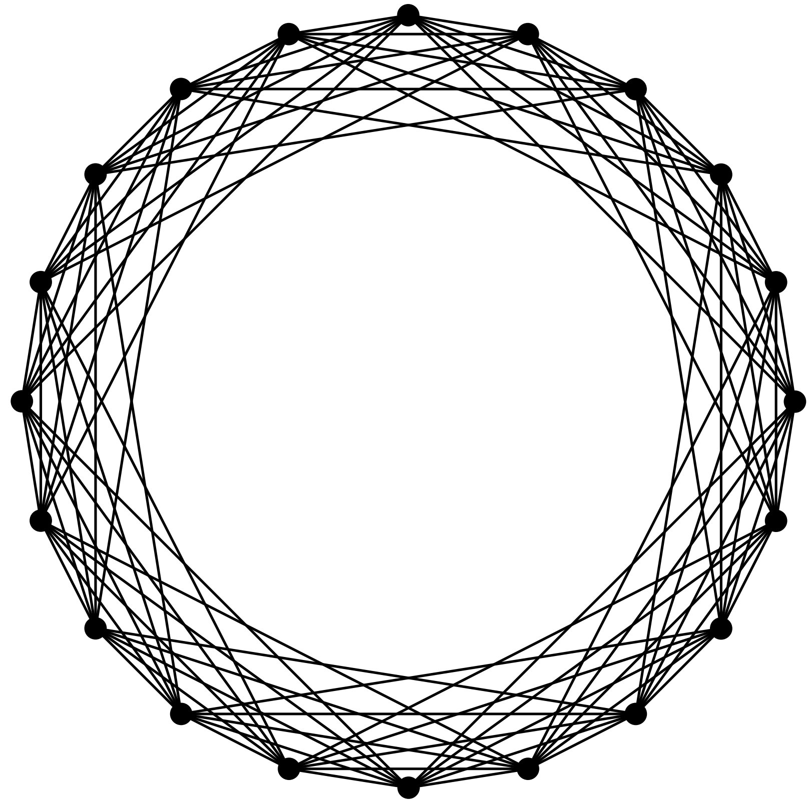
Control and Estimation of Multi-Agent Systems with Bearing-Only Sensing: Rigidity Theory for SE(2)



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Faculty of Aerospace Engineering
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Kolloquium Technische Kybernetik
Stuttgart, Germany

Challenges in Multi-Robot Systems

Solutions to coordination problems in multi-robot systems are *highly* dependent on the sensing and communication mediums available!

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<u>Sensing</u>

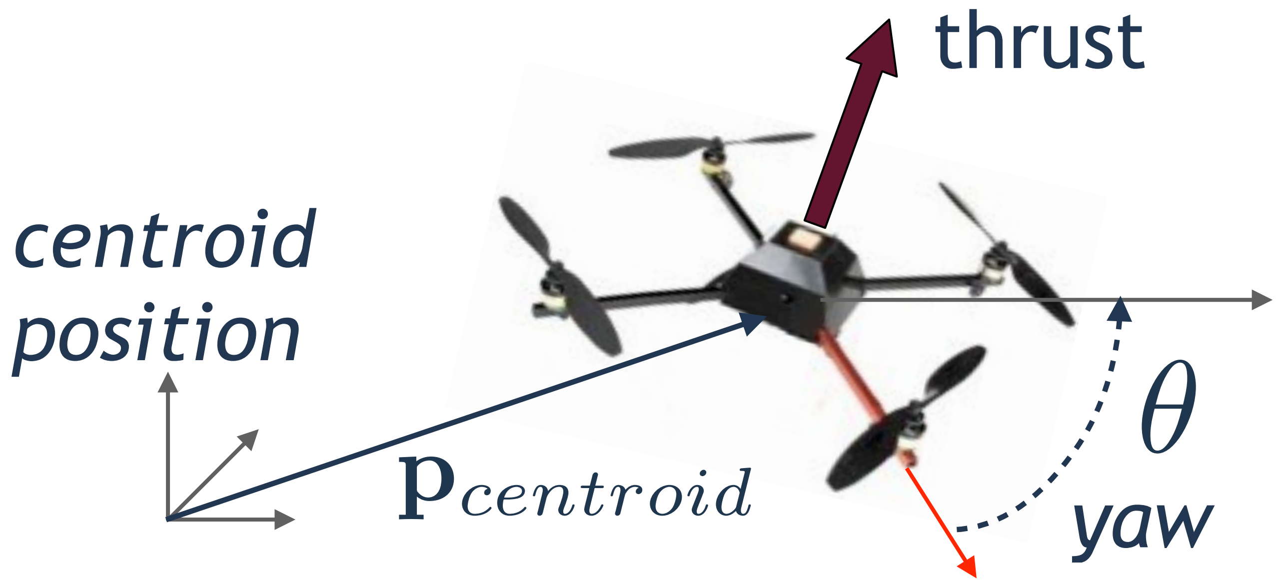
- GPS
- Relative Position Sensing
- Range Sensing
- Bearing Sensing

<u>Communication</u>

- Internet
- Radio
- Sonar
- MANet

selection criteria depends on mission requirements, cost, environment…

Challenges in Multi-Robot Systems



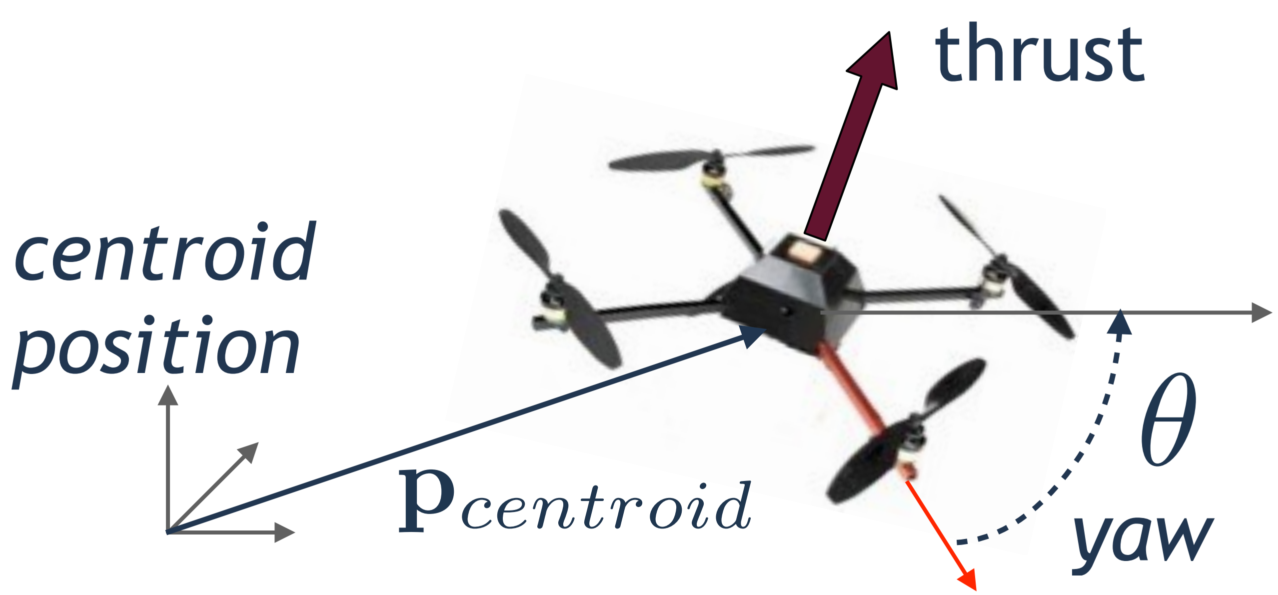
$$J_i w_i + S(w_i) J_i w_i = \gamma_i + \zeta_i$$

fully-actuated rotational dynamics

$$m_i \ddot{x}_i = -\lambda_i R_i e_3 + m_i g e_3 + \delta_i$$

under-actuated translational dynamics

Challenges in Multi-Robot Systems



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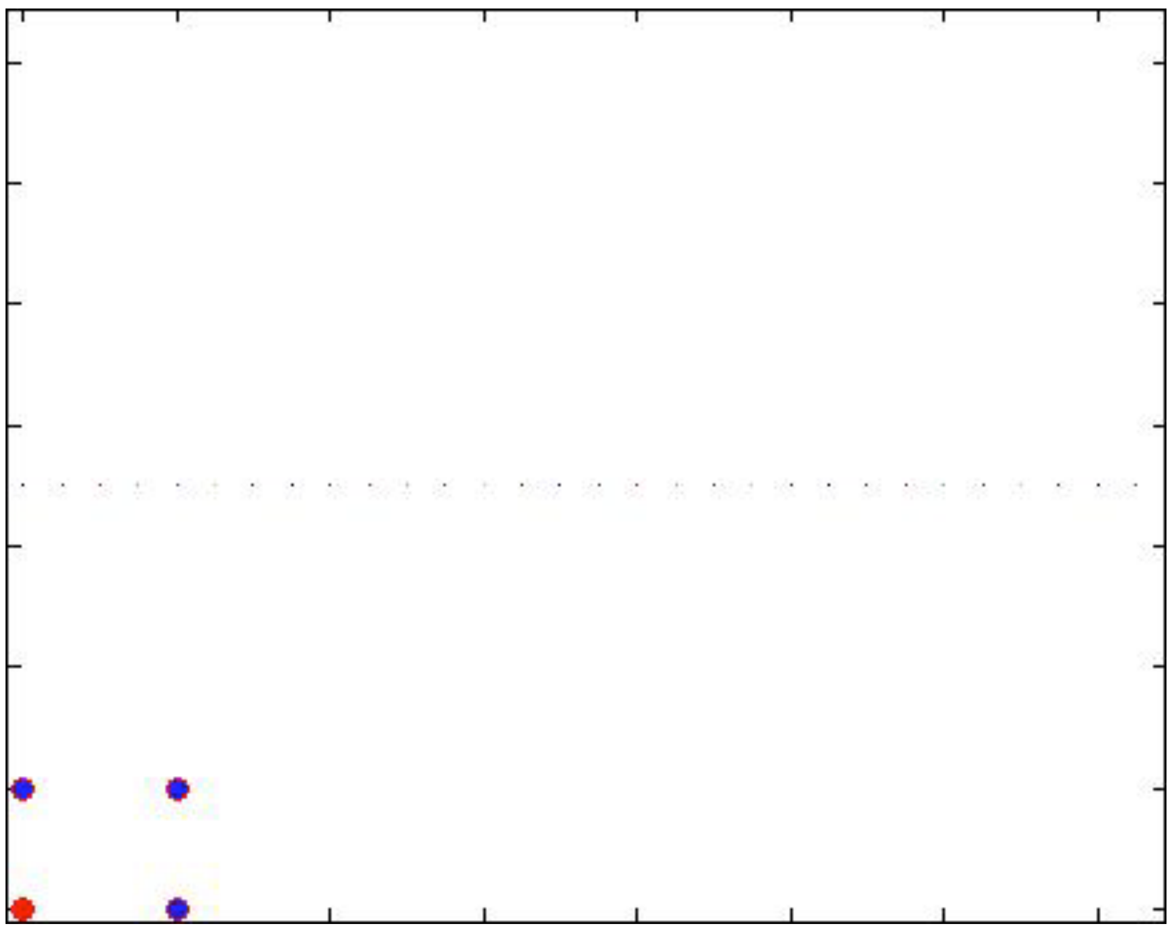
under-actuated translational dynamics

sensed information depends *both* on sensor type and how it is physically attached to the robot

Outline

- Introduction
- Rigidity Theory - a short review
- Bearing-Only Sensing and Formation control
  - Parallel Rigidity
  - Stability of Bearing-Only Formation Control
- Bearing-Only Sensing with No Common Reference
  - Rigidity in SE(2)
  - Distributed Estimation of a Common Reference
- Conclusions and Outlook

Formation Control: Distance-Based Approaches



robots modeled as integrators

$$\dot{p}_i = u_i$$

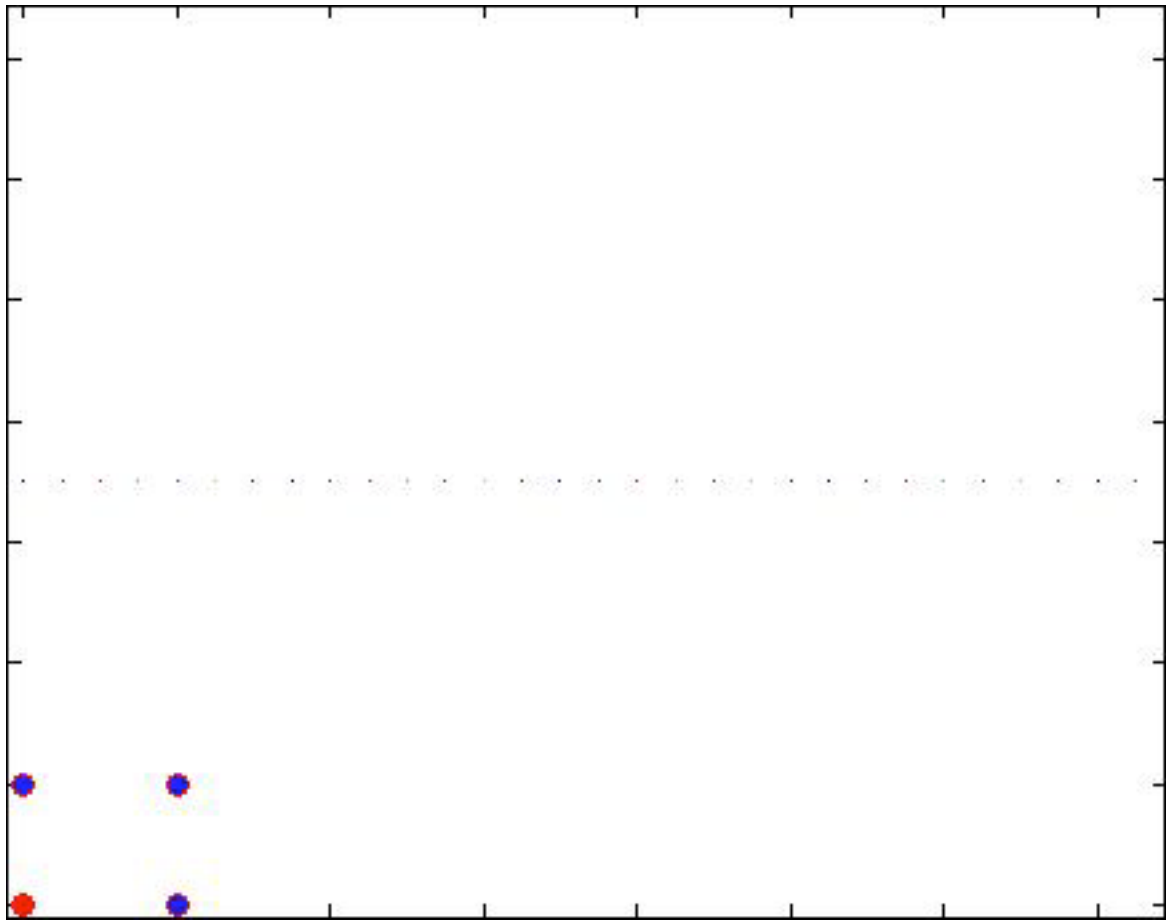
agents can sense range to neighbors determined by a (fixed) sensing graph

$$\|p_i - p_j\|^2$$

desired formation is specified by a vector of distances

$$d_{ij}^2$$

Formation Control: Distance-Based Approaches



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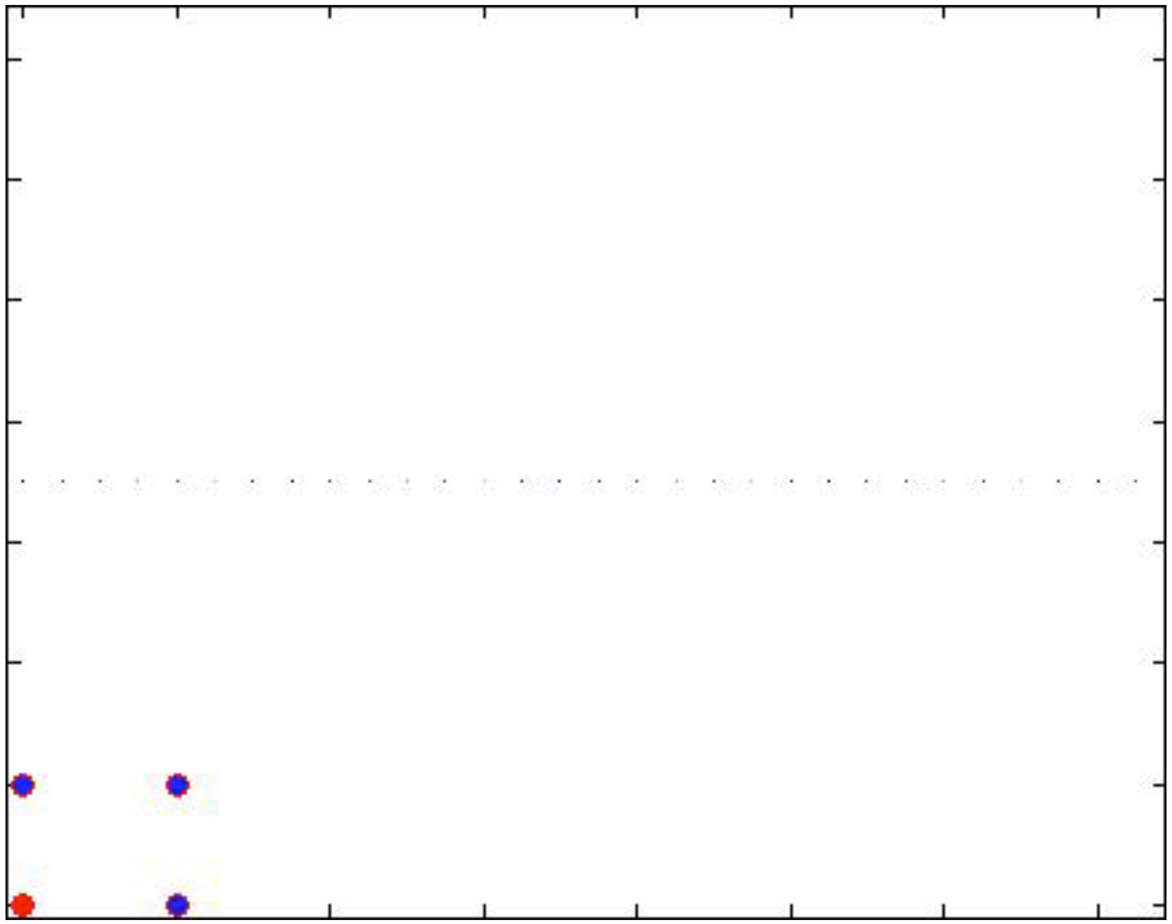

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$$\dot{p}_i = \sum_{j \sim i} \left(\|p_i - p_j\|^2 - d_{ij}^2 \right) (p_j - p_i)$$

desired formation is (locally) asymptotically stable if the sensing graph is ***infinitesimally rigid***

[Krick2007, Anderson2008, Dimarogonas2008, Dörfler2010]

Rigidity Theory

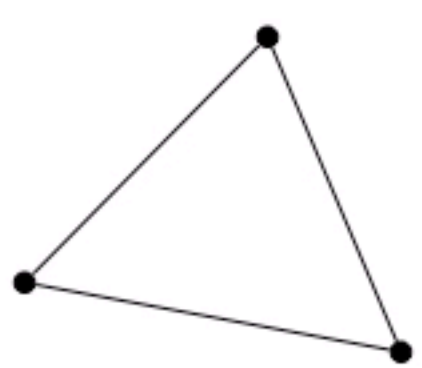
Rigidity is a combinatorial theory for characterizing the "stiffness" or "flexibility" of structures formed by rigid bodies connected by flexible linkages or hinges.

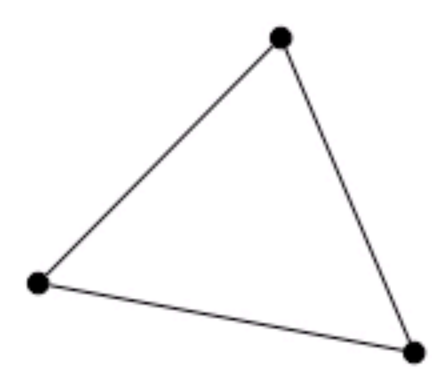
Distance Rigidity

- maintain distance pairs
- rigid body rotations and translations

Parallel Rigidity

- maintain angles (shape)
- rigid body translations and dilations





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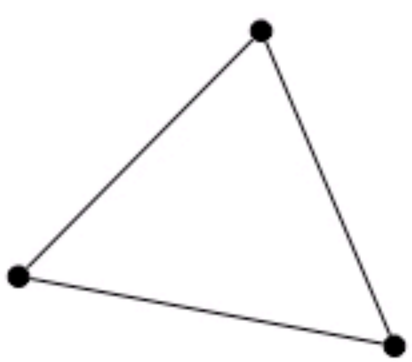
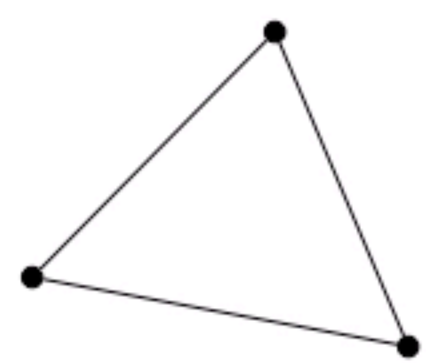


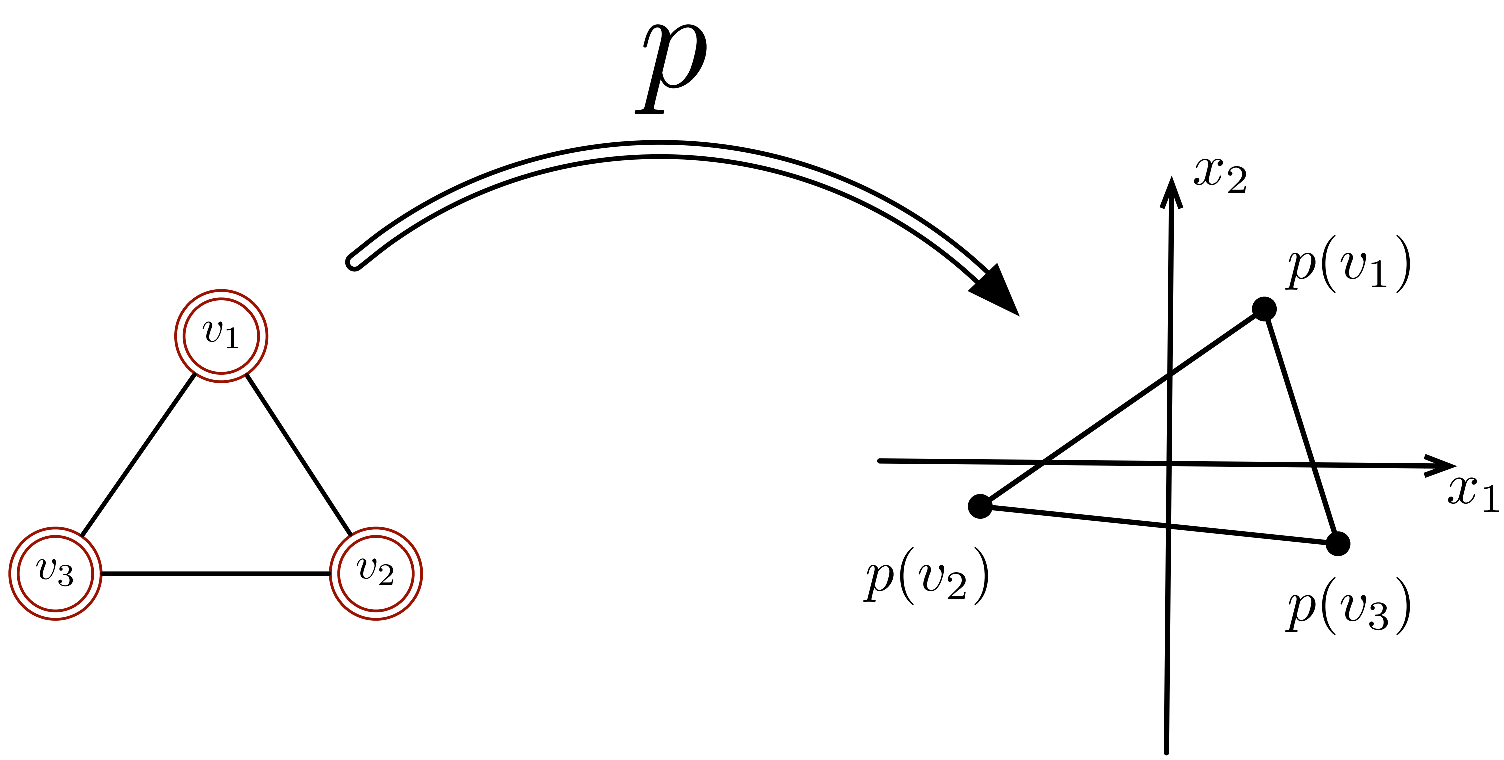


Rigidity Theory

bar-and-joint frameworks

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$
$$p : \mathcal{V} \to \mathbb{R}^2$$

maps every vertex to a
point in the plane



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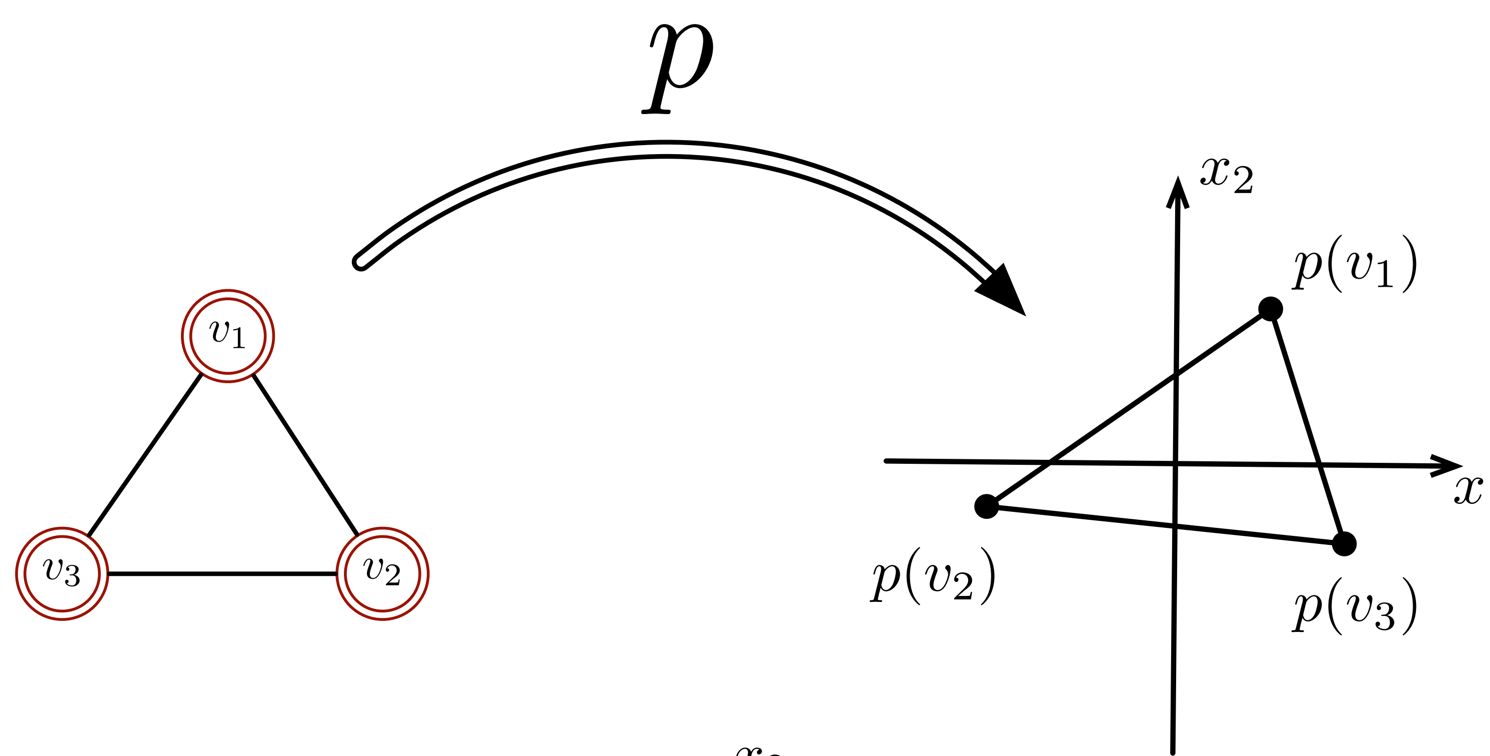
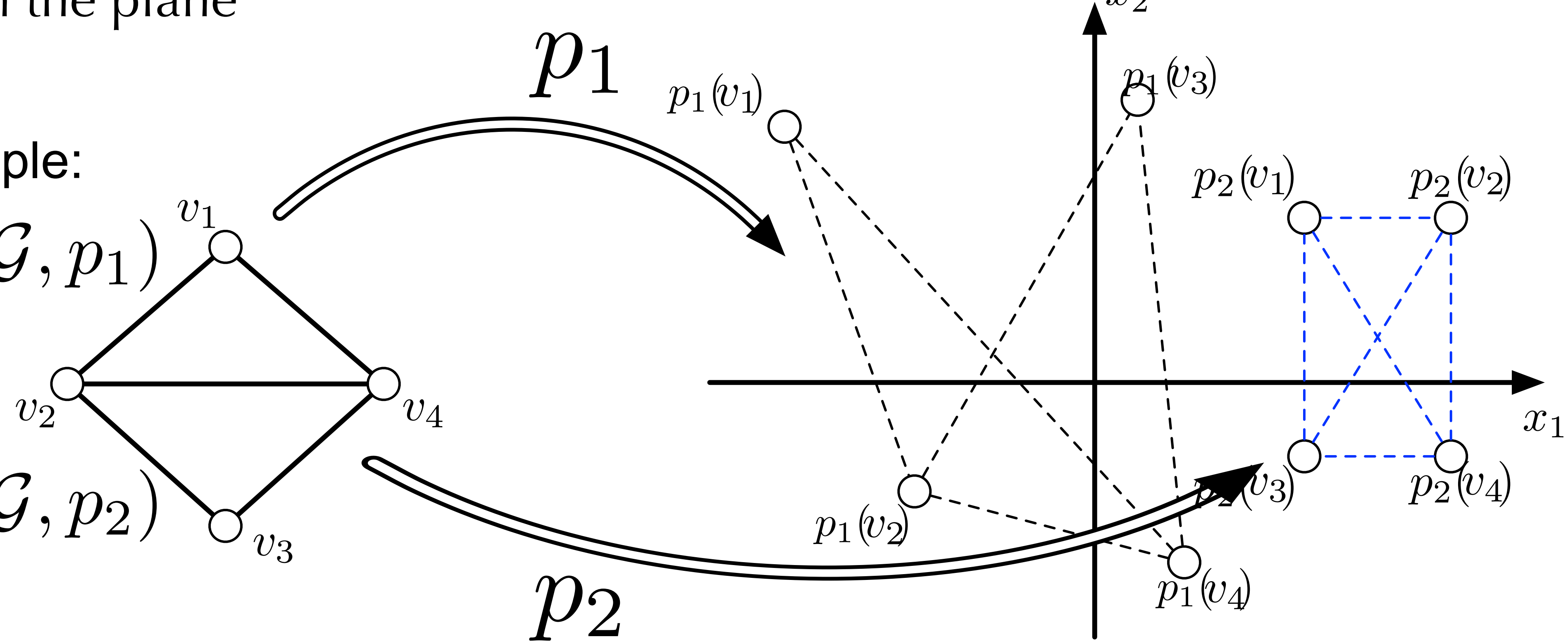


maps every vertex to a point in the plane

example:

$$\mathcal{F}_1 = (\mathcal{G}, p_1)$$

$$\mathcal{F}_2 = (\mathcal{G}, p_2)$$



Rigidity Theory

Rigidity is a combinatorial theory for characterizing the "stiffness" or "flexibility of structures formed by rigid bodies connected by flexible linkages or hinges.

Distance Rigidity

infinitesimal motions

$$(p(u) - p(v))^T (\xi(u) - \xi(v)) = 0$$

Rigidity Matrix

$$R(p)\xi = 0$$

Parallel Rigidity

infinitesimal motions

$$\left((p(u) - p(v))^\perp\right)^T (\xi(u) - \xi(v)) = 0$$

Parallel Rigidity Matrix

$$R_{||}(p)\xi = 0$$

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Parallel Rigidity Matrix

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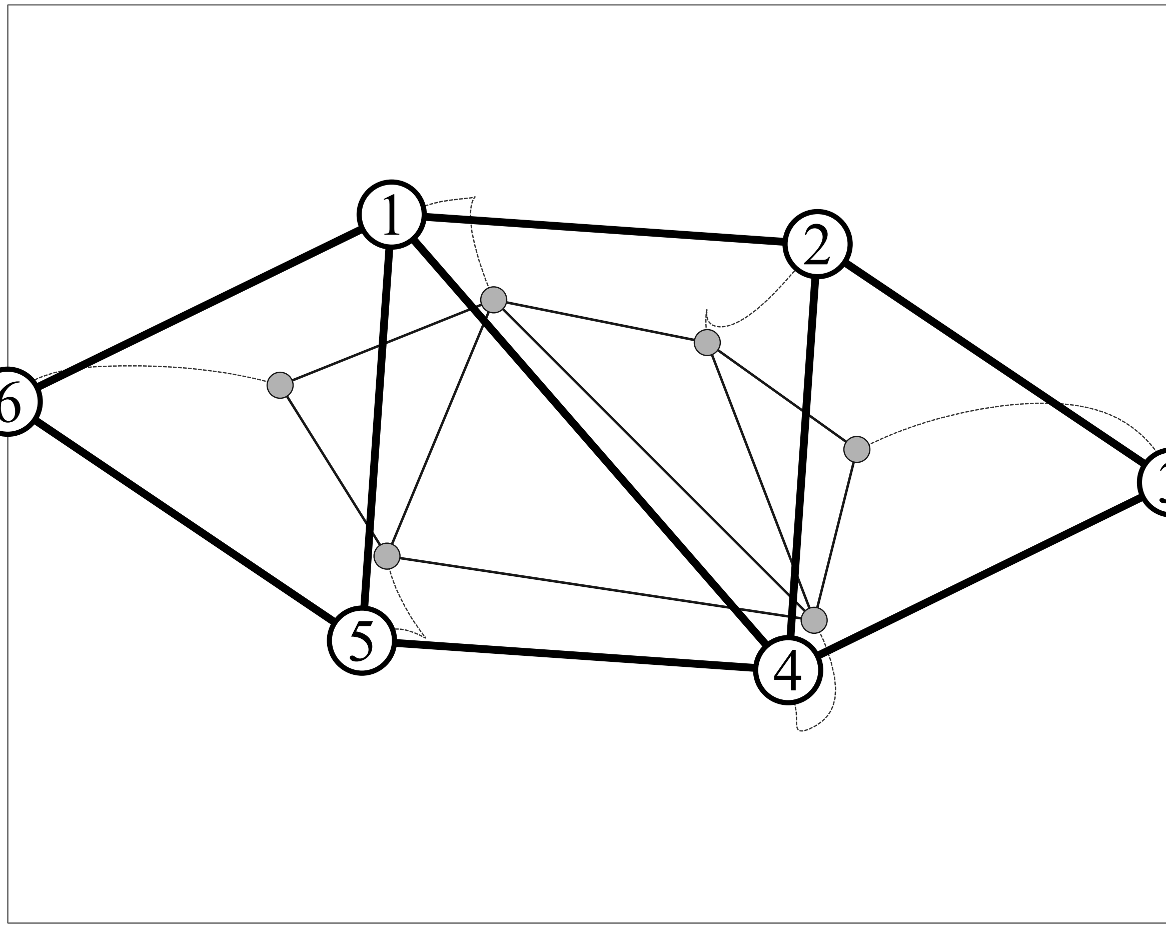
Theorem

A framework is infinitesimally rigid if and only if the rank of the rigidity matrix is $2|\mathcal{V}| - 3$

Formation Control: Distance-Based Approaches



$$\dot{p}_i = \sum_{j\sim i} \left(\|p_i - p_j\|^2 - d_{ij}^2\right)(p_j - p_i)$$

Important Assumptions

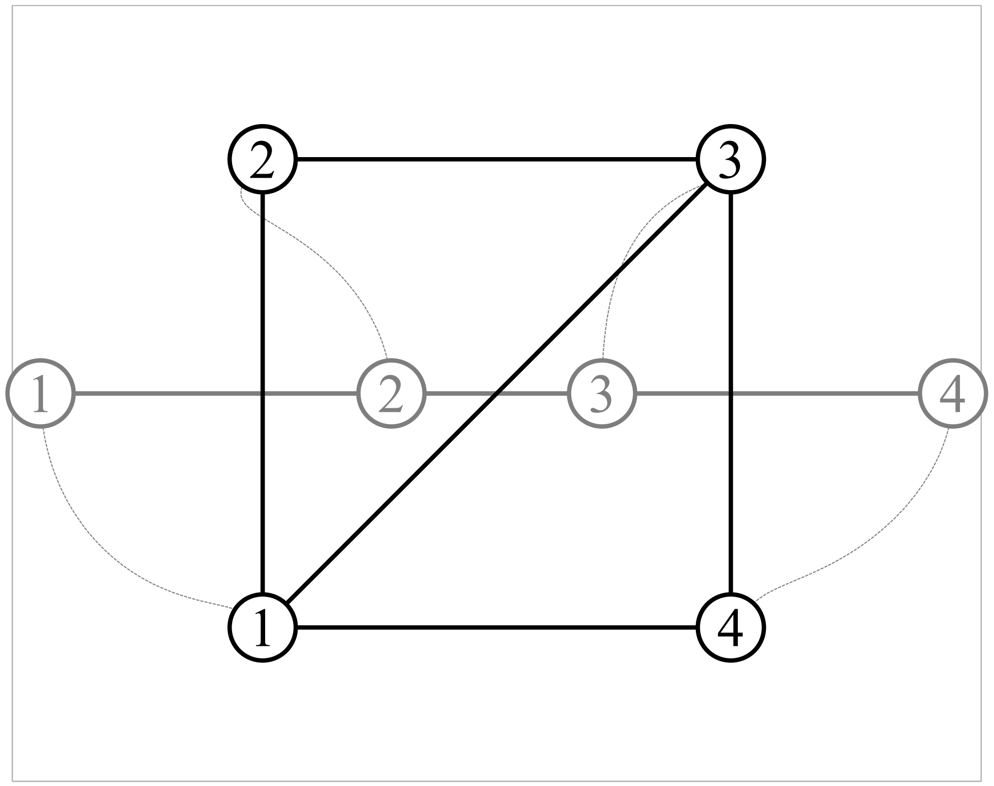
- point masses
- bidirectional sensing
- range measurements*
- *common reference frame is implicit*

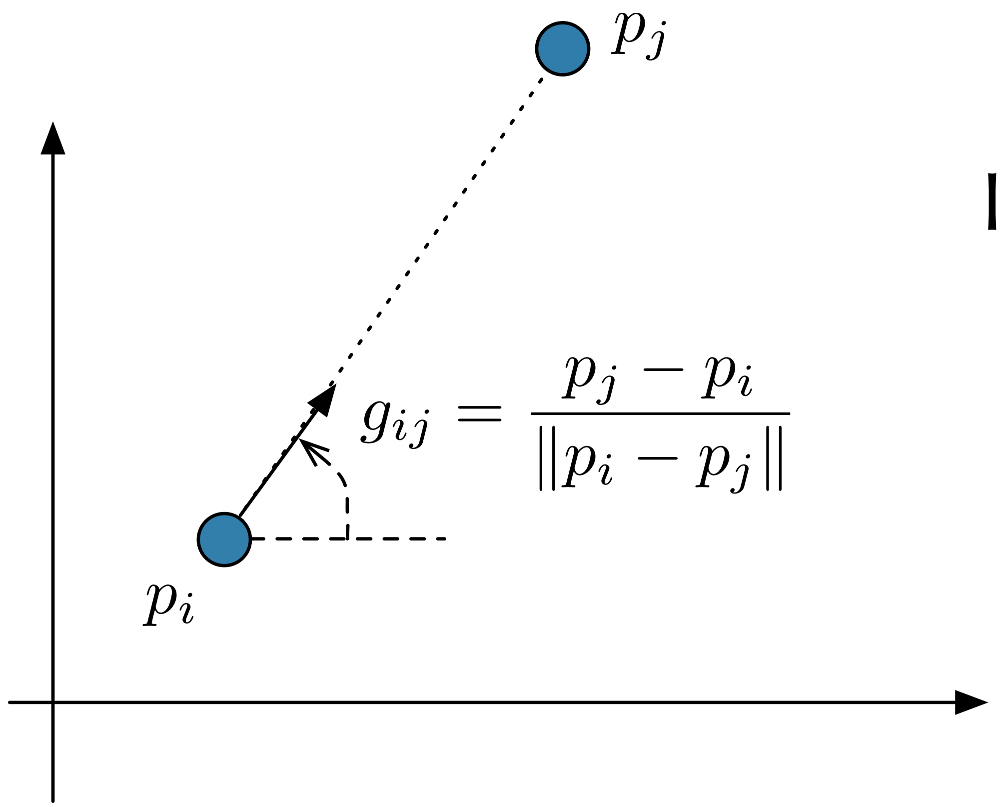
A Gradient Control Law

$$J(p) = \frac{1}{4}\sum_{i\sim j}\left(\|p_i - p_j\|^2 - d_{ij}^2\right)^2$$

$$\dot{p} = -\nabla J(p) = -R(p)^T R(p) p + R(p)^T d$$

Formation Control: Bearing-Constrained Formations





Formation specified by desired *bearing* constraints

$$g^*_{12} = -g^*_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad g^*_{13} = -g^*_{31} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$g^*_{23} = -g^*_{32} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad g^*_{14} = -g^*_{41} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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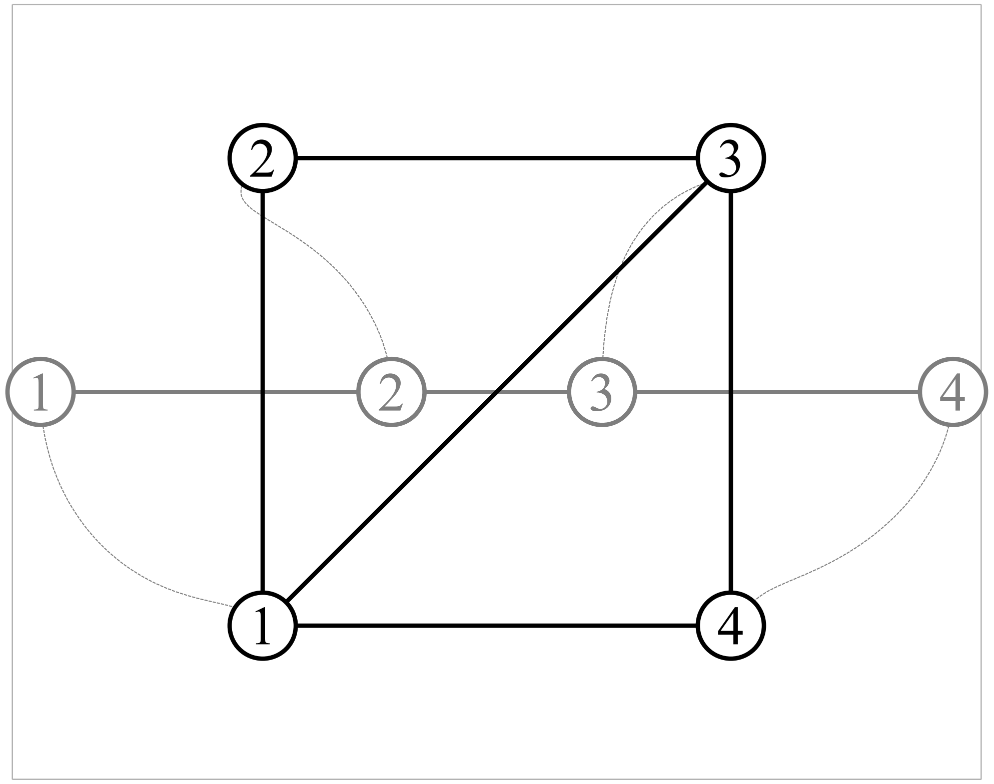
Important Assumptions

- point masses
- bidirectional sensing
- bearing sensing
- *common reference frame is implicit* (i.e., a compass)

Formation Control: Bearing-Constrained Formations



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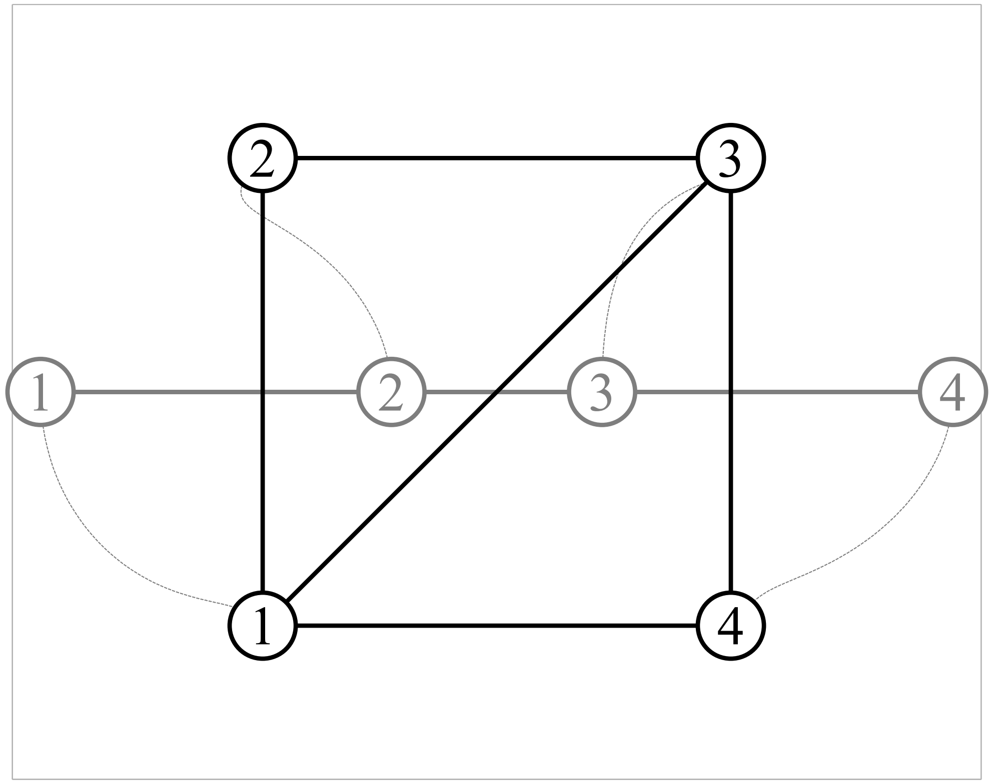
A Gradient Control Law?

$$J(g) = \sum_{i \sim j} \|g_{ij} - g_{ij}^*\|^2$$

$$\dot{p}_i = -\sum_{j \sim i} \frac{1}{\|p_i - p_j\|} \left(I_2 - \frac{(p_j - p_i)(p_j - p_i)^T}{\|p_i - p_j\|^2} \right) g_{ij}^*$$

Formation Control: Bearing-Constrained Formations



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not a bearing-only control law!

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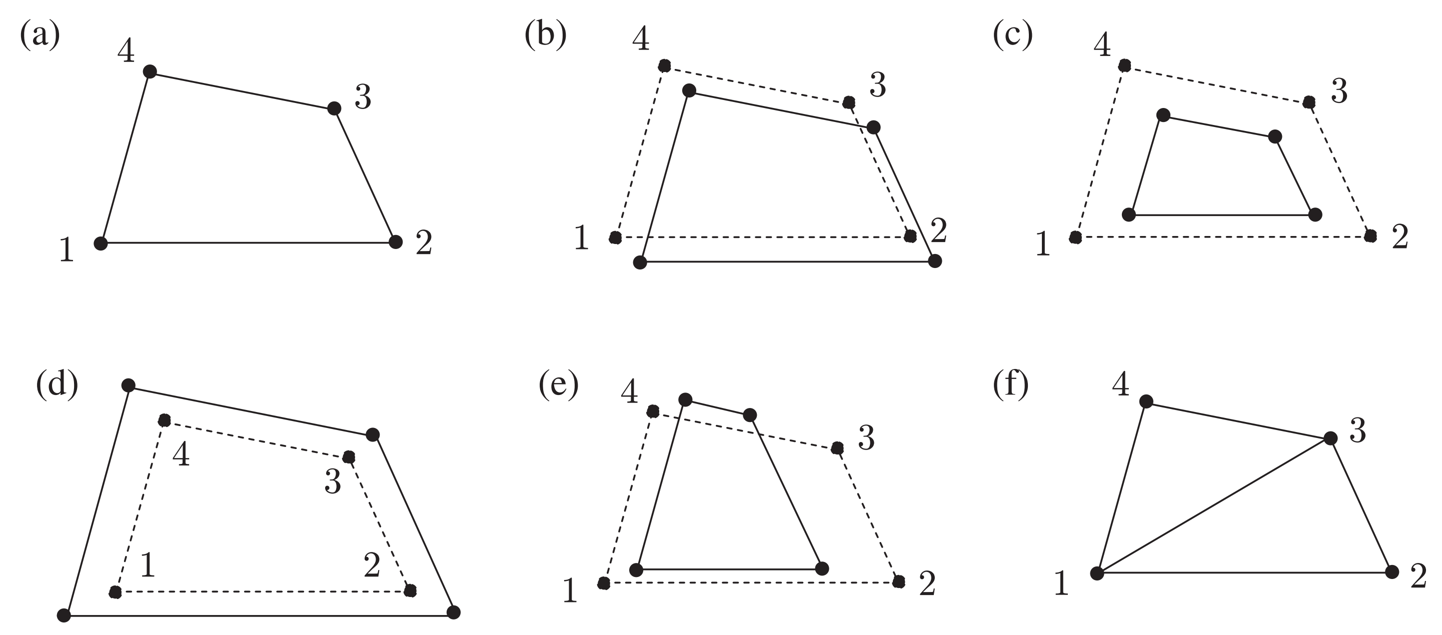
Parallel Rigidity in Arbitrary Dimension

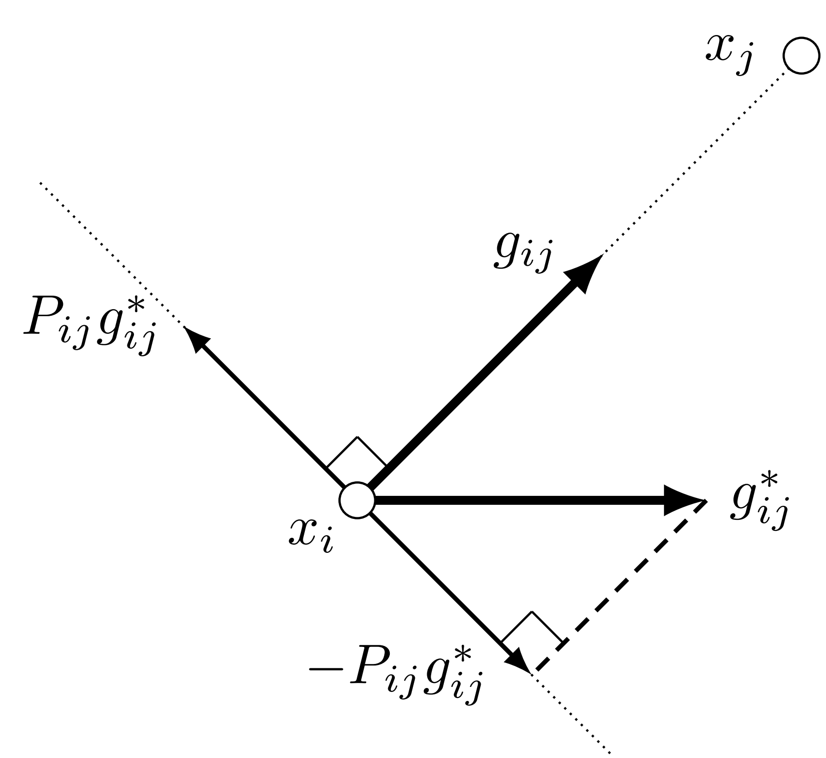
bar-and-joint frameworks

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$
$$p : \mathcal{V} \rightarrow \mathbb{R}^2$$



Parallel Drawings

$$((p_i - p_j)^{\perp})^T (p_k - p_l) = 0 \quad \mathbb{R}^2$$



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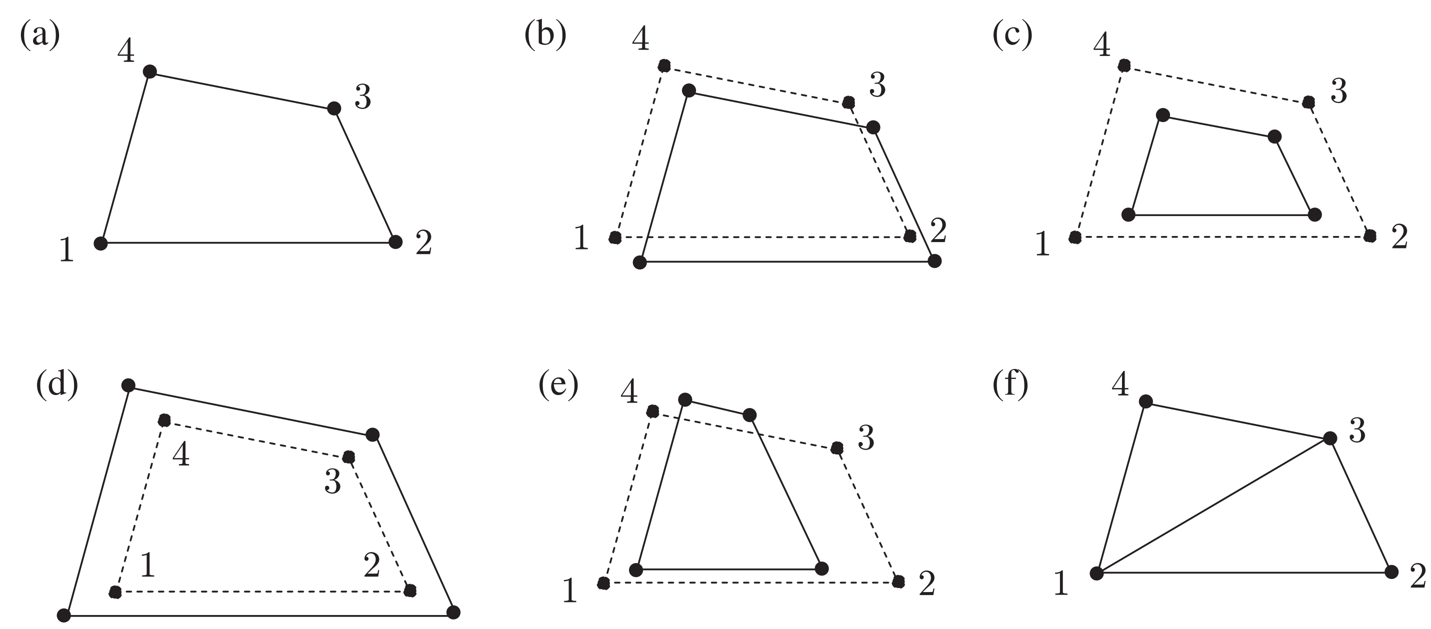


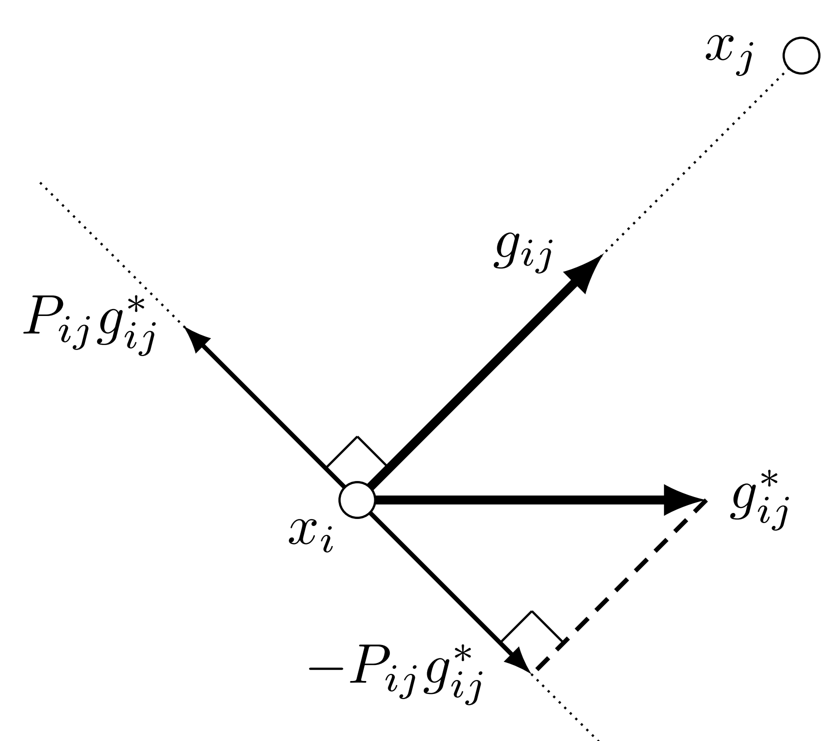
Parallel Drawings

$$((p_i - p_j)^{\perp})^T (p_k - p_l) = 0 \quad \mathbb{R}^2$$

or…

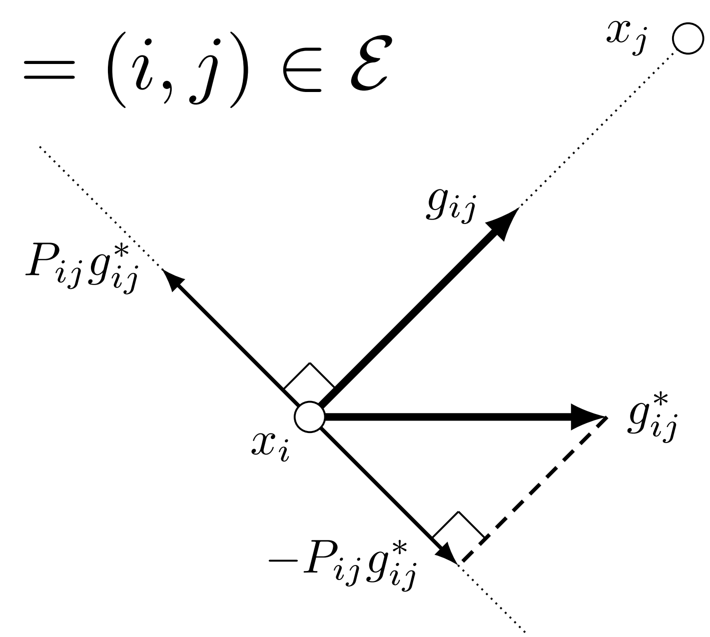
$$P_v u = 0, \quad v = p_i - p_j, \; u = p_k - p_l$$

$$P_v = I_d - \frac{v}{\|v\|}\frac{v}{\|v\|}^T$$ projection matrix $\mathbb{R}^d$



Parallel Rigidity in Arbitrary Dimension

$$e_k = p_j - p_i, \; k = (i,j) \in \mathcal{E}$$



bar-and-joint frameworks

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$
$$p : \mathcal{V} \to \mathbb{R}^2$$

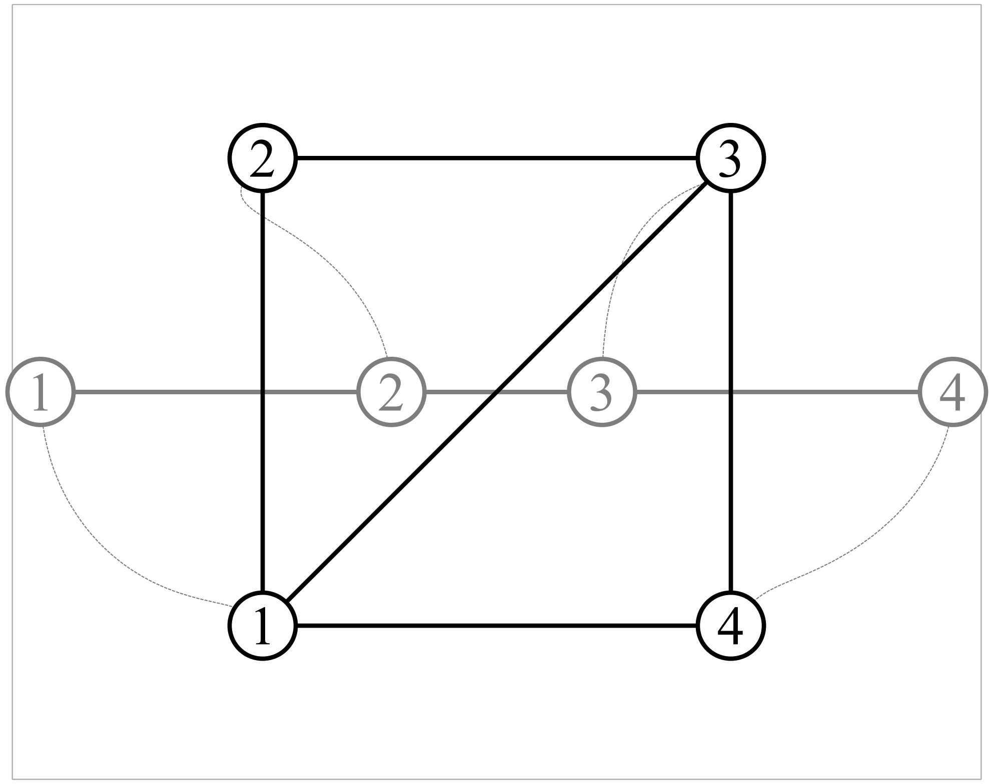
Bearing-Edge Function

$$f(p) = \begin{bmatrix} \vdots \\ \frac{p_j - p_i}{\|p_i - p_j\|} \\ \vdots \end{bmatrix}$$

Parallel Rigidity Matrix (arbitrary dimension)

$$R_{\|}(p) = \frac{\partial f(p)}{\partial p} \in \mathbb{R}^{d|\mathcal{E}| \times d|\mathcal{V}|}$$
$$= \mathbf{diag}\left(\frac{P_{e_k}}{\|e_k\|}\right)\left(E(\mathcal{G})^T \otimes I_d\right)$$

Formation Control: Bearing-Constrained Formations



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A Gradient Control Law?

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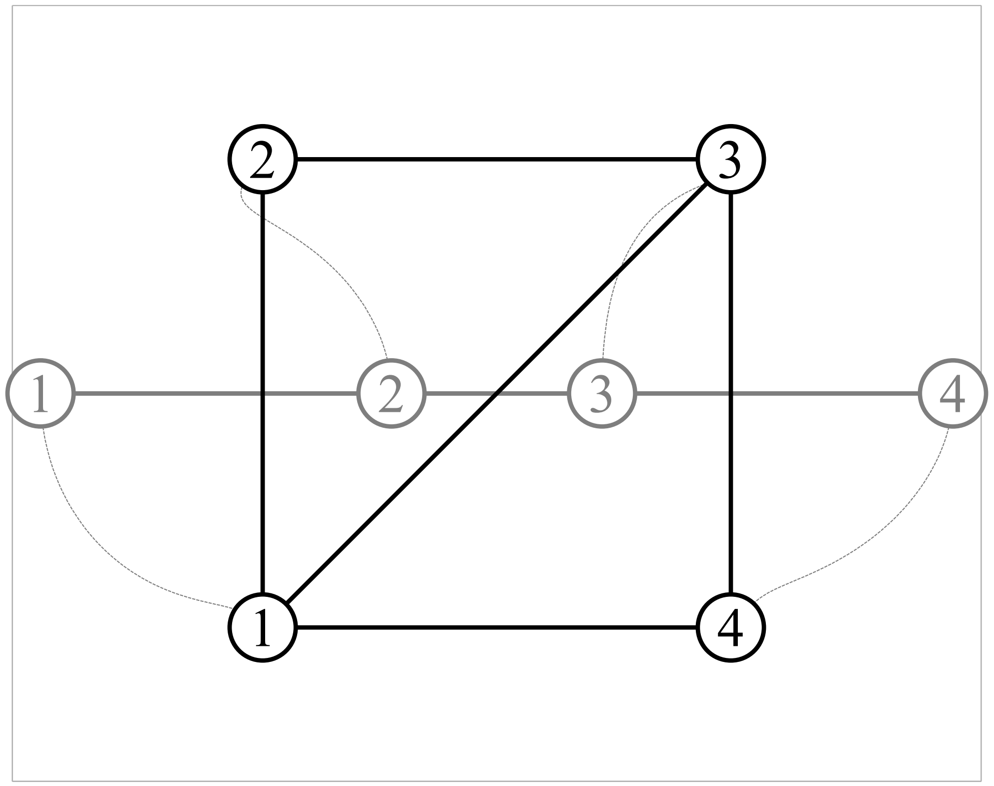
$$\dot{p} = -R_{\|}(p)^T g^*$$

not a bearing-only control law!

Formation Control: Bearing-Constrained Formations



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A Bearing-Only Control Law

$$\dot{p} = -\sum_{j \sim i} P_{g_{ij}} g_{ij}^*$$

Formation Control: Bearing-Constrained Formations

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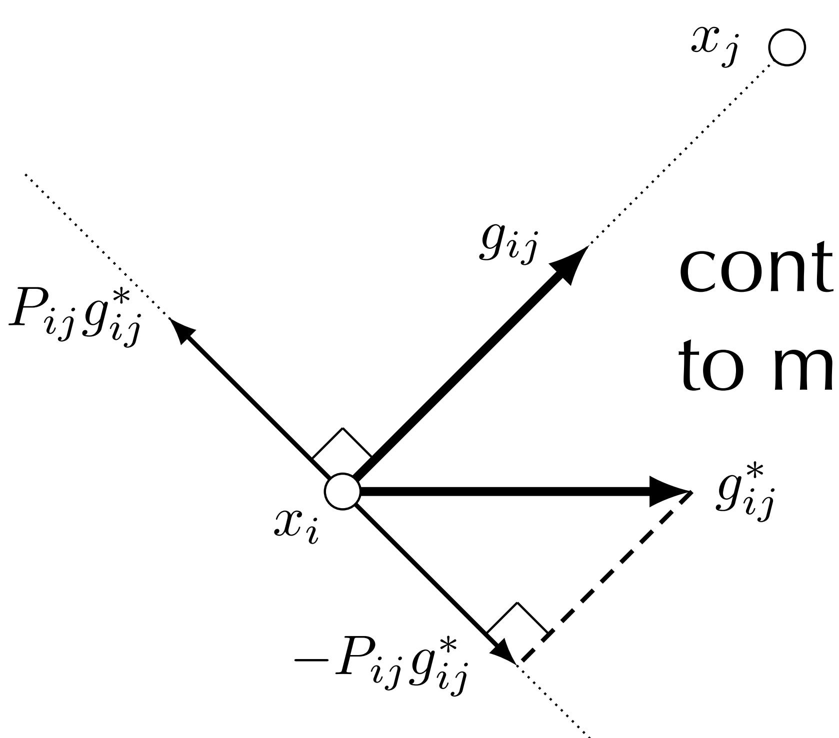
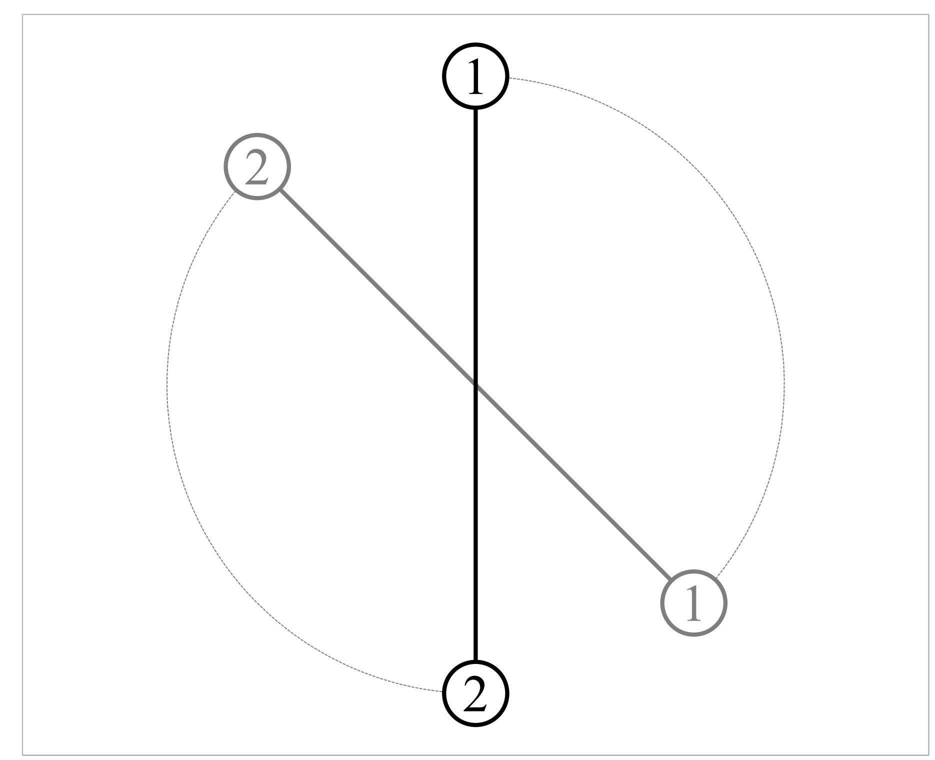


control is orthogonal to measured bearing

trajectories evolve on circle of constant radius



Formation Control: Bearing-Constrained Formations

A Bearing-Only Control Law

$$\dot{p} = -\sum_{j\sim i} P_{g_{ij}} g_{ij}^*$$

Theorem

If the desired bearing formation is feasible and infinitesimally parallel rigid, then the bearing-only control law converges exponentially to the desired formation.

Lyapunov function: $V(p) = \frac{1}{2}(p-p^*)^T(p-p^*)$

centroid of formation is invariant

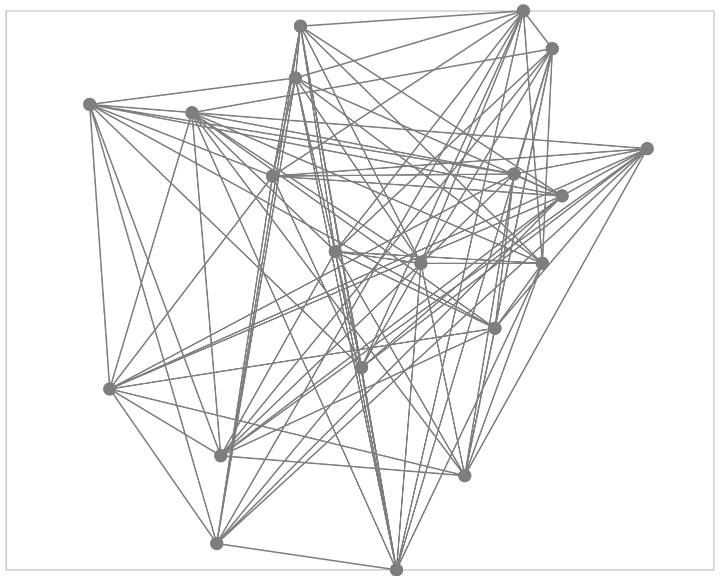
$$\bar{p} = \frac{1}{|\mathcal{V}|}\sum_{i=1}^{|\mathcal{V}|} p_i$$

scale of formation is invariant

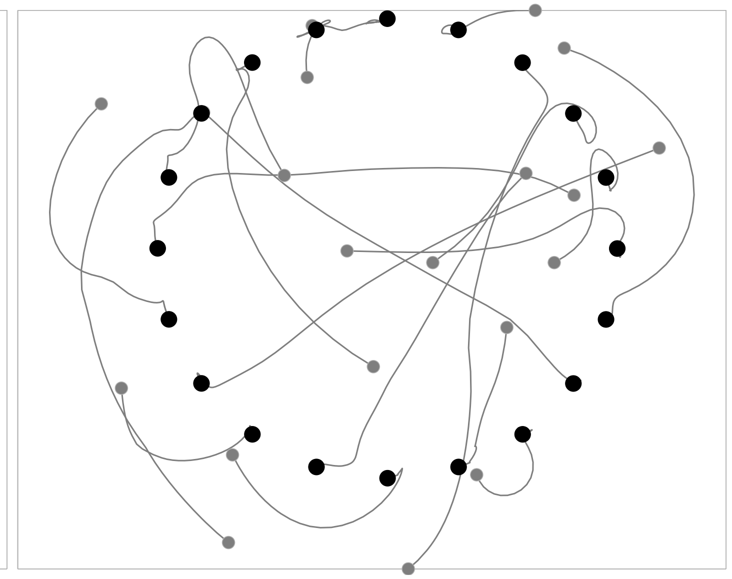
$$s = \sqrt{\frac{1}{n}\sum_{i=1}^{|\mathcal{V}|} \|p_i - p\|^2}$$

collision avoidance guaranteed (under assumptions of theorem)

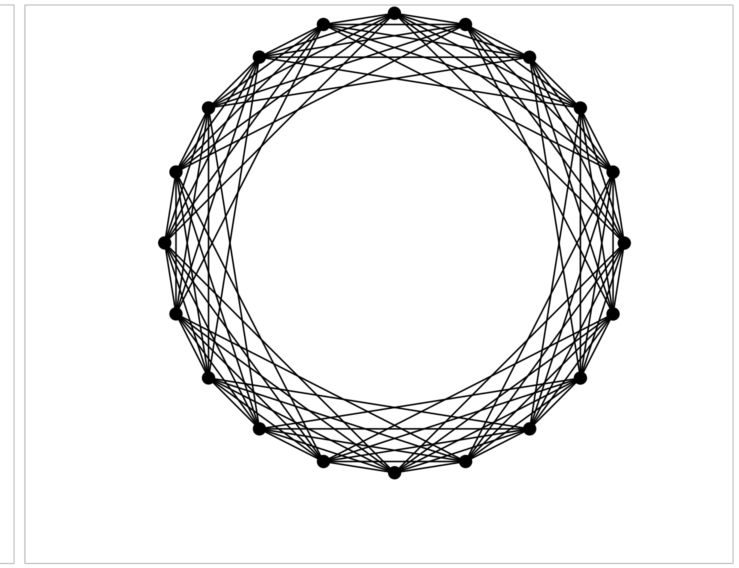
Formation Control: Bearing-Constrained Formations



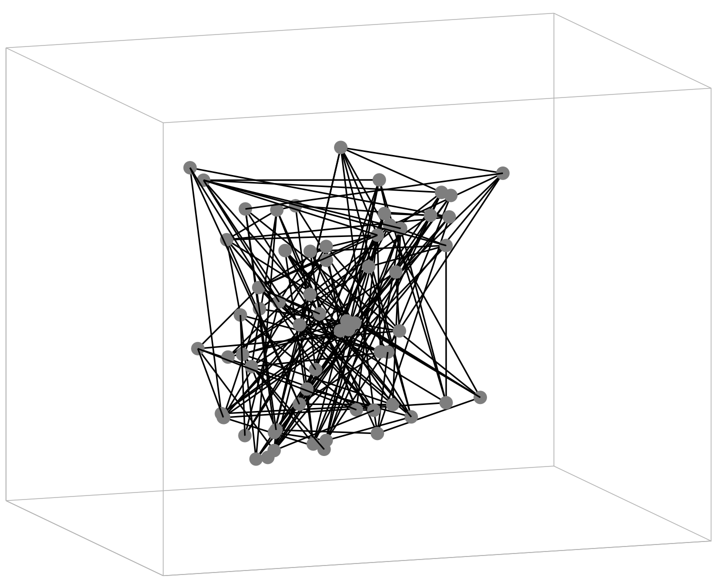
(a) Randomly generated initial formation



(b) Agent trajectory



(c) Final formation



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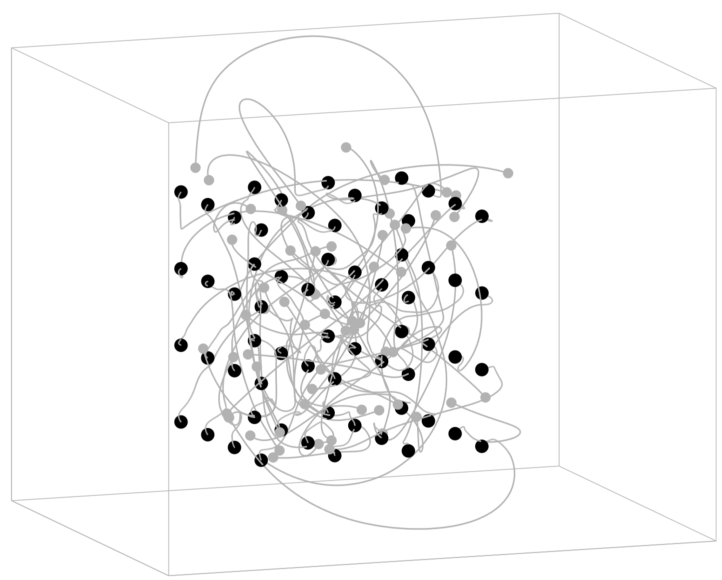

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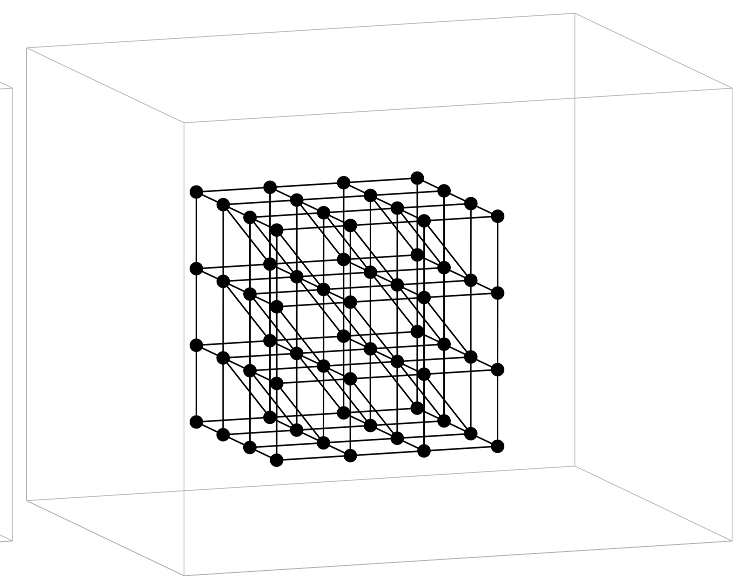

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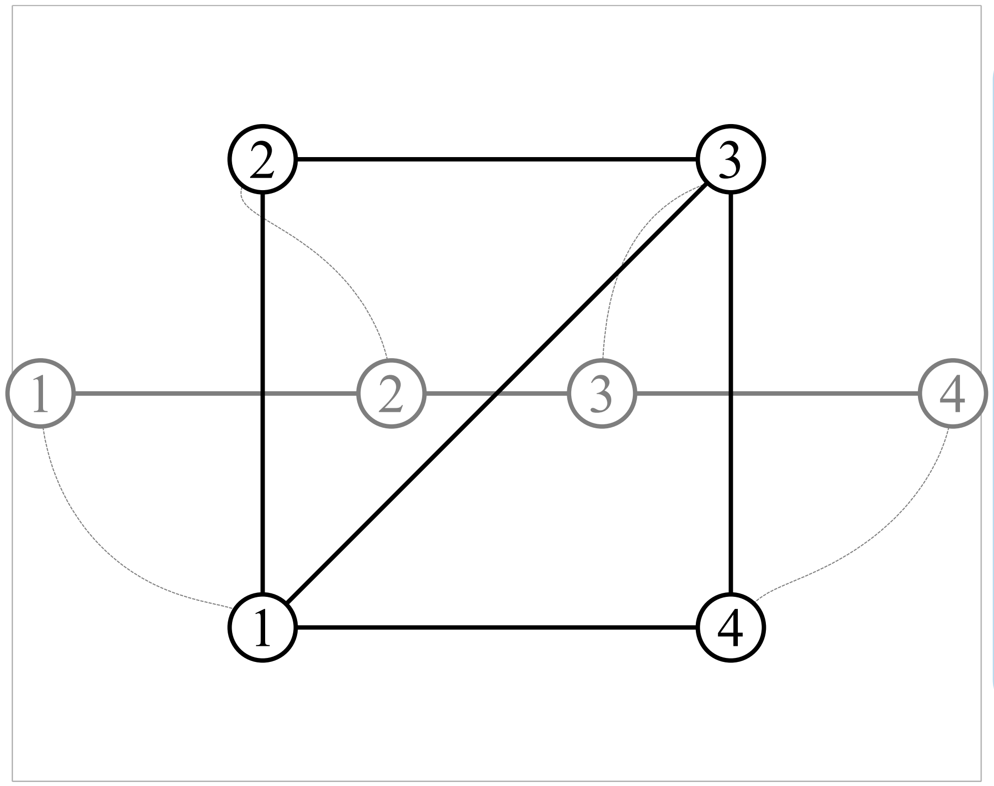
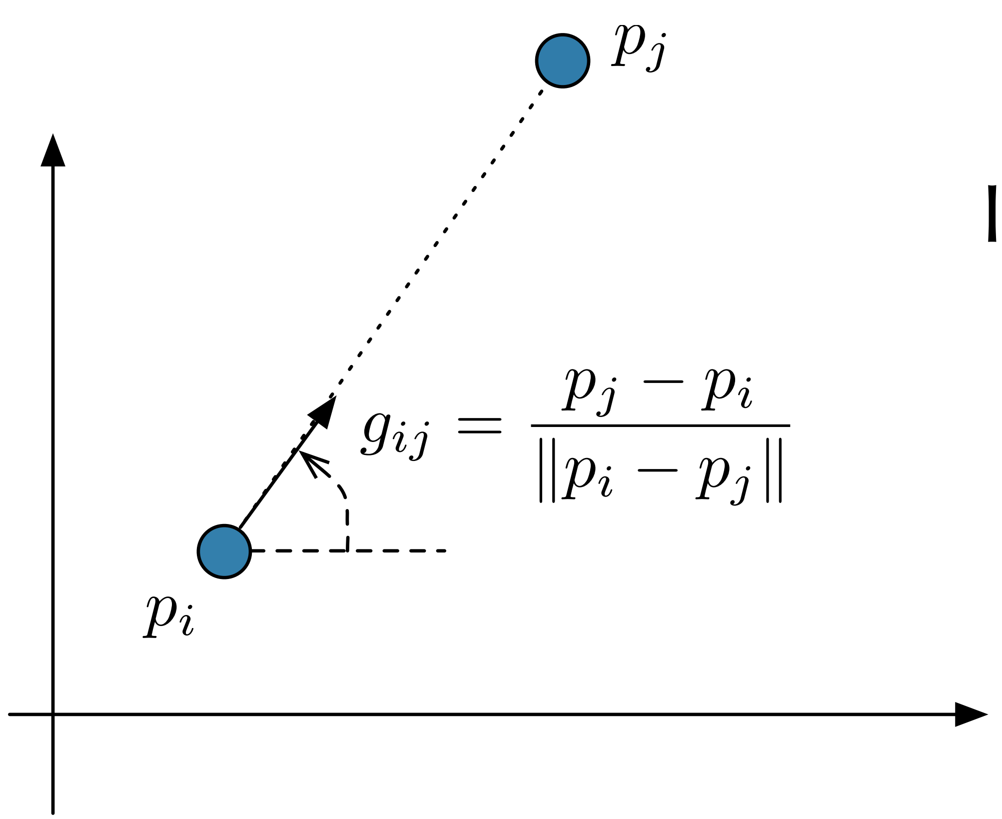


A Bearing-Only Control Law

$$\dot{p} = -\sum_{j \sim i} P_{g_{ij}} g_{ij}^*$$

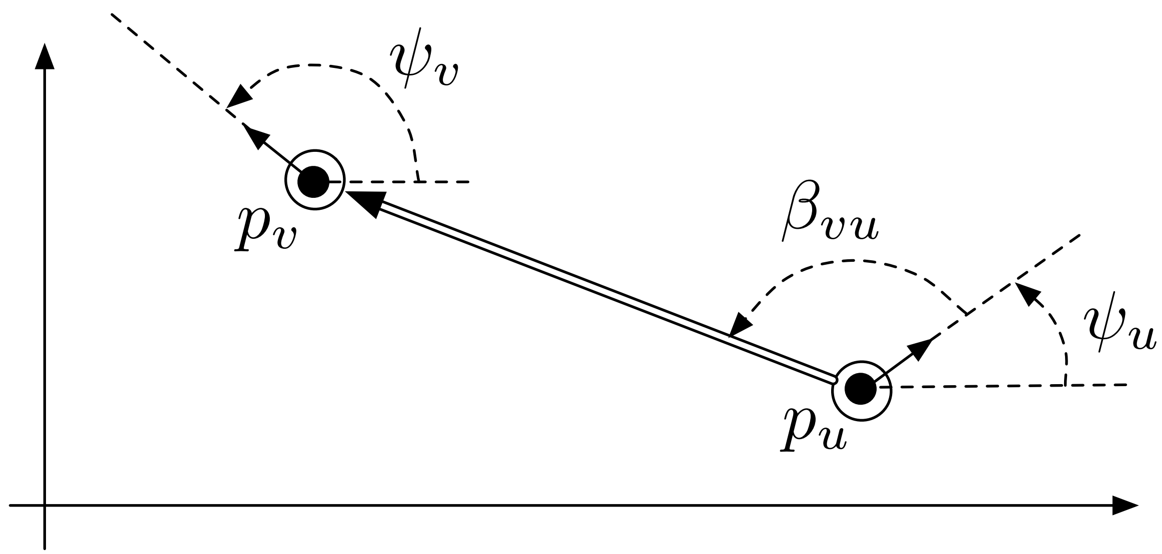


Important Assumptions

- point masses
- bidirectional sensing
- bearing sensing
- *common reference frame is implicit* (i.e., a compass)

Formation Control: Distance-Based Approaches

A more "practical" approach…



- agents represented by points in SE(2) (position and orientation)
- bearing measurements with respect to *body-frame*
- unidirectional sensing

Rigidity Theory in SE(2)

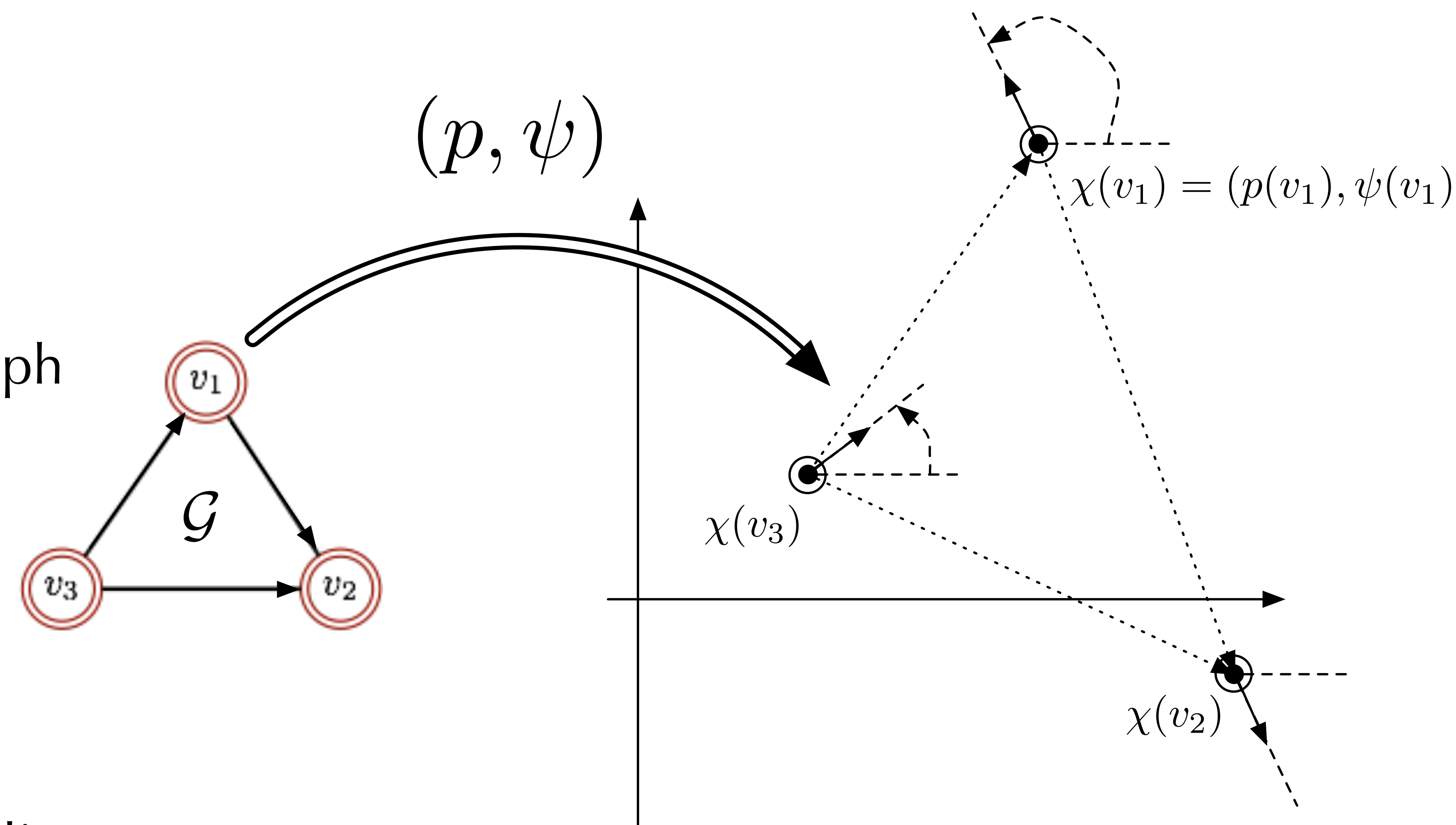
bar-and-joint frameworks in SE(2)

$$(\mathcal{G}, p, \psi)$$

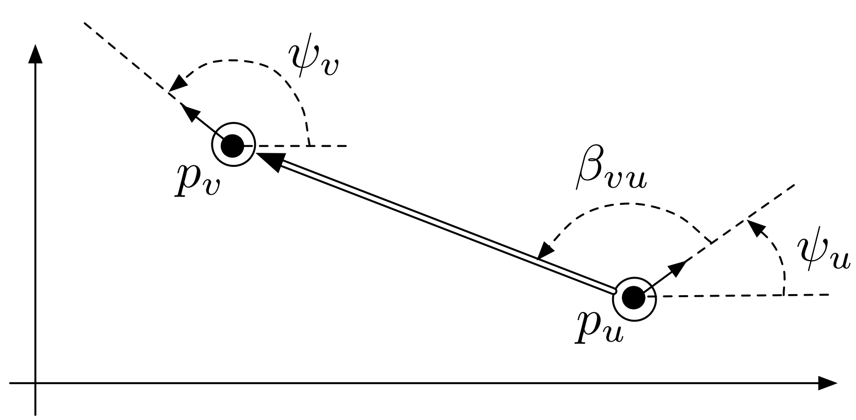
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a *directed* graph

$p : \mathcal{V} \to \mathbb{R}^2$

$\psi : \mathcal{V} \to \mathcal{S}^1$



a directed edge indicates availability of relative bearing measurement



stacked vector of entire framework

$$\chi_p = p(\mathcal{V}) \in \mathbb{R}^{2|\mathcal{V}|}$$

$$\chi_\psi = \psi(\mathcal{V}) \in \mathcal{S}^{1^{|\mathcal{V}|}}$$

Rigidity Theory in SE(2)

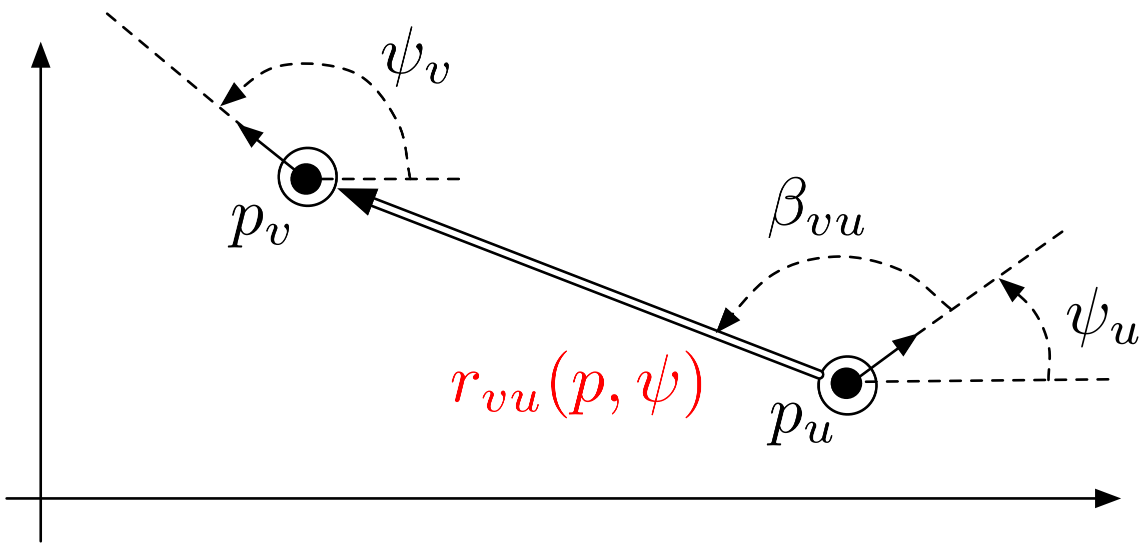
bar-and-joint frameworks in SE(2)

$$(\mathcal{G}, p, \psi)$$

directed bearing rigidity function

$$b_{\mathcal{G}} : SE(2)^{|\mathcal{V}|} \to \mathcal{S}^{1\,|\mathcal{E}|}$$

$$b_{\mathcal{G}}(\chi(\mathcal{V})) = \left[\begin{array}{ccc} \beta_{e_1} & \cdots & \beta_{e_{|\mathcal{E}|}} \end{array}\right]^T$$



bearing can be expressed as a unit vector

$$r_{vu}(p,\psi) = \left[\begin{array}{c} r_{vu}^x \\ r_{vu}^y \end{array}\right] = \left[\begin{array}{c} \cos(\beta_{vu}) \\ \sin(\beta_{vu}) \end{array}\right]$$

$$= \underbrace{\left[\begin{array}{cc} \cos(\psi(v)) & \sin(\psi(v)) \\ -\sin(\psi(v)) & \cos(\psi(v)) \end{array}\right]}_{T(\psi(v))} \frac{(p(u)-p(v))}{\|p(v)-p(u)\|}$$

Rigidity Theory in SE(2)

Definition (Rigidity in SE(2))

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph and $K_{|\mathcal{V}|}$ be the complete directed graph on $|\mathcal{V}|$ nodes. The $SE(2)$ framework $(\mathcal{G}, p, \psi)$ is *rigid* in $SE(2)$ if there exists a neighborhood S of $\chi(\mathcal{V}) \in SE(2)^{|\mathcal{V}|}$ such that

$$b_{K_{|\mathcal{V}|}}^{-1}(b_{K_{|\mathcal{V}|}}(\chi(\mathcal{V}))) \cap S = b_{\mathcal{G}}^{-1}(b_{\mathcal{G}}(\chi(\mathcal{V}))) \cap S,$$

where $b_{K_{|\mathcal{V}|}}^{-1}(b_{K_{|\mathcal{V}|}}(\chi(\mathcal{V}))) \subset SE(2)$ denotes the pre-image of the point $b_{K_{|\mathcal{V}|}}(\chi(\mathcal{V}))$ under the directed bearing rigidity map.

The $SE(2)$ framework $(\mathcal{G}, p, \psi)$ is *roto-flexible* in $SE(2)$ if there exists an analytic path $\eta : [0,\, 1] \to SE(2)^{|\mathcal{V}|}$ such that $\eta(0) = \chi(\mathcal{V})$ and

$$\eta(t) \in b_{\mathcal{G}}^{-1}(b_{\mathcal{G}}(\chi(\mathcal{V}))) - b_{K_{|\mathcal{V}|}}^{-1}(b_{K_{|\mathcal{V}|}}(\chi(\mathcal{V})))$$

for all $t \in (0,\, 1]$.

Rigidity Theory in SE(2)

Definition (Equivalent and Congruent SE(2) Frameworks)

Frameworks $(\mathcal{G}, p, \psi)$ and $(\mathcal{G}, q, \phi)$ are *bearing equivalent* if

$$T(\psi(u))^T \overline{p}_{uv} = T(\phi(u))^T \overline{q}_{uv},$$

for all $(u, v) \in \mathcal{E}$ and are *bearing congruent* if

$$\begin{aligned} T(\psi(u))^T \overline{p}_{uv} &= T(\phi(u))^T \overline{q}_{uv} \text{ and} \\ T(\psi(v))^T \overline{p}_{vu} &= T(\phi(v))^T \overline{q}_{vu}, \end{aligned}$$

for all $u, v \in \mathcal{V}$.

Definition (Global Rigidity of SE(2) Frameworks)

A framework $(\mathcal{G}, p, \psi)$ is *globally rigid* in $SE(2)$ if every framework which is bearing equivalent to $(\mathcal{G}, p, \psi)$ is also bearing congruent to $(\mathcal{G}, p, \psi)$.

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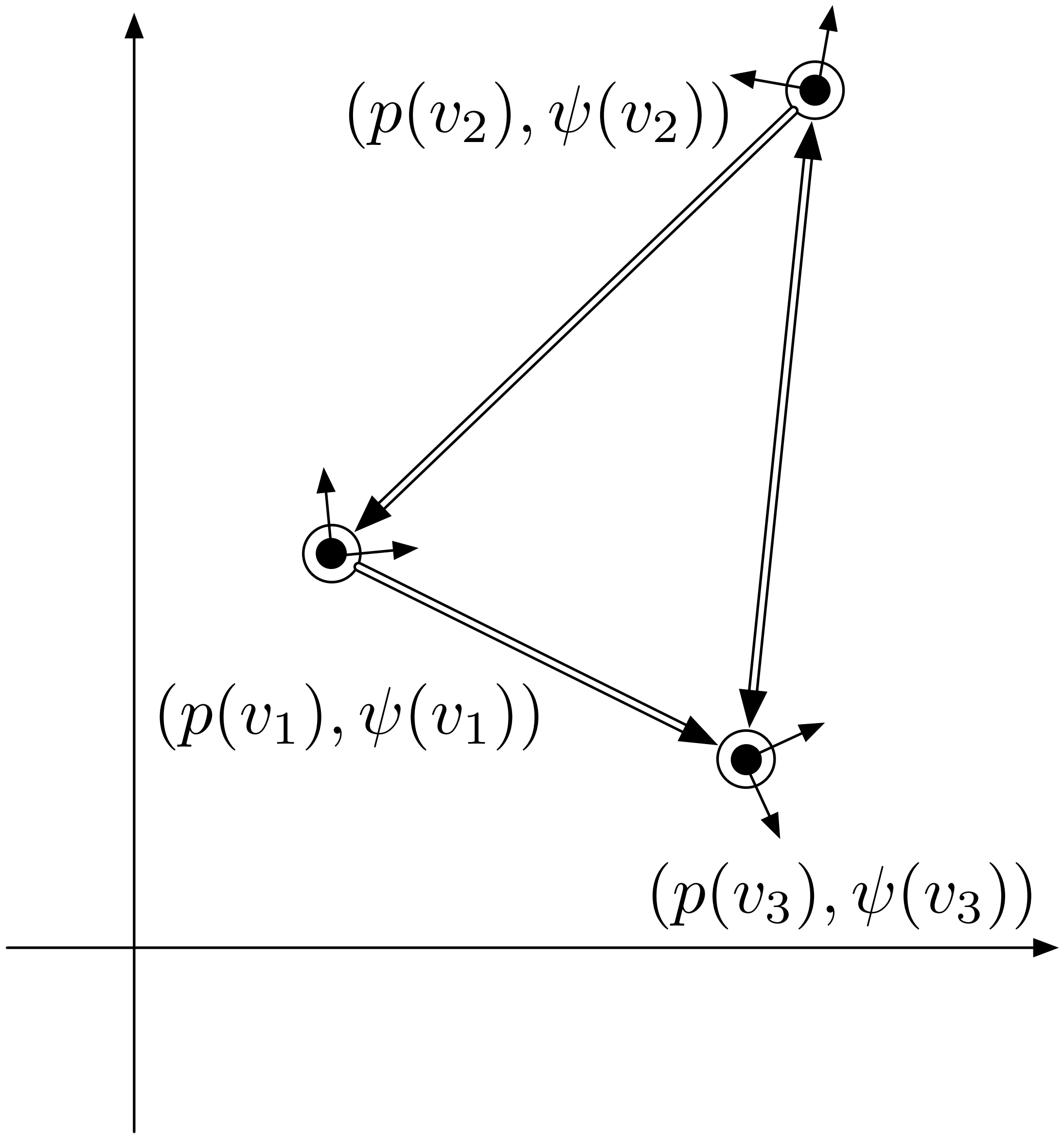
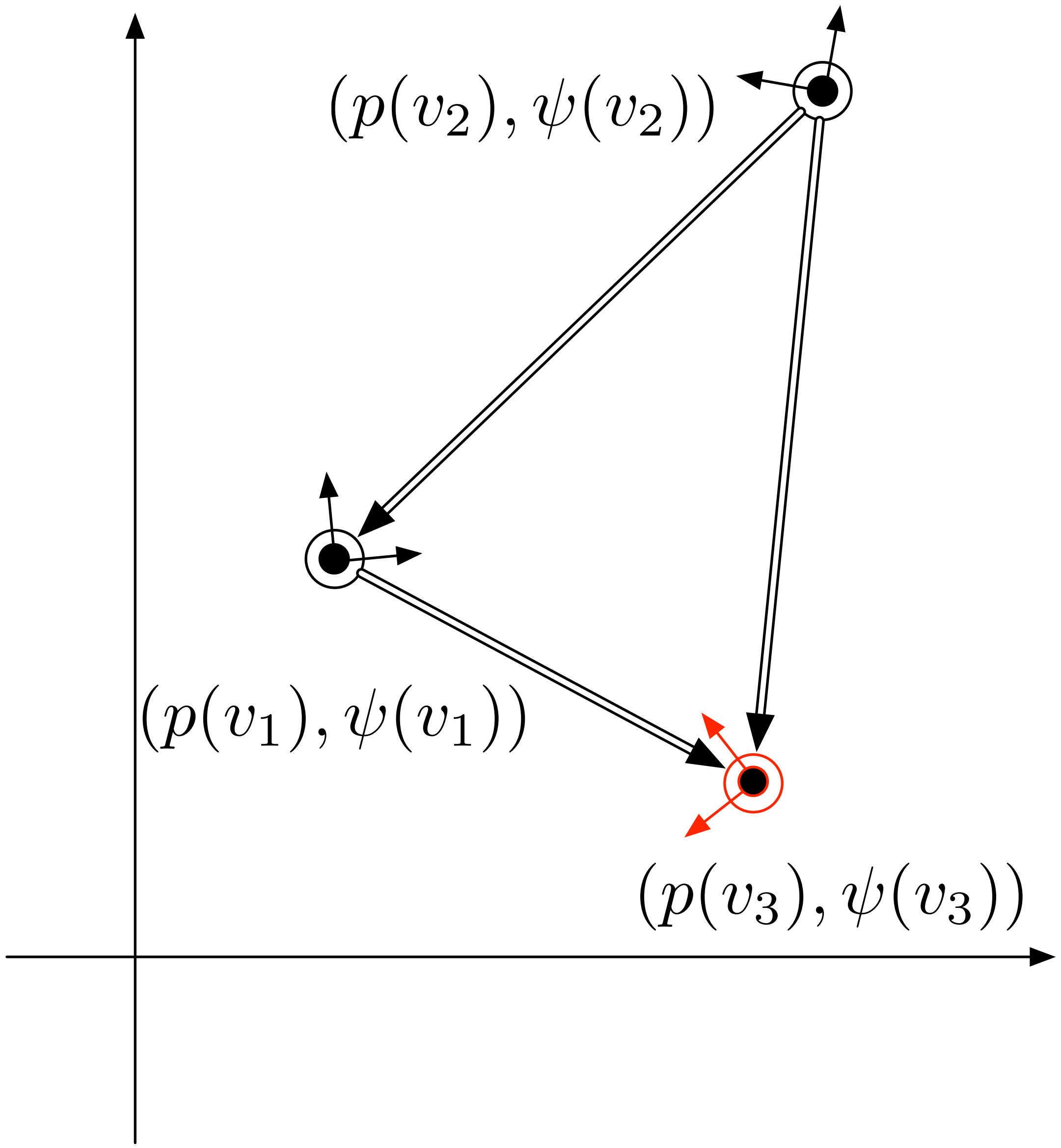
Rigidity Theory in SE(2)





both frameworks are *parallel rigid* (i.e., internal angles are fixed)

agent 3 maintains no bearing angles and is free to "spin" —> framework is *not* globally rigid in SE(2)!

Rigidity Theory in SE(2)

a "linearized" version of bearing rigidity

$$b_{\mathcal{G}}(\chi(\mathcal{V}) + \delta\chi) = b_{\mathcal{G}}(\chi(\mathcal{V})) + (\nabla_{\chi} b_{\mathcal{G}}(\chi(\mathcal{V})))\,\delta\chi + h.o.t.$$

Directed Bearing Rigidity Matrix

$$\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V})) := \nabla_{\chi} b_{\mathcal{G}}(\chi(\mathcal{V})) \in \mathbb{R}^{|\mathcal{E}|\times 3|\mathcal{V}|}$$

Theorem

An $SE(2)$ framework is infinitesimally rigid if and only if

$$\mathbf{rk}[\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V}))] = 3|\mathcal{V}| - 4$$

Rigidity Theory in SE(2)

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$$\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V})) = \left[\begin{array}{cc} D_{\mathcal{G}}^{-1}(\chi_p)R_{\|}(\chi_p) & \overline{E}(\mathcal{G})^T \end{array}\right]$$

$$D_{\mathcal{G}}(\chi_p) = \mathbf{diag}\{\ldots, \|p(u) - p(v)\|^2, \ldots\}$$

$$[\overline{E}(\mathcal{G})]_{ik} = \left\{\begin{array}{ll} 1, & \text{if } e_k = (v_i, v_j) \in \mathcal{E} \\ 0, & \text{o.w.} \end{array}\right.$$

Infinitesimal Motions in SE(2)

recall…

Distance Rigidity

- maintain distance pairs
- rigid body rotations and translations

$$R(p)\xi = 0$$

Parallel Rigidity

- maintain angles (shape)
- rigid body translations and dilations

$$R_{||}(p)\xi = 0$$

Theorem

Every infinitesimal motion $\delta\chi \in \mathcal{N}\left[\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V}))\right]$ satisfies

$$R_{||}(\chi_p)\delta\chi_p = -D_{\mathcal{G}}(\chi_p)\overline{E}^T(\mathcal{G})\delta\chi_\psi$$

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What are the infinitesimal motions in SE(2)?

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Infinitesimal Motions in SE(2)

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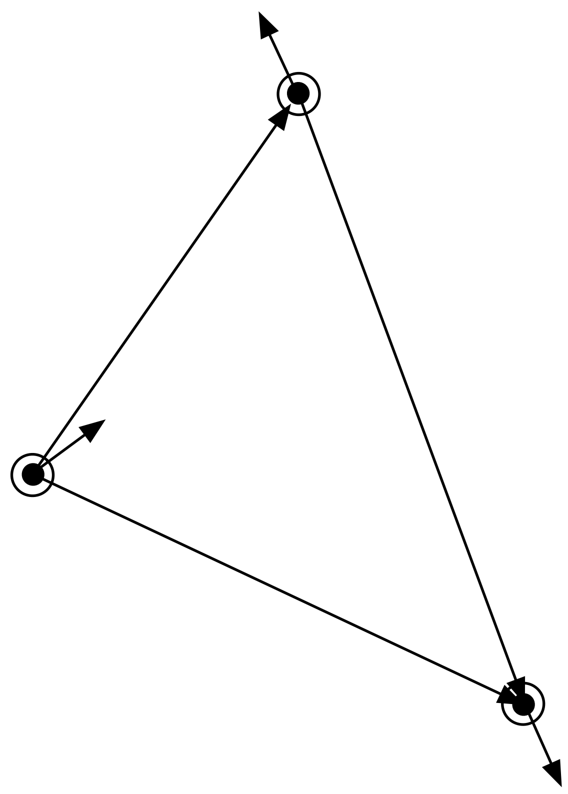
if all agents maintain attitude, infinitesimal motions are the ***translations*** and ***dilations*** of the framework



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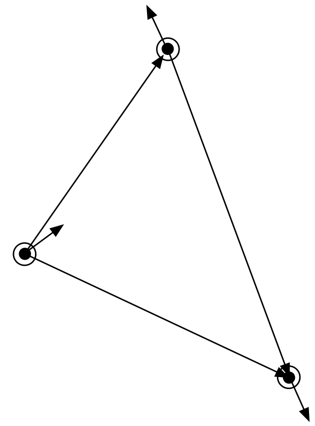

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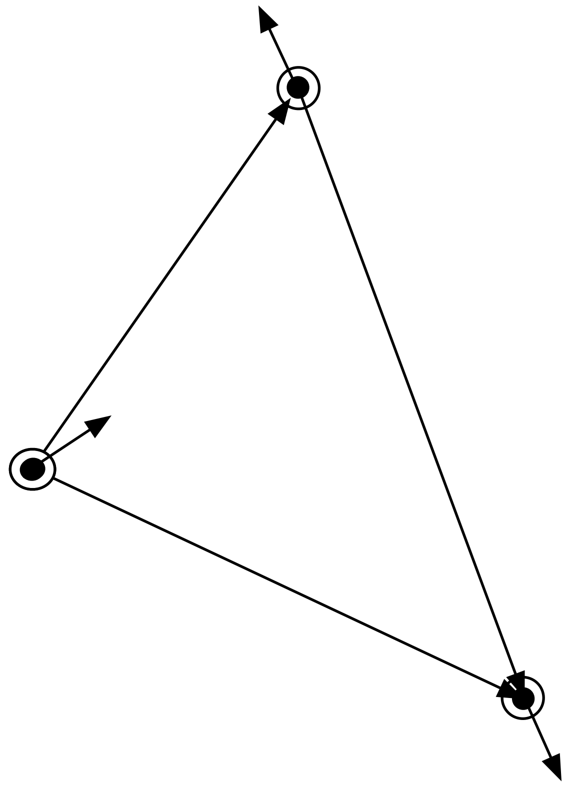

coordinated rotation subspace

$$\mathcal{R}_{\circlearrowleft}(\mathcal{G}) = \mathrm{IM}\left\{R_{\|,\mathcal{G}}(\chi_p)\right\} \cap \mathrm{IM}\left\{-D_{\mathcal{G}}(\chi_p)\overline{E}^T(\mathcal{G})\right\}$$

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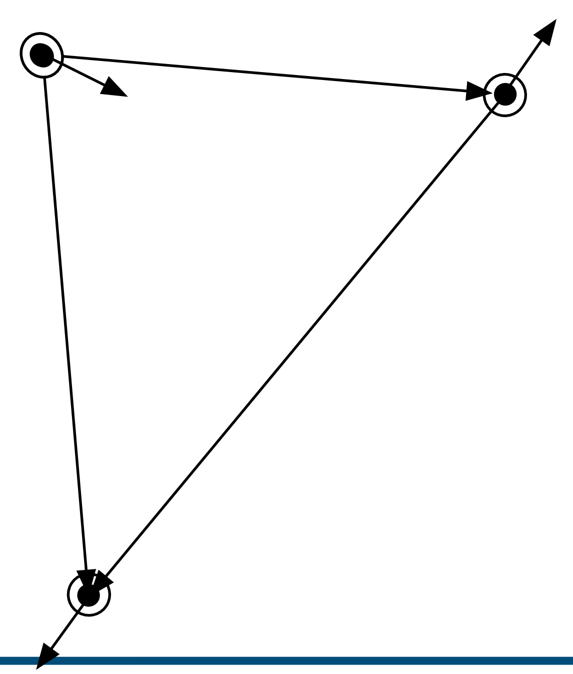

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Infinitesimal Motions in SE(2)

Proposition

The coordinated rotation subspace is non-trivial.

$$\dim \mathcal{R}_{\circlearrowright}(\mathcal{G}) \geq 1$$

For the complete directed graph, one has

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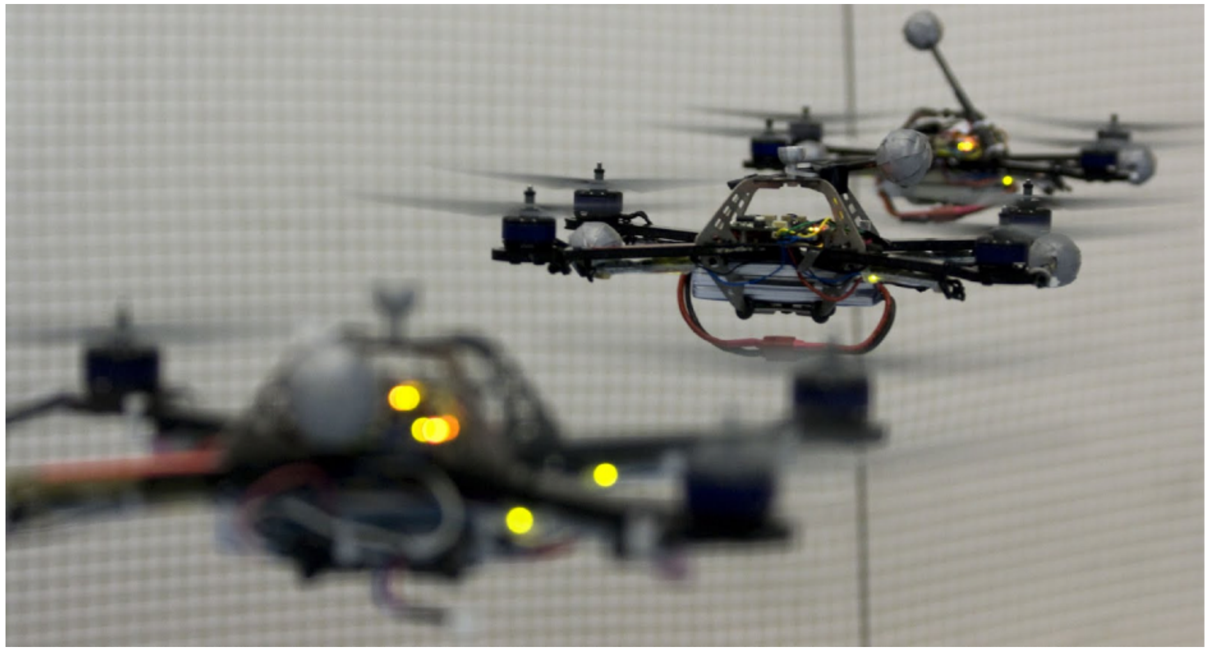
Corollary

An $SE(2)$ framework is infinitesimally rigid in $SE(2)$ if and only if

1. $\mathrm{rk}[R_{||,\mathcal{G}}(\chi_p)] = 2|\mathcal{V}| - 3$ and
2. $\dim\{\mathcal{R}_{\circlearrowright}(\mathcal{G})\} = 1.$

Estimation of Relative Positions



high level coordination objectives (formation keeping, localization, sensor fusion) require robots to know the transformation between local body frames - **relative positions** and **relative orientation**

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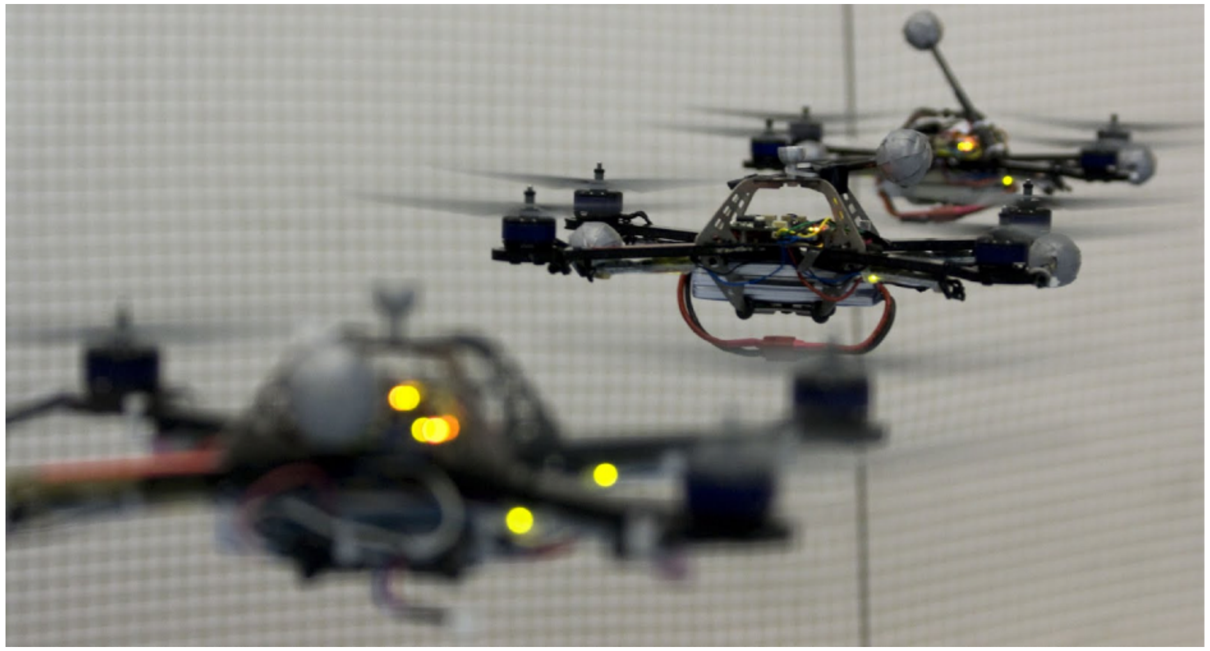

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A distributed gradient descent estimator

Bearing Error:

$$e(\hat{\xi}, \hat{\vartheta}, p, \psi) = b_{\mathcal{G}}(\chi(\mathcal{V})) - \hat{b}_{\mathcal{G}}(\hat{\xi}, \hat{\vartheta})$$

Cost Function:

$$J(e) = \frac{1}{2}\left(k_e\|e(\hat{\xi}, \hat{\vartheta}, p, \psi)\|^2 + k_1\|\hat{\xi}_{\iota\iota}\|^2 + k_2(\|\hat{\xi}_{\iota\kappa}\|^2 - 1)^2 + k_3(1 - \cos\hat{\vartheta}(\iota))\right)$$

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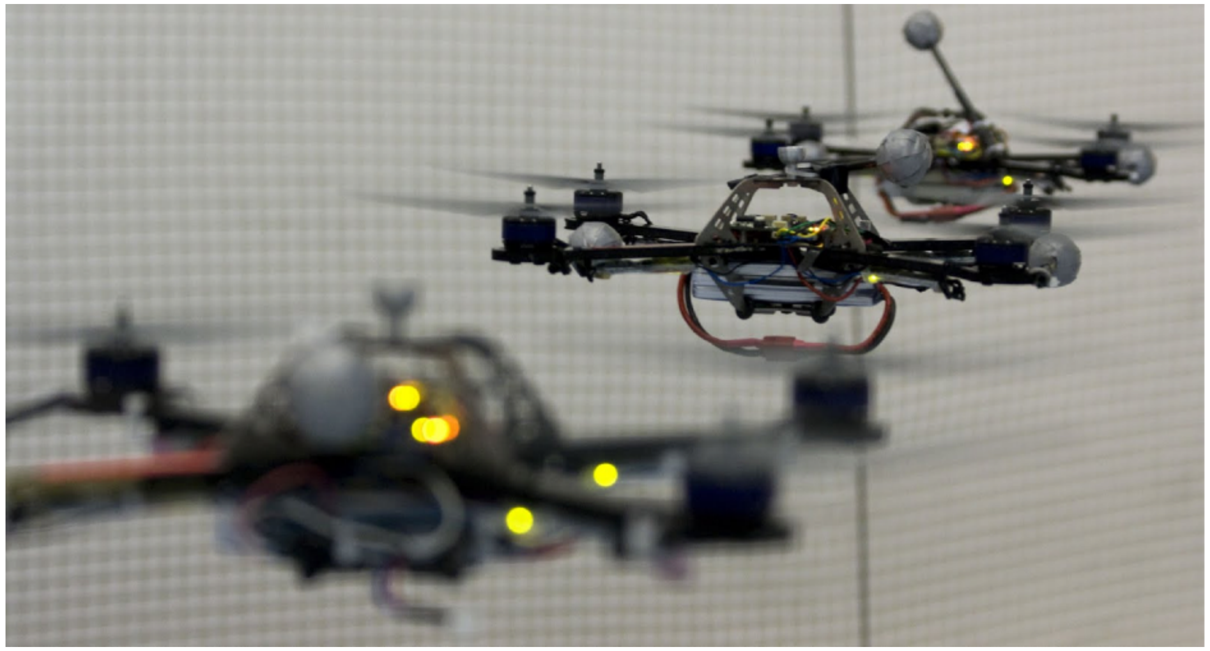

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"unscaled"

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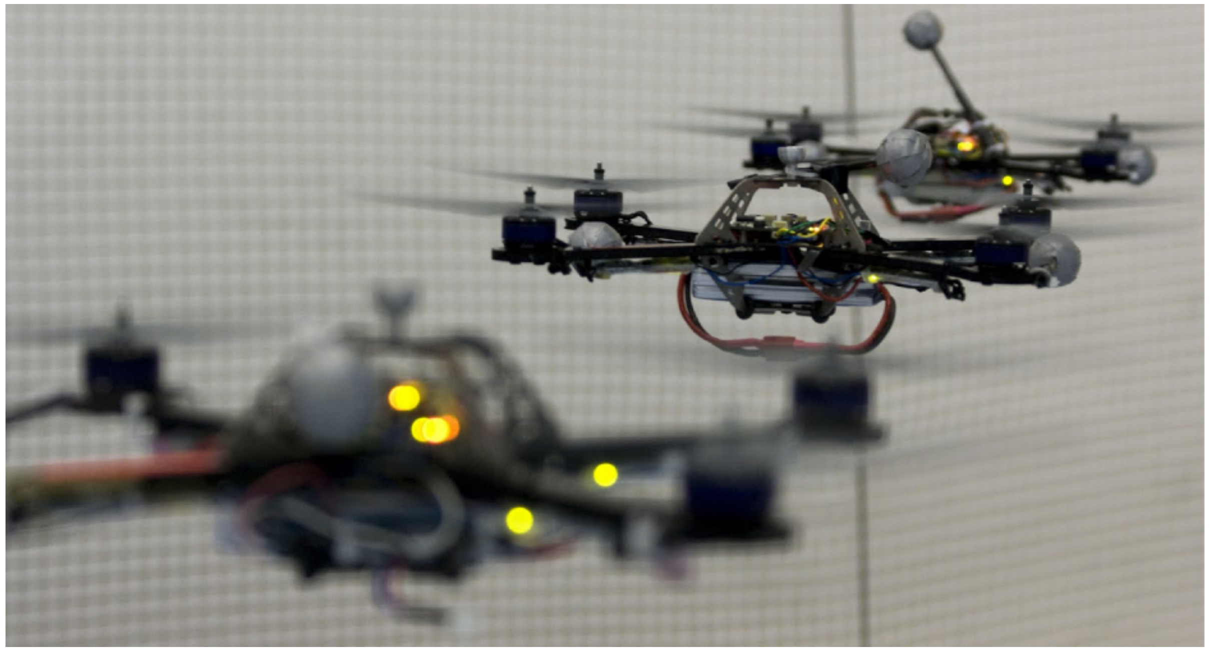

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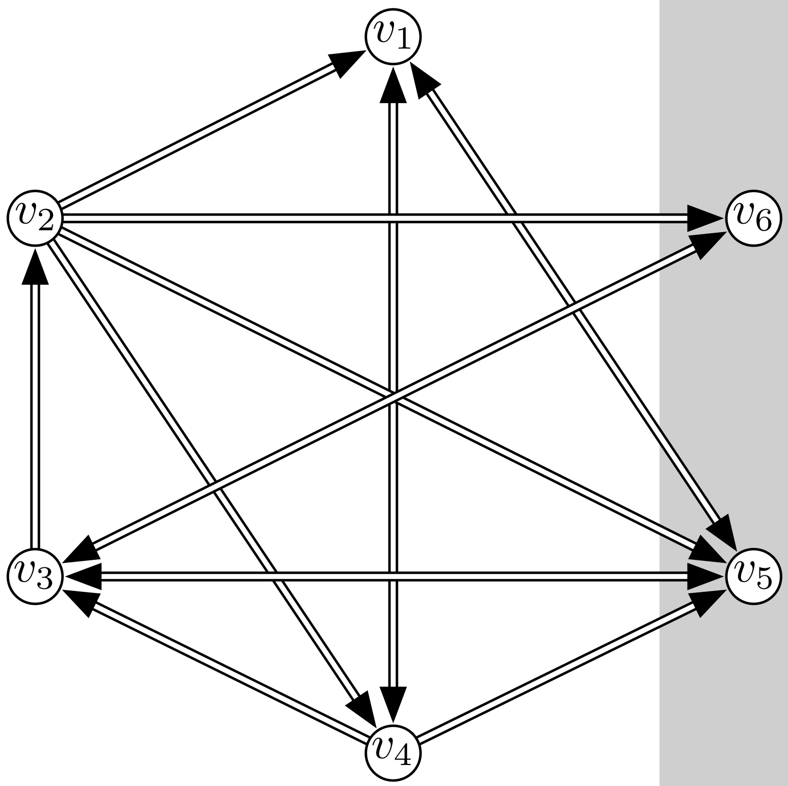
Theorem

If the framework is infinitesimally rigid in SE(2) then the estimator

$$\begin{bmatrix} \dot{\hat{\chi}} \\ \dot{\hat{\vartheta}} \end{bmatrix} = -\nabla J(e)$$

converges to a local minimum of the bearing error function.

Estimation of Relative Positions



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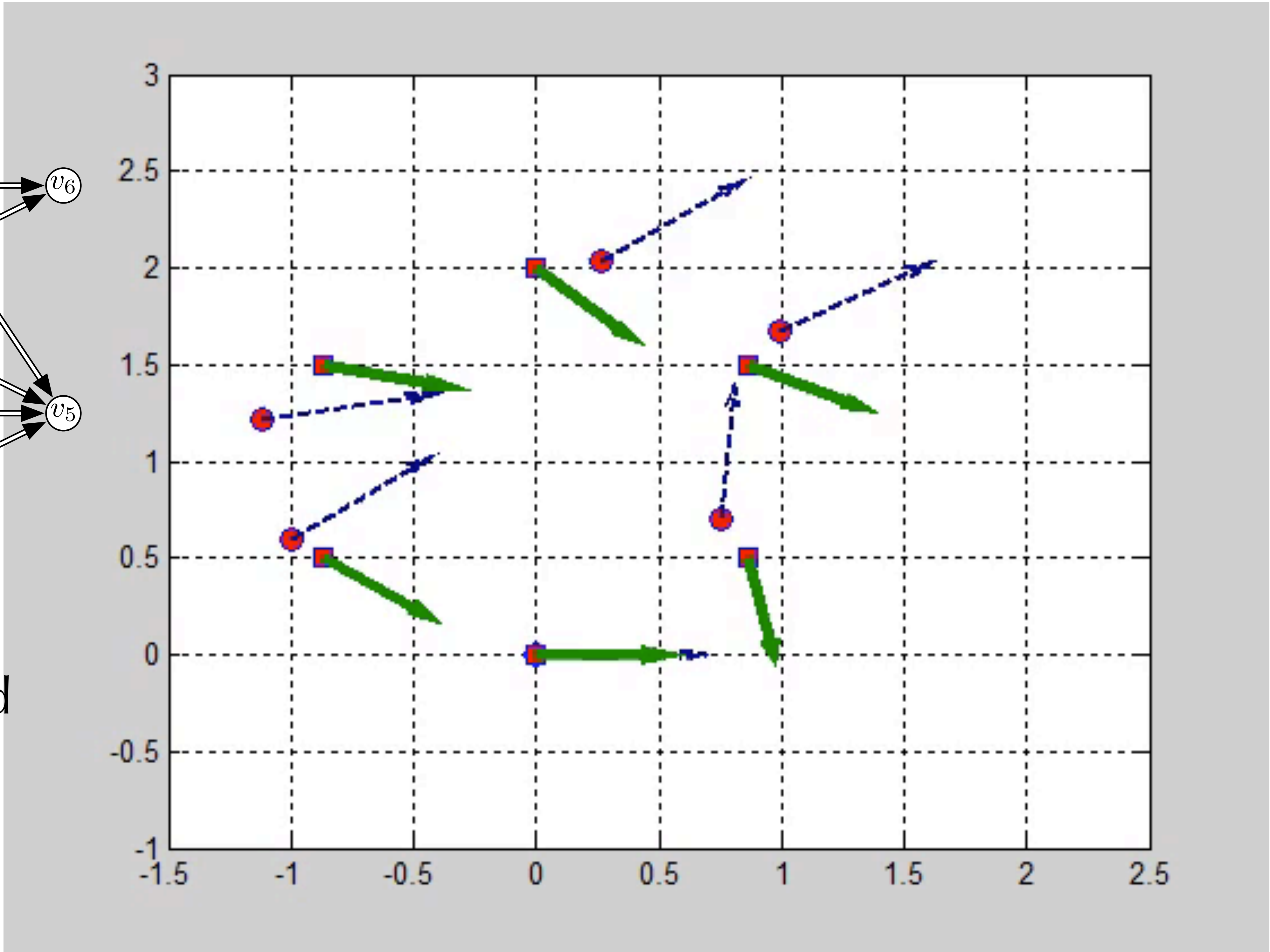

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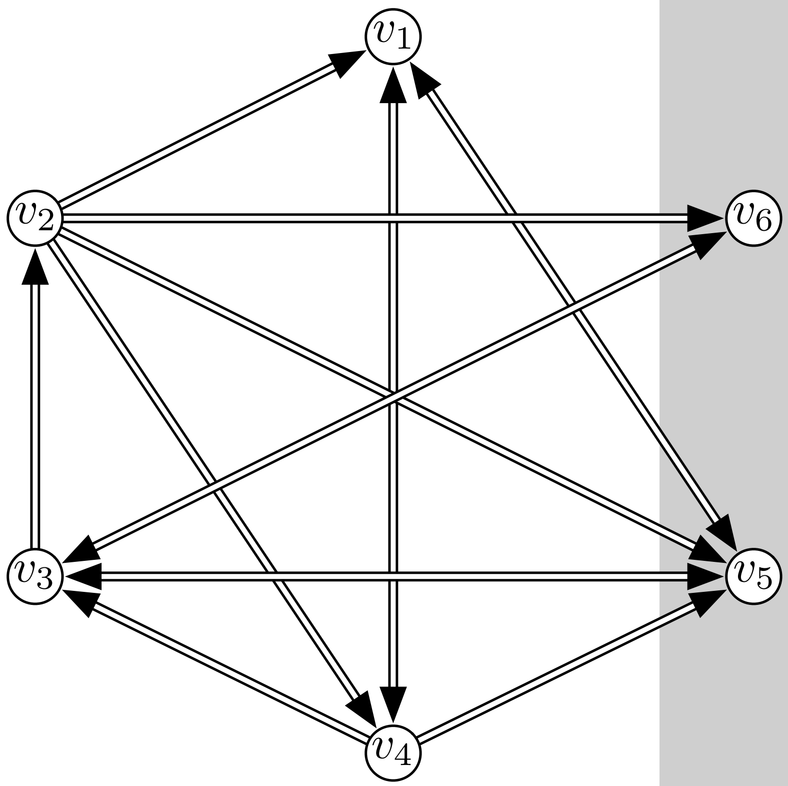

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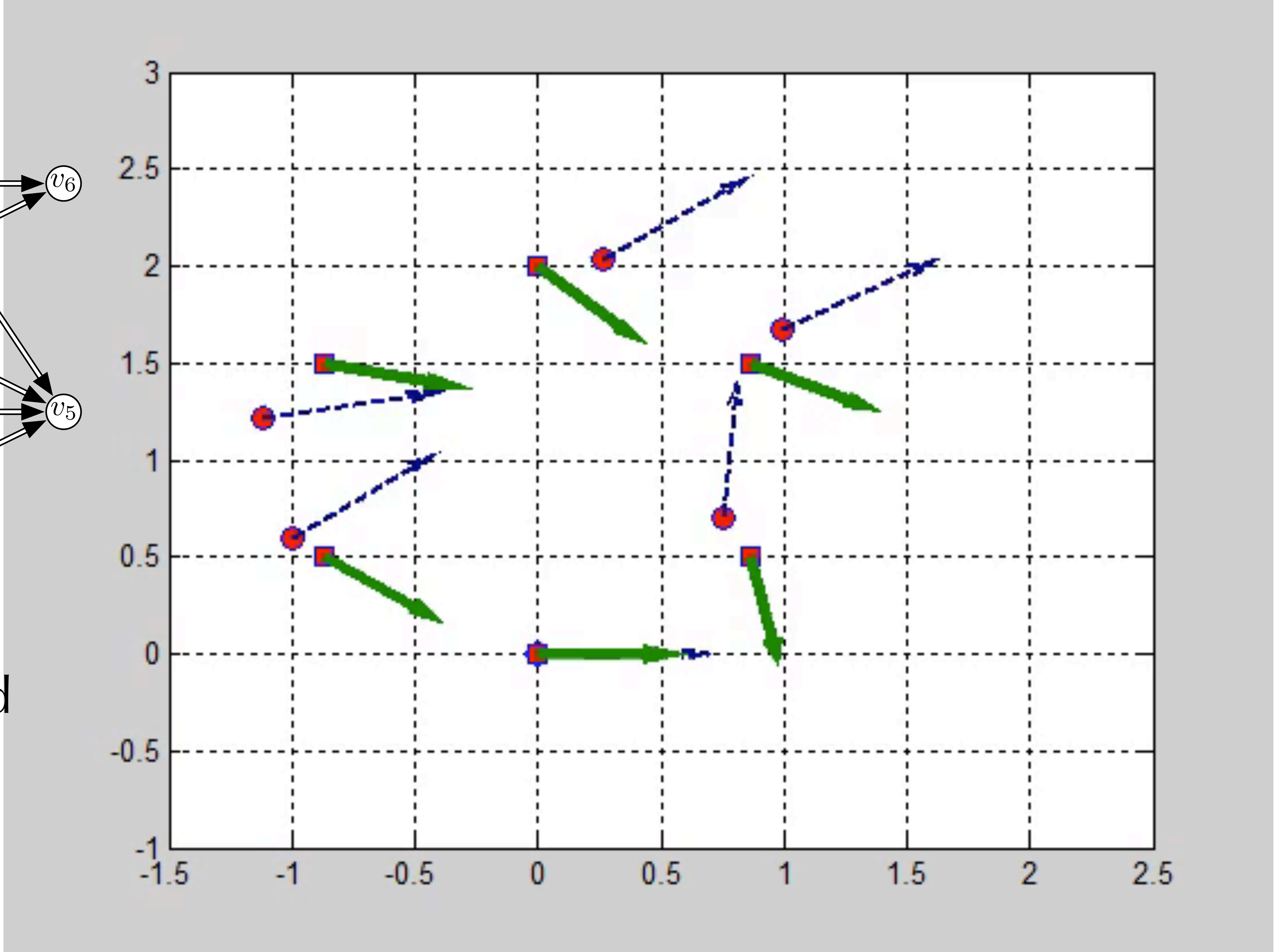

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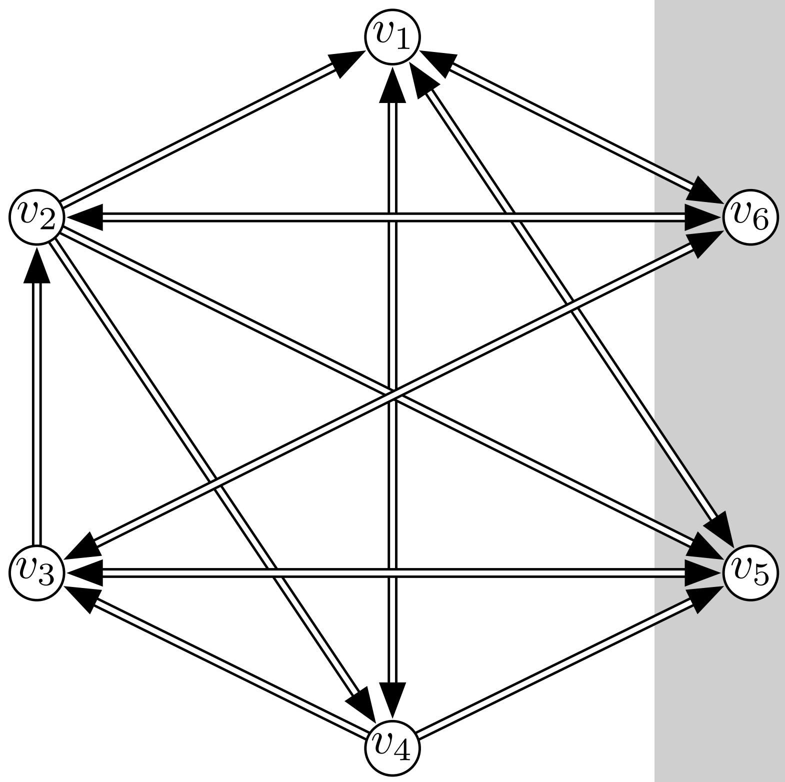

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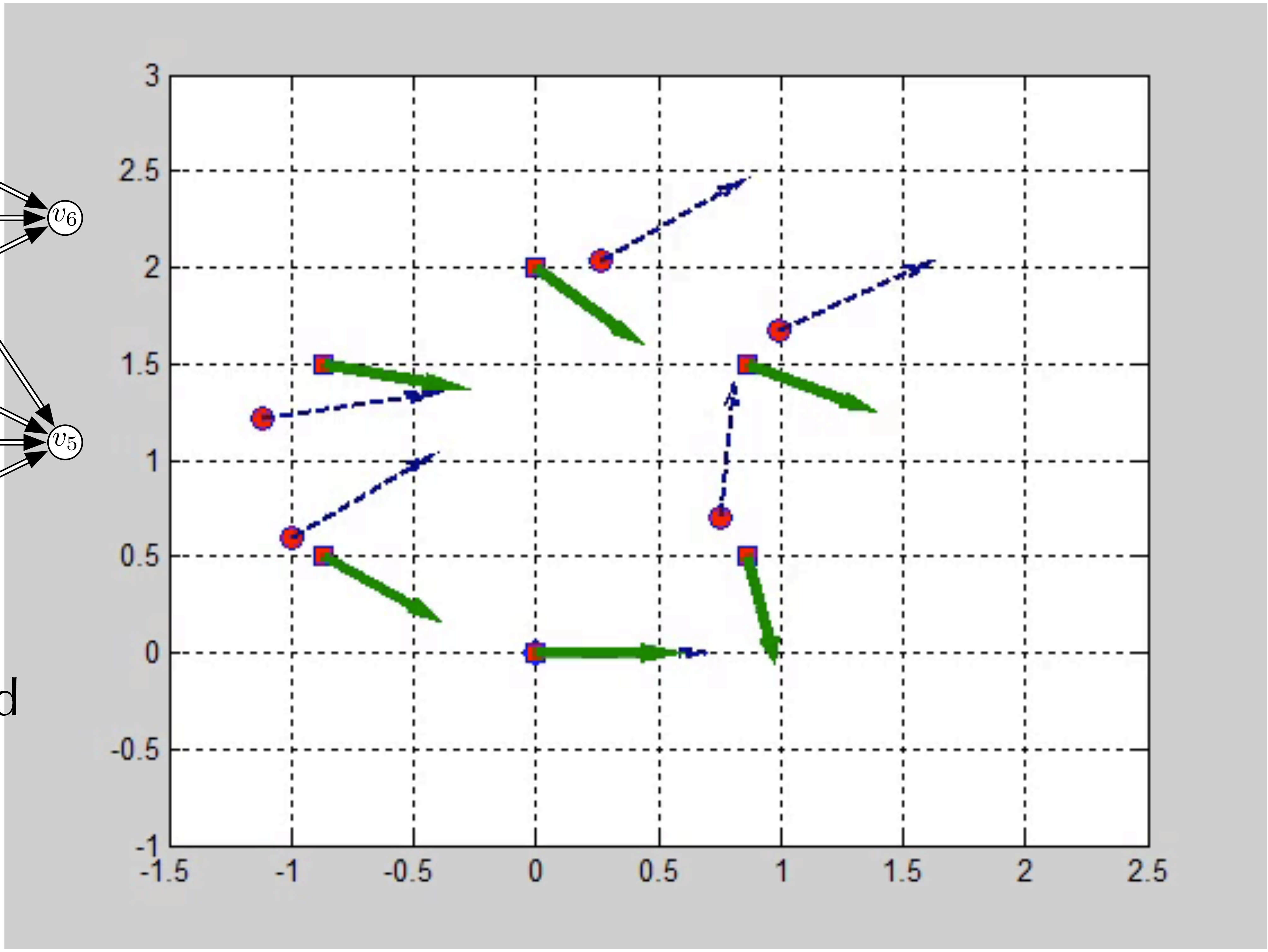

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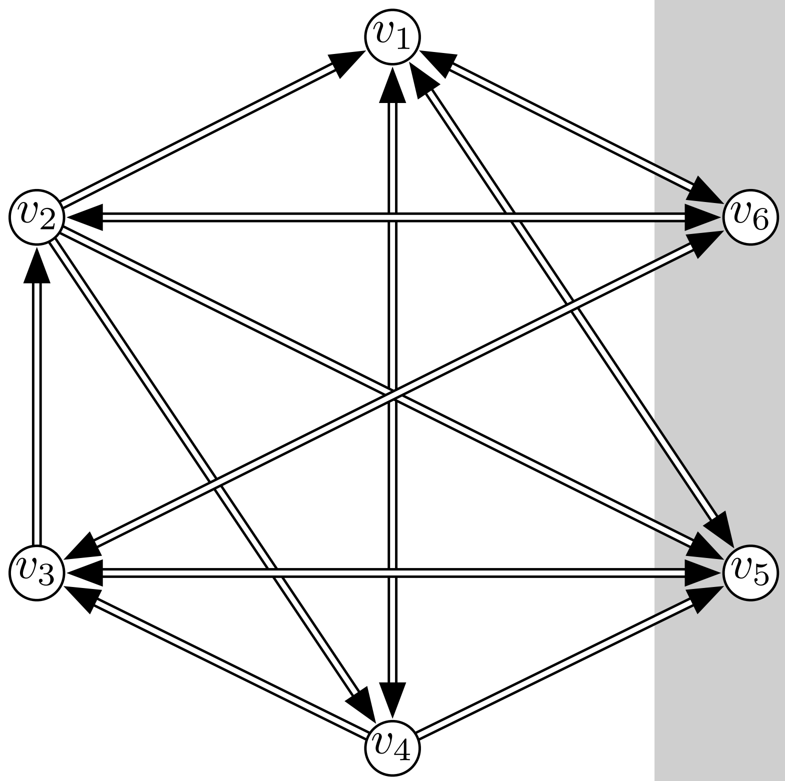

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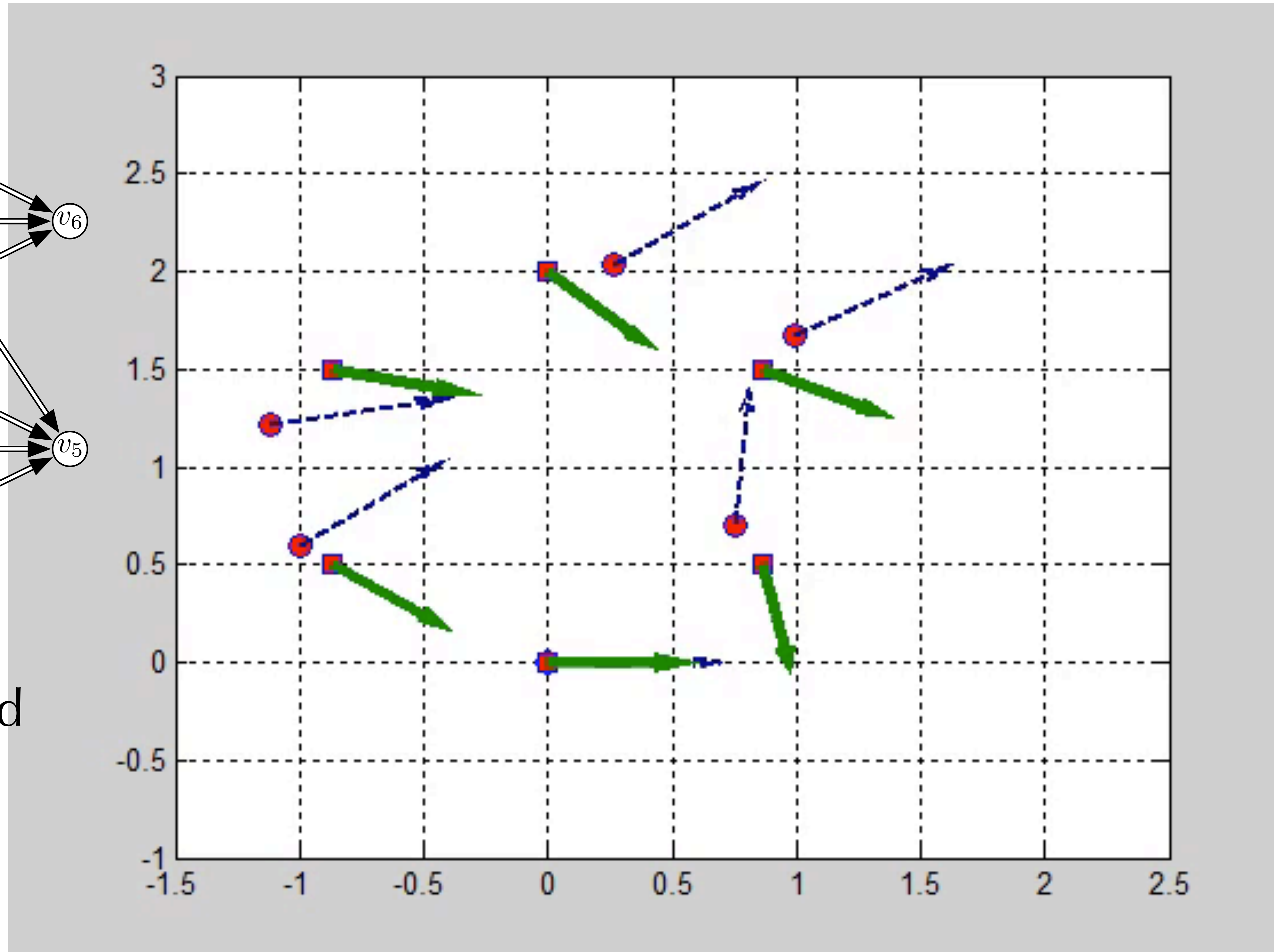

Conclusions and Outlook

- coordination methods for multi-agent systems depend on sensing and communication mediums
- systems with *bearing* only sensing is a practical solution for many multi-agent systems

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- parallel rigidity in arbitrary dimension
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- extension of rigidity to concepts to frameworks in SE(2)
- SE(2) rigidity used to distributedly estimate relative positions from only bearing measurements

Conclusions and Outlook

- deeper results for bearing rigidity
- extensions to SE(3)
- estimation filter combined with higher-level tasks (formation keeping)
- control and estimation with field-of-view constraints

Acknowledgements

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Questions?