ and for every $j \in \Gamma_i$, there is a $\gamma_j \in \Gamma$ such that $\tau(\gamma_j)p_j = p_i$ for all $j \in \Gamma_i$

isometries of configuration p coincide with symmetries of the automorphisms of $\mathcal{G}$

- in $\tau(\Gamma)$-symmetric frameworks, the configurations p are in a special geometric position (not necessarily generic)
- symmetry can lead to unexpected infinitesimal flexibility/rigidity

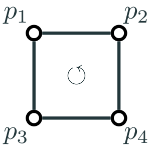
Definition

An infinitesimal motion u of a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric* if

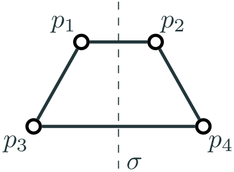
$$\tau(\gamma)(u_i) = u_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}. \tag{1}$$

We say that $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric infinitesimally rigid* if every $\tau(\Gamma)$-symmetric infinitesimal motion is trivial.

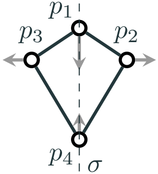
- infinitesimal motions are all the distance-preserving motions of a framework
- characterized by the kernel of the rigidity matrix
- we can find a subspace of the kernel that is isomorphic to the space of 'fully-symmetric' infinitesimal motions
- velocity assignments to the points of $(\mathcal{G}, p)$ that exhibit exactly the same symmetry as the configuration p



(a)



(b)



(c)

- $\mathcal{C}_{4v}$-symmetric (and hence $\tau(\Gamma)$-symmetric for any subgroup $\tau(\Gamma)$ of $\mathcal{C}_{4v}$)
- $\tau(\Gamma)$-symmetric infinitesimally rigid

- $\mathcal{C}_s$-symmetric (with respect to the reflection σ)
- $\tau(\Gamma)$-symmetric infinitesimally rigid

- $\mathcal{C}_s$-symmetric (with respect to the reflection σ) with a non-trivial $\mathcal{C}_s$-symmetric infinitesimal motion
- $\tau(\Gamma)$-symmetric infinitesimally flexible

Symmetric Formation Control Objective

Consider a group of n integrator agents that interact over the Γ-symmetric sensing graph $\mathcal{G}$. Let $p^\star \in \mathbb{R}^{dn}$ be a configuration such that $(\mathcal{G}, p^\star)$ is $\tau(\Gamma)$-symmetric for some desired point group $\tau(\Gamma)$, and let $\mathcal{V}_0$ be a set of representatives of the vertex orbits of $\mathcal{G}$ under Γ. Design a control $u_i(t)$ for each agent i such that

(i) $\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = \|p_i^\star - p_j^\star\| = d_{ij}^\star$ for all $ij \in \mathcal{E}$; (distance constraints)

(ii) $\lim_{t\to\infty} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\| = 0$ for all $u, v \in \Gamma_i$, $i \in \mathcal{V}_0$. (symmetry constraints)

A GRADIENT APPROACH

- the formation potential

$$F_f(p(t)) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2 \right)^2$$

A GRADIENT APPROACH

- the formation potential

$$F_f(p(t)) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2 \right)^2$$

- the symmetry potential

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu}) p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the formation potential

$$F_f(p(t)) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2 \right)^2$$

- the symmetry potential

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu}) p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the symmetric formation potential

$$F(p(t)) = F_f(p(t)) + F_s(p(t))$$

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- closed-loop dynamics

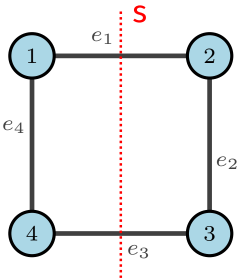
$$\dot{p}(t) = -R(p(t))^T \left(R(p(t))p(t) - (d^\star)^2\right) - Qp(t)$$

where Q is symmetric and a block-diagonal matrix with

$$[Q_i]_{uv} = \begin{cases} d_{\Gamma_i}(u)I, & u = v,\ u \in \Gamma_i \\ -\tau(\gamma_{uv}), & uv \in \mathcal{E}, u, v \in \Gamma_i \\ 0, & \text{o.w.} \end{cases}.$$

- $Q_i \in \mathbb{R}^{|\Gamma_i|d \times |\Gamma_i|d}$
- $[Q]_{uv} \in O(\mathbb{R}^d)$ (orthogonal group)
- $\tau(\gamma_{uv})^{-1} = \tau(\gamma_{uv})^T$

- symmetric formation potential makes no assumption on relation between the graph $\mathcal{G}$ and the point group $\tau(\Gamma)$
- we restrict our study to graphs where communication required by symmetric potential use same edges as $\mathcal{G}$



- $\Gamma = \{\mathrm{Id}, \psi_4\} \subseteq \mathrm{Aut}(\mathcal{G})$
- $\Gamma_1 = \Gamma_2 = \{1, 2\},\ \Gamma_3 = \Gamma_4 = \{3, 4\}$
- $\mathcal{V}_0 = \{1, 4\}$
- isometry $\tau(\gamma) : (a, b) \mapsto (-a, b)$

satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$ and for each $i \in \mathcal{V}_0$ and $j \in \Gamma_i \setminus \{i\}$, the edge ij is in $\mathcal{E}$ (i.e. $\mathcal{G}(\Gamma_i)$ is connected)

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- closed-loop dynamics

$$\dot{p}(t) = -R(p(t))^T \left(R(p(t))p(t) - (d^\star)^2\right) - Qp(t)$$

- dynamics at for each agent

$$\dot{p}_i(t) = \sum_{ij \in \mathcal{E}} (\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2)(p_j(t) - p_i(t)) + \sum_{\substack{ij \in \mathcal{E} \\ i,j \in \Gamma_u}} (\tau(\gamma_{ij})p_j(t) - p_i(t)).$$

Theorem

Consider a team of n integrator agents interacting over a Γ-symmetric graph $\mathcal{G}$ satisfying Assumption 1 that can be drawn with maximum point group symmetry $\mathcal{S}$ in $\mathbb{R}^d$, and let

$$\mathcal{F}_f = \{p \in \mathbb{R}^{dn} \,|\, \|p_i - p_j\| = d^\star_{ij}\; ij \in \mathcal{E}\}, \text{ and } \mathcal{F}_s = \{p \in \mathbb{R}^{dn} \,|\, \tau(\gamma)(p_i) = p_{\gamma(i)}\; \forall \gamma \in \Gamma,\; i \in \mathcal{V}\}.$$

Then for initial conditions $p_i(0)$ satisfying

$$\sum_{ij \in \mathcal{E}} (\|p_i(0) - p_j(0)\| - d^\star_{ij})^2 \leq \epsilon_1, \text{ and } \|p_i(0) - \tau(\gamma_{ij})p_j(0)\|^2 \leq \epsilon_2$$

for all $i, j \in \Gamma_u$ and $u \in \mathcal{V}_0$, for a sufficiently small and positive constant ϵ_1 and ϵ_2, the control

$$u = -\nabla F(p(t)),$$

renders the set $\mathcal{F}_f \cap \mathcal{F}_s$ exponentially stable, i.e.

$$\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = d^\star_{ij} \text{ and } \lim_{t\to\infty} \tau(\gamma)(p_i(t)) = \lim_{t\to\infty} p_{\gamma(i)}(t) \quad \text{for all } \gamma \in \Gamma, i \in \mathcal{V}.$$

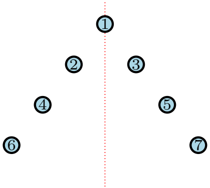
Proof Sketch

- observe the invariant quantity (group average)

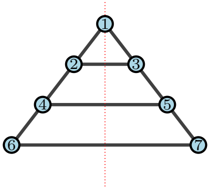
$$z(t) = \sum_{v \in \mathcal{V}} \sum_{\gamma \in \Gamma} \tau(\gamma) p_v(t)$$

- combine with stability properties of gradient dynamical systems

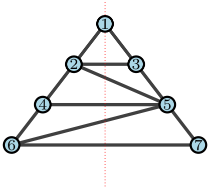
- formation flight for aircraft originated in WWI
- Vic formation used by pilots to improve visual communication and defensive advantages



Vic formation with symmetry mirror

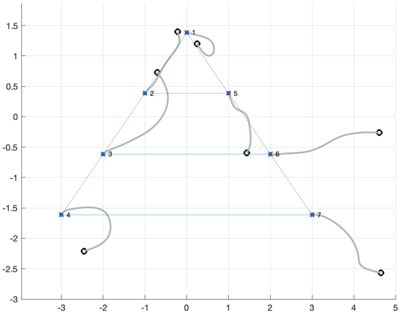


Flexible framework (9 edges; satisfies Assumption 1)

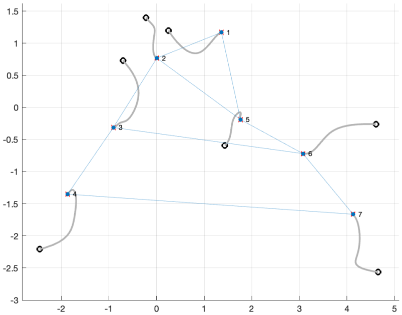


Minimally Rigid framework (11 edges)

EXAMPLE: THE VIC FORMATION



- symmetry constraints force agents to correct formation
- requires less agent communication than standard formation control with MIR requirement



- with flexible framework and only formation potential can not guarantee convergence to correct shape

EXPLOIT MORE SYMMETRY

- proposed strategy does not take advantage of the full power of symmetry

EXPLOIT MORE SYMMETRY

- proposed strategy does not take advantage of the full power of symmetry
- can we find redundant information between the symmetry constraints and the distance constraints?

Definition

An infinitesimal motion u of a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric* if

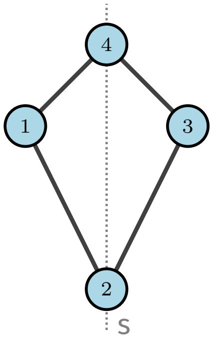
$$\tau(\gamma)(u_i) = u_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}. \tag{2}$$

We say that $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric infinitesimally rigid* if every $\tau(\Gamma)$-symmetric infinitesimal motion is trivial.

infinitesimal motions can also be studied in this framework

- $\tau(\gamma)(u_i) = u_{\theta(\gamma)(i)}$
- understanding symmetry structure means we only need to find infintesimal motion for one representative vertex in each vertex orbit

EXAMPLE



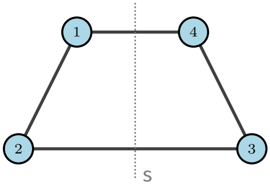
$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (0, c)^T$
- $p_3 = (-a, b)^T$
- $p_4 = (0, d)^T$

$$R(p) = \begin{bmatrix} (a\ b-c) & (-a\ c-b) & (0\ 0) & (0\ 0) \\ (a\ b-d) & (0\ 0) & (0\ 0) & (-a\ d-b) \\ (0\ 0) & (a\ c-b) & (-a\ b-c) & (0\ 0) \\ (0\ 0) & (0\ 0) & (-a\ b-d) & (a\ d-b) \end{bmatrix}$$

- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by
$(-1\ 0\ 0\ \frac{a}{c-b}\ 1\ 0\ 0\ \frac{a}{d-b})^T$
flex is symmetric with respect to S
($\tau(\gamma) : (a, b) \mapsto (-a, b)$)



$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-c, d)^T$
- $p_4 = (-a, b)^T$

Rigidity matrix

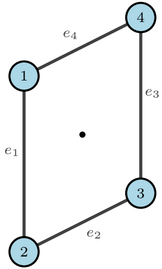
$$R(p) = \begin{bmatrix} (a-c \; b-d) & (c-a \; d-b) & (0 \; 0) & (0 \; 0) \\ (2a \; 0) & (0 \; 0) & (0 \; 0) & (-2a \; 0) \\ (0 \; 0) & (2c \; 0) & (-2c \; 0) & (0 \; 0) \\ (0 \; 0) & (0 \; 0) & (a-c \; d-b) & (c-a \; b-d) \end{bmatrix}$$

- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by

$\left(-1 \;\; -1 \;\; -1 \;\; \frac{2(c-a)+b-d}{d-b} \;\; -1 \;\; -\frac{2(c-a)+b-d}{d-b} \;\; 1 \;\; 1\right)^T$

flex is not symmetric with respect to s



$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-a, -b)^T$
- $p_4 = (-c, -d)^T$

$$R(p) = \begin{bmatrix} (a-c \;\; b-d) & (c-a \;\; d-b) & (0 \;\; 0) & (0 \;\; 0) \\ (a+c \;\; b+d) & (0 \;\; 0) & (0 \;\; 0) & (-a-c \;\; -b-d) \\ (0 \;\; 0) & (0 \;\; 0) & (c-a \;\; d-b) & (a-c \;\; b-d) \\ (0 \;\; 0) & (a+c \;\; b+d) & (-a-c \;\; -b-d) & (0 \;\; 0) \end{bmatrix}$$

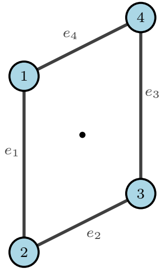
- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by

$$(-1 \;\; 0 \;\; \tfrac{cd-ab}{ad-bc} \;\; \tfrac{a^2-c^2}{ad-bc} \;\; 1 \;\; 0 \;\; -\tfrac{cd-ab}{ad-bc} \;\; -\tfrac{a^2-c^2}{ad-bc})^T$$

flex is symmetric with respect to $180°$ rotation ($\mathcal{C}_2$)

EXAMPLE



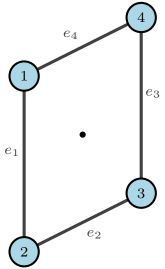
$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-a, -b)^T$
- $p_4 = (-c, -d)^T$

$$R(p) = \begin{bmatrix} (a-c \;\; b-d) & (c-a \;\; d-b) & (0 \;\; 0) & (0 \;\; 0) \\ (a+c \;\; b+d) & (0 \;\; 0) & (0 \;\; 0) & (-a-c \;\; -b-d) \\ (0 \;\; 0) & (0 \;\; 0) & (c-a \;\; d-b) & (a-c \;\; b-d) \\ (0 \;\; 0) & (a+c \;\; b+d) & (-a-c \;\; -b-d) & (0 \;\; 0) \end{bmatrix}$$

- $180°$ rotation of points corresponds to $\psi_2 \in \mathrm{Aut}(\mathcal{G})$
- recall: vertex orbits : $\{1, 3\}$, $\{2, 4\}$, edge orbits: $\{e_1, e_3\}$, $\{e_2, e_4\}$

EXAMPLE



$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-a, -b)^T$
- $p_4 = (-c, -d)^T$

$$R(p) = \begin{bmatrix} (a-c \;\; b-d) & (c-a \;\; d-b) & (0\;\;0) & (0\;\;0) \\ (a+c \;\; b+d) & (0\;\;0) & (0\;\;0) & (-a-c \;\; -b-d) \\ (0\;\;0) & (0\;\;0) & (c-a \;\; d-b) & (a-c \;\; b-d) \\ (0\;\;0) & (a+c \;\; b+d) & (-a-c \;\; -b-d) & (0\;\;0) \end{bmatrix}$$

symmetries make certain rows and columns of the rigidity matrix redundant

ORBIT RIGIDITY MATRIX

symmetries make certain rows and columns of the rigidity matrix redundant

$$
R(p) = \begin{array}{c} \\ e_1 \\ e_4 \\ \mathcal{C}_2(e_1) \\ \mathcal{C}_2(e_4) \end{array}
\begin{array}{cccc} 1 & 2 & 3 = \mathcal{C}_2(1) & 4 = \mathcal{C}_2(2) \\
\left(\begin{array}{c} (a - c \;\; b - d) \\ (a + c \;\; b + d) \\ (0 \;\; 0) \\ (0 \;\; 0) \end{array} \right. &
\begin{array}{c} (c - a \;\; d - b) \\ (0 \;\; 0) \\ (0 \;\; 0) \\ (a + c \;\; b + c) \end{array} &
\begin{array}{c} (0 \;\; 0) \\ (0 \;\; 0) \\ (c - a \;\; d - b) \\ (-a - c \;\; -b - d) \end{array} &
\left. \begin{array}{c} (0 \;\; 0) \\ (-a - c \;\; -b - d) \\ (a - c \;\; b - d) \\ (0 \;\; 0) \end{array} \right)
\end{array}
$$

ORBIT RIGIDITY MATRIX

symmetries make certain rows and columns of the rigidity matrix redundant

$$
R(p) = \begin{matrix} \\ e_1 \\ e_4 \\ \mathcal{C}_2(e_1) \\ \mathcal{C}_2(e_4) \end{matrix}
\begin{matrix} 1 & 2 & 3 = \mathcal{C}_2(1) & 4 = \mathcal{C}_2(2) \\
\end{matrix}
\begin{pmatrix}
(a-c \;\; b-d) & (c-a \;\; d-b) & (0 \;\; 0) & (0 \;\; 0) \\
(a+c \;\; b+d) & (0 \;\; 0) & (0 \;\; 0) & (-a-c \;\; -b-d) \\
(0 \;\; 0) & (0 \;\; 0) & (c-a \;\; d-b) & (a-c \;\; b-d) \\
(0 \;\; 0) & (a+c \;\; b+c) & (-a-c \;\; -b-d) & (0 \;\; 0)
\end{pmatrix}
$$

Orbit Rigidity Matrix

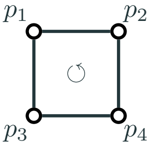
$$
\begin{matrix} \\ e_1 \\ e_4 \end{matrix}
\begin{matrix} 1 & 2 \end{matrix}
\begin{pmatrix}
(p_1 - p_2)^T & (p_2 - p_1)^T \\
(p_1 - \mathcal{C}_2(p_2))^T & (p_2 - \mathcal{C}_2^{-1}(p_1))^T
\end{pmatrix}
=
\begin{matrix} 1 & 2 \end{matrix}
\begin{pmatrix}
(a-c,\; b-d) & (c-a,\; d-b) \\
(a+c,\; b+d)) & (c+a,\; d+b)
\end{pmatrix}
$$

- 2 rows - one for each representative of edge orbits under action of $\mathcal{C}_2$
- 4 columns - nodes p_1, p_2 each have two dof; nodes $p_3 = \mathcal{C}_2(p_1)$ and $p_4 = \mathcal{C}_2(p_2)$ are uniquely determined by the symmetries

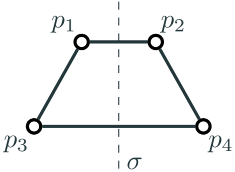
- relation between vertices within vertex orbits and between vertex orbits (through edge orbits) captured by quotient gain graph of a Γ-symmetric graph
  - node set is representative vertex set $\mathcal{V}_0$
  - edge set is representative edge set $\mathcal{E}_0$: choose edge of form $i\gamma(j)$ with $i, j \in \mathcal{V}_0$

    it is ok for $i = j$

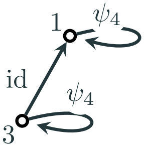
    edges are directed with 'edge gain' being the group action $\gamma \in \Gamma$

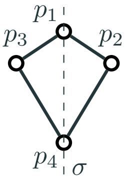


- $\Gamma = \{\mathrm{Id}, \psi_1\}$ (rotation)
- $\Gamma_i = \{1, 2, 3, 4\}$
- $\mathcal{V}_0 = \{1\}$, $\mathcal{E}_0 = \{e_1\}$

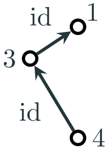


- $\Gamma = \{\mathrm{Id}, \psi_4\}$ (reflection)
- $\Gamma_{1,2} = \{1, 2\}$, $\Gamma_{3,4} = \{3, 4\}$
- $\mathcal{V}_0 = \{1, 3\}$,
  $\mathcal{E}_0 = \{12,\ 13,\ 24\}$





- $\Gamma = \{\mathrm{Id}, \psi_6\}$ (reflection)
- $\Gamma_1 = \{1\}$, $\Gamma_4 = \{4\}$,
  $\Gamma_{2,3} = \{2, 3\}$
- $\mathcal{V}_0 = \{1, 3, 4\}$, $\mathcal{E}_0 = \{13,\ 14\}$



Definition [Shulze 2011]

For a Γ-symmetric framework $(\mathcal{G}, p)$ with quotient gain Γ-gain graph $(\mathcal{G}_0, w)$, the orbit rigidity matrix, $\mathcal{O}(\mathcal{G}_0, w, p)$, is the $|\mathcal{E}_0| \times d|\mathcal{V}_0|$ matrix defined as follows. Choose a representative vertex $\tilde{i}$ for each vertex Γ_i in $\mathcal{V}_0$. The row corresponding to the edge $\tilde{e} = (\tilde{i}, \tilde{j})$ with gain $w(\tilde{e})$ in $\mathcal{E}_0$ is given by

$$(0 \;\cdots\; 0 \;\underbrace{p(\tilde{i}) - \tau(w(\tilde{e}))p(\tilde{j})}_{\tilde{i}}\; 0 \;\cdots\; 0 \;\underbrace{p(\tilde{j}) - \tau(w(\tilde{e}))^{-1}p(\tilde{j})}_{\tilde{i}}\; 0 \;\cdots\; 0).$$

If $\tilde{e} = (\tilde{i}, \tilde{i})$ is a loop at $\tilde{i}$, then the row corresponding to $\tilde{e}$ is given by

$$(0 \;\cdots\; 0 \;\underbrace{2p(\tilde{i}) - \tau(w(\tilde{e}))p(\tilde{i}) - \tau(w(\tilde{e}))^{-1}p(\tilde{i})}_{\tilde{i}}\; 0 \;\cdots\; 0\; 0\; 0 \;\cdots\; 0).$$

Theorem [Shulze 2011]

The kernel of the orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, w, p)$ is the space of (w, Γ)-symmetric infinitesimal motions of $(\mathcal{G}, p)$ restricted to the set of vertex orbits Γ_i of $\mathcal{G}$.

- Orbit rigidity matrix can be used to identify symmetric infinitesimal flexes
- full-rank $\mathcal{O}(\mathcal{G}_0, w, p)$ implies none exist
- size of $\mathcal{O}(\mathcal{G}_0, w, p)$ does not depend on p, but only the graph and symmetry constraints

Theorem [Shulze 2011]

The kernel of the orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, w, p)$ is the space of (w, Γ)-symmetric infinitesimal motions of $(\mathcal{G}, p)$ restricted to the set of vertex orbits Γ_i of $\mathcal{G}$.

- Orbit rigidity matrix can be used to identify symmetric infinitesimal flexes
- full-rank $\mathcal{O}(\mathcal{G}_0, w, p)$ implies none exist
- size of $\mathcal{O}(\mathcal{G}_0, w, p)$ does not depend on p, but only the graph and symmetry constraints

key point: quotient gain graph and orbit rigidity matrix suggests a further way to exploit symmetry in formation control

- representative edges used to maintain distances
- symmetry within vertex orbits have no need for distance constraints

A MODIFIED FORMATION POTENTIAL

- the representative edge formation potential

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij \in \mathcal{E}_0} \left(\|p_i - \tau(\gamma_{ij}) p_j\|^2 - (d_{ij}^{\star})^2 \right)^2 .$$

- the representative edge formation potential

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij \in \mathcal{E}_0} \left(\|p_i - \tau(\gamma_{ij})p_j\|^2 - (d_{ij}^\star)^2 \right)^2 .$$

- the symmetry potential

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the representative edge formation potential

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij \in \mathcal{E}_0} \left(\|p_i - \tau(\gamma_{ij}) p_j\|^2 - (d_{ij}^\star)^2 \right)^2 .$$

- the symmetry potential

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu}) p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the symmetric formation potential

$$F(p(t)) = F_e(p(t)) + F_s(p(t))$$

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

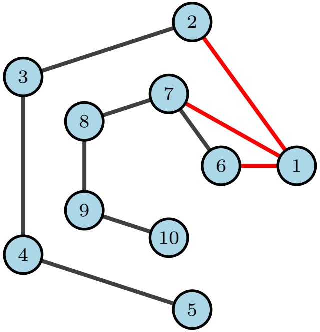
- closed-loop dynamics

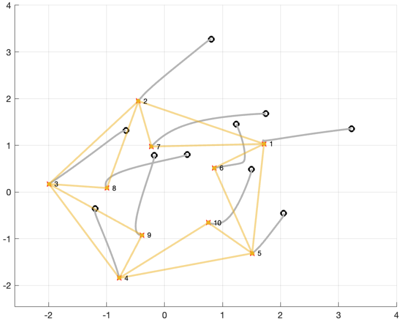
$$\begin{bmatrix} \dot{p}_0(t) \\ \dot{p}_f(t) \end{bmatrix} = \begin{bmatrix} -\mathcal{O}^T(\mathcal{G}_0, p_0(t))\left(\mathcal{O}(\mathcal{G}_0, p_0(t))p_0(t) - \mathbf{d}_0^2\right) \\ 0 \end{bmatrix} + Q \begin{bmatrix} p_0(t) \\ p_f(t) \end{bmatrix}.$$

- structure idea
    - representative vertices in $\mathcal{V}_0$ take care of distances
    - other vertices just maintain symmetry constraints

EXAMPLE

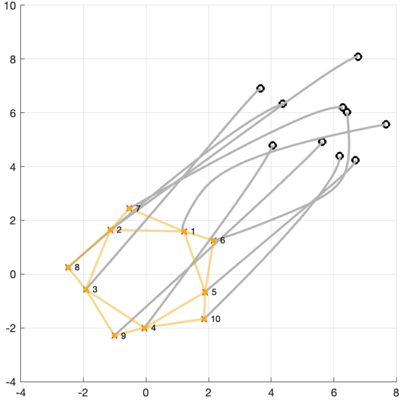
- $\mathcal{V}_0 = \{1, 6\}$
- $\mathcal{E}_0 = \{16, 17, 12\}$



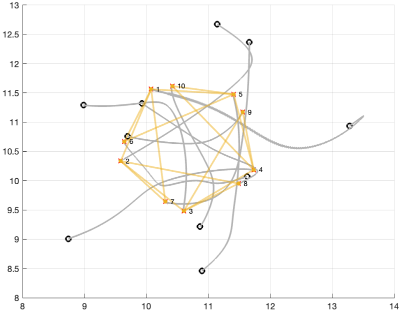


- strategy requires only 3 distance constraints and 8 symmetry constraints
- compared to 17 distance constraint for MIR classic approach

CENTROID CONSENSUS



- symmetry relies on a fixed inertial frame



- can add consensus term to agree on arbitrary centroid

Symmetry preserving motion coordination aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body and preserving symmetry of configuration.

Symmetry preserving motion coordination aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body and preserving symmetry of configuration.

- can we maneuver a symmetric formation in space?
- if we relax rigidity requirement, can you introduce symmetry-preserving motions?

Theorem - Distance Constrained Formation Control

Consider the potential function

$$V(p) = \frac{1}{4} \sum_{i \sim j} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2 \right)^2$$

and assume the desired distances $d_{ij}^\star$ correspond to a feasible formation. Then the gradient dynamical system

$$\dot{p} = -\nabla_p V(p) = -R^T(p)R(p)p + R^T(p)(d^\star)^2$$

asymptotically converges to the critical points of the potential function, i.e., $\frac{\partial V(p)}{\partial p} = 0$.

- $R(p)$ is the *rigidity matrix* for the framework $(\mathcal{G}, p)$
- rigidity theory used here to understand more about the equilibrium sets

PROOF SKETCH

(following De Queiroz ’18)

Define some notations...

- relative positions: $\tilde{p}_{ij} = p_i - p_j$
- distance error: $e_{ij} = \|\tilde{p}_{ij}\| - d^\star_{ij}$
- intermediate variable: $z_{ij} = \|\tilde{p}_{ij}\|^2 - (d^\star_{ij})^2 = e_{ij}(e_{ij} + 2d^\star_{ij})$

PROOF SKETCH

Define some notations...

- relative positions: $\tilde{p}_{ij} = p_i - p_j$
- distance error: $e_{ij} = \|\tilde{p}_{ij}\| - d^\star_{ij}$
- intermediate variable: $z_{ij} = \|\tilde{p}_{ij}\|^2 - (d^\star_{ij})^2 = e_{ij}(e_{ij} + 2d^\star_{ij})$

introduce Lyapunov candidate:

$$V(e) = \frac{1}{4}\sum_{i\sim j} z_{ij}^2 = z^T z$$

PROOF SKETCH

Define some notations...

- relative positions: $\tilde{p}_{ij} = p_i - p_j$
- distance error: $e_{ij} = \|\tilde{p}_{ij}\| - d^\star_{ij}$
- intermediate variable: $z_{ij} = \|\tilde{p}_{ij}\|^2 - (d^\star_{ij})^2 = e_{ij}(e_{ij} + 2d^\star_{ij})$

introduce Lyapunov candidate:

$$V(e) = \frac{1}{4}\sum_{i\sim j} z_{ij}^2 = z^T z$$

time-derivative of Lyapunov function along trajectories

$$\boxed{\dot{V} = z^T R(p) u}$$

IDEA: Design control u to ensure Lyapunov function is decreasing!

- Formation acquisition: $u = -R(p)^T z$
  ensures stable formation dynamics
  "classic" distance-constrained formation controller

FORMATION MANEUVERING

Formation maneuvering aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body.

...recall our earlier Lyapunov function

$$\dot{V} = z^T R(p) u$$

choose $u = u_a + u_m$

- $u_a = -R(p)^T z$: used to attain desired formation
- $u_m = \mathbb{1} \otimes v_0 + \begin{bmatrix} \vdots \\ \omega_0 \times \tilde{q}_i \\ \vdots \end{bmatrix}$: rigid body translation (v_0) and rotation about a point ($\omega_0 \times \tilde{q}_i$)

Main Idea: rigid body rotations and translations are in the Kernel of the rigidity matrix!

BACK TO MOTION COORDINATION

...recall our earlier Lyapunov function

$$\dot{W} = z^T R(p) u$$

choose $u = u_a + u_m + u_s$

- $u_a = -R(p)^T z$: used to attain desired formation
- $u_m = \mathbb{1} \otimes v_0 + \begin{bmatrix} \vdots \\ \omega_0 \times \tilde{q}_i \\ \vdots \end{bmatrix}$: rigid body translation (v_0) and rotation about a point ($\omega_0 \times \tilde{q}_i$)
- u_s obtained from kernel of Orbit rigidity matrix

Summary

- exploit notions of symmetry in formation control
- $\tau(\Gamma)$-symmetric graphs captures symmetry of configurations and graphs
- symmetric formation potential used to design distributed control law with less edges compared to “traditional” formation control strategies
- opportunities for more sophisticated motion coordination

Future Work

- formation maneuvering requires time-varying point group symmetries
- is it possible to distributedly decide on certain symmetries?
- can we eliminate need for requiring self-state in protocol?
- more?

Questions?