GRAPH THEORY IN SYSTEMS AND CONTROL

A TUTORIAL

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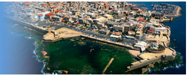
27th Mediterranean Conference on Control and Automation

1 – 4 JULY 2019 AKKO, ISRAEL



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TOPICS OF INTEREST

Adaptive Control	Guidance	Optimal Control
Aerospace Control	Health Related Topics	Optimization
Automotive Control	Human Machine Interaction	Power Systems
Autonomous Systems	Hybrid Systems	Predictive Control
Biologically Inspired Systems	Industrial Automation	Process Control
Cognitive Systems	Internet of Things	Prognostics & Diagnostics
Control Theory	Intelligent Transportation Systems	Real-Time Control
Communication Systems	Linear Systems	Robotics Applications
Cyber-Physical systems	Manufacturing	Robust Control
Deep Learning in Control	Multi-Agent Systems	Sampled-Data Control
Discrete Event Systems	Marine Control	Smart Grids & Smart Cities
Economic Control	Mechatronic Systems	Societal Challenges
Education & Training	Modeling & Simulation	Soft Computing
Embedded Control Systems	Navigation	System Identification
Fault-Tolerant Control	Network Controlled Systems	Uncertain Systems
Fuzzy Cognitive Maps	Nonlinear Control	Virtual/Augmented Reality

IMPORTANT DATES

22 January 2019: Invited sessions, tutorial proposals, contributed papers due.

18 April 2019: Notification of acceptance/rejection.

15 May 2019: Final submissions due.

SUBMISSION DETAILS

PAPERS: Papers must be submitted electronically by January 22, 2019. The paper format must follow IEEE paper submission rules: 2 column layout, 10 pt. Times New Roman font. The maximum number of pages per paper is 6. Additional pages, up to a maximum 2, will be charged. All papers will be peer reviewed and acceptance notified by April 18, 2019. Accepted papers are to be uploaded electronically by May 15, 2019. All submissions are via **https://controls.papercept.net**.

INVITED SESSIONS: A statement describing the motivation and relevance of the proposed session, invited paper titles and author names must be submitted electronically for review by January 22, 2019. Authors must submit electronically full versions of the invited papers marked as "Invited Session Paper".

TUTORIALS/WORKSHOPS: Proposals for tutorials or workshops should contain the title of the session, the list of speakers, and extended summaries (2000 words) of the presentations. Proposals for review must be sent to the Tutorial and Workshop Chair by e-mail (giuseppe.notarstefano@unibo.it) by January 22, 2019.

For information visit the website: **https://med19.technion.ac.il** or write to: **medcon19@gmail.com**

MCA CSS IEEE IEEE Robotics & Automation Society TECHNION Israel Institute of Technology IAAC Israeli Association for Automatic Control

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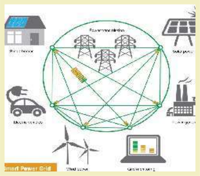

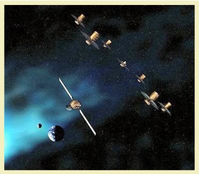

Networks of dynamical systems are one of the enabling technologies of the future.

SOME IMMEDIATE OBSERVATIONS

- networked systems are coupled through information exchange
- inter-agent information exchange is through sensing and communication
- the collective dynamics is a function of ”agent” dynamics and the information-induced coupling
- we can synthesize collective behavior by making the control action on each agent a function of the information available to the agent (sense, communicated, etc.)

a powerful abstraction for encoding “interactions” in a network is that of a graph

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Introduction to Networked Dynamic Systems

Basic Graph Theory

Protocols on Graphs

Structural Stability of Linear Time-Invariant Systems

Graphs and Input-Output Properties of Network Systems

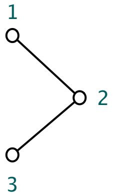
Unexplored Opportunities

- a finite, undirected, simple graph, or a graph for short, is built upon a finite set of nodes, or the vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$
- the edge set is a subset of the two-element subsets of $\mathcal{V}$, i.e., $\mathcal{E} \subseteq [\mathcal{V}]^2$
- the graph is then specified by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

for example, we can have $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where

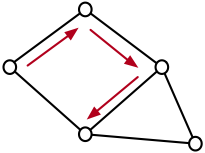
$$\mathcal{V} = \{1, 2, 3\} \quad \text{and} \quad \mathcal{E} = \{\{1, 2\}, \{2, 3\}\}$$

a simpler representation however would be

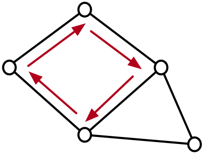


Some natural constructs based on the correspondence between set theoretic and graph-theoretic representation can now be defined – examples: paths, walks, cycles, etc.

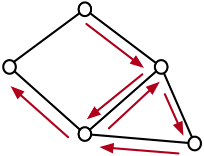
SIMPLE CONSTRUCTS ON GRAPHS



a path



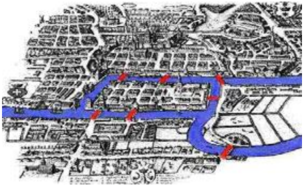
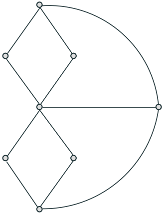
a cycle



a walk

graphs can be used in general to encode relations between objects, e.g., existence of communication or sensing links, routes, etc.

bridges of Konigsberg and Euler's abstraction:

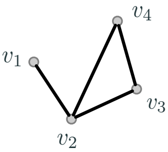
Abstract away all particular details related to the Konigsberg bridges that are not relevant to the problem! This leads to a graph! We want to find out if there is a closed walk traversing all edges of the graph exactly once - a Eulerian Graph.

Theorem

A connected graph $\mathcal{G}$ is Eulerian if and if only every vertex has an even degree.

GRAPHS AND MATRICES

As we aim to embed graph/networks in dynamic systems, it is natural to work with linear algebraic representation. For example, a graph can be represented as,



$$A(\mathcal{G}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

the adjacency matrix for the n-node graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the $n \times n$ matrix:

$$[A(\mathcal{G})]_{ij} = \begin{cases} 1 & \text{if } \; v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

note that the adjacency for the graph is symmetric by construction

there are other matrices associated with the graph, for example, let $d(v)$ be the number of neighbors of vertex v (its degree) and define the degree matrix as,

$$\Delta(\mathcal{G}) = \begin{pmatrix} d(v_1) & 0 & \cdots & 0 \\ 0 & d(v_2) & \cdots & 0 \\ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d(v_n) \end{pmatrix}$$

note that the adjacency and the degree matrices are both square, say, $n \times n$, where n is the number of nodes

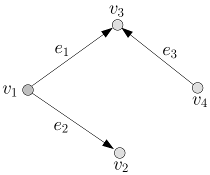
Another useful matrix representation is the Laplacian:

$$L(\mathcal{G}) = \Delta(\mathcal{G}) - A(\mathcal{G})$$

graph Laplacian has been very popular in multiagent networks!

Yet another matrix representation can in fact capture the orientation of the edge as well: suppose the graph has n nodes and m edges: the $n \times m$ *incidence matrix* $E(\mathcal{G})$ is defined as

$$E(\mathcal{G}) = [E_{ij}], \text{ where } E_{ij} = \begin{cases} -1 \text{ if } v_i \text{ is the tail of } e_j, \\ 1 \text{ if } v_i \text{ is the head of } e_j, \\ 0 \text{ otherwise.} \end{cases}$$



$$E(\mathcal{G}) = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

note that for different orientations on the edges we get a different incidence matrix (of same dimension)!

Let us see what happens when we consider $E(\mathcal{G})E(\mathcal{G})^T$ for some arbitrary orientation. First notice that the resulting matrix will be $n \times n$.

A compact formula for matrix multiplication is of course:

$$[AB]_{ij} = \sum_k A_{ik} B_{kj}$$

$$[E(\mathcal{G})E(\mathcal{G})^T]_{ij} = \sum_k E(\mathcal{G})_{ik} E(\mathcal{G})_{jk}$$

which is -1 when i and j are incident on the same edge k, that is if they are neighbors! Moreover,

$$[E(\mathcal{G})E(\mathcal{G})^T]_{ii} = \sum_k E(\mathcal{G})_{ik} E(\mathcal{G})_{ik}$$

counts the number of edges incident on node i, i.e., its degree! Therefore,

$$L(\mathcal{G}) = E(\mathcal{G})E(\mathcal{G})^T$$

is independent of the orientation!

This also shows that $L(\mathcal{G})$ is positive semi-definite, since for all $x \in \mathbf{R}^n$:

$$x^T L(\mathcal{G})x = x^T E(\mathcal{G})E(\mathcal{G})^T x = \|E(\mathcal{G})^T x\|^2 \geq 0$$

i.e., the eigenvalues of the Laplacian are real numbers (as the Laplacian is symmetric) and non-negative. We can order the eigenvalues as follows,

$$0 \leq \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \ldots \lambda_n(\mathcal{G});$$

in this case, λ_k refers to the kth smallest eigenvalue of the (graph) Laplacian ...

- By construction, $L(\mathcal{G})\mathbf{1} = 0$ for any graph (why?). So $\lambda_1(\mathcal{G}) = 0$.
- A natural question (with many consequences) is whether $\lambda_2(\mathcal{G}) > 0$?

NULL SPACE OF THE LAPLACIAN

We need to characterize the null space of $L(\mathcal{G})$:

$$\mathcal{N}(L(\mathcal{G})) = \{z \in \mathbf{R}^n \mid L(\mathcal{G})z = 0\}$$

In order to answer this question, notice that if $z \in \mathcal{N}(L(\mathcal{G}))$, then

$$L(\mathcal{G})z = E(\mathcal{G})E(\mathcal{G})^T z = 0$$

that is,

$$z^T E(\mathcal{G})E(\mathcal{G})^T z = 0$$

or $\|E(\mathcal{G})^T z\|^2 = 0$ or $E(\mathcal{G})^T z = 0$ or $z^T E(\mathcal{G}) = 0$. This means that if $ij \in E$, then $z_i = z_j$; so if the graph is connected,

$$z_1 = z_2 = \cdots = z_n$$

that is $z = \alpha\mathbf{1}$ for some α! And in fact, if we think of z as

$$z : \mathcal{V}(\mathcal{G}) \to \mathbf{R}^n$$

then z is constant on each (connected) component of $\mathcal{G}$. For each component we get one extra dimension for the null space of $L(\mathcal{G})$.

RANK, λ_2, AND CONNECTIVITY

Lemma

Let $\mathcal{G}$ have c connected components (when $c = 1$ the graph is connected). Then **rank** $L(\mathcal{G})$ is $n - c$.

and in fact, **rank** $L(\mathcal{G}) = n - 1$ if and only if $\mathcal{G}$ is connected! this is our first encounter with how the “linear algebra” of the Laplacian tells us something about the structure of the graph.

Corollary

$\mathcal{G}$ is connected if and only if $\lambda_2(\mathcal{G}) > 0$

a natural question now is whether a more positive λ_2 captures some qualitative notion of ”more” connectivity?

For example, we can define the node connectivity of $\mathcal{G}$, denoted by $\kappa_0(\mathcal{G})$ as the minimum number of nodes that needs to be removed from the graph before the graph becomes disconnected.

Courant-Fisher

$$\lambda_2(\mathcal{G}) = \min_{x \perp \mathbf{1}, \|x\|=1} x^\top L(\mathcal{G}) x$$

So this means that

$$\lambda_2(\mathcal{G}) \leq x^\top L(\mathcal{G}) x \quad \text{for all } x \perp \mathbf{1}, \|x\| = 1$$

Let us consider removing $S \subset \mathcal{V}$ (subset of nodes) from the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$; we denote the Laplacian of this new graph as $L(\mathcal{G} \backslash S)$. Let y be the normalized eigenvector corresponding to $\lambda_2(\mathcal{G} \backslash S)$:

$$L(\mathcal{G} \backslash S) y = \lambda_2(\mathcal{G} \backslash S) y; \quad \|y\| = 1, y \perp \mathbf{1}$$

Now define the vector

$$z = \begin{bmatrix} y \\ 0 \end{bmatrix};$$

note that $\|z\| = 1$ and $z \perp 1$; as such $\lambda_2(\mathcal{G}) \leq z^\top L(\mathcal{G})z$. That is,

$$\lambda_2(\mathcal{G}) \leq \sum_{uv \in E(\mathcal{G} \backslash S)} (y_u - y_v)^2 + \underbrace{\sum_{uv \in E(S)} (z_u - z_v)^2}_{0} + \sum_{u \in S} \sum_{v \in \mathcal{G} \backslash S} (\underbrace{z_u}_{0} - z_v)^2$$

so,

$$\lambda_2(\mathcal{G}) \leq \lambda_2(\mathcal{G} \backslash S) + \sum_{u \in S} 1 = \lambda_2(\mathcal{G} \backslash S) + |S|$$

Okay! Now suppose that S is chosen as the cutset corresponding to $\kappa_0(\mathcal{G})$. Then $\lambda_2(\mathcal{G} \backslash S) = 0$ and

$$\lambda_2(\mathcal{G}) \leq \kappa_0(\mathcal{G})$$

Upshot: $\lambda_2(\mathcal{G})$ is a lower bound for node connectivity!

The bound is actually tight, for example $\lambda_2(C_4) = \kappa_0(C_4) = 2$

summary so far:

- $L(\mathcal{G}) = E(\mathcal{G})E(\mathcal{G})^\top = \Delta(\mathcal{G}) - A(\mathcal{G})$
- $L(\mathcal{G})$ is positive semidefinite
- $\lambda_2(\mathcal{G}) > 0$ iff $\mathcal{G}$ is connected
- $\lambda_2(\mathcal{G})$ is a measure of connectivity

Oh ... one last thing: trace of any matrix is the sum of its eigenvalues, so

$$\textbf{trace}\, L(\mathcal{G}) = \sum_i d(v_i) = 2\,|\mathcal{E}(\mathcal{G})|$$

It would be good to develop some intuition for spectra of graphs, and in particular their dependencies on n, if any.

$$L(K_n) = \begin{bmatrix} n-1 & -1 & \cdots & -1 & -1 \\ -1 & n-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & n-1 \end{bmatrix} = nI - \mathbf{1}\mathbf{1}^T$$

as always, $\lambda_1(K_n) = 0$ and $u_1 = \mathbf{1}/\sqrt{n}$. The other eigenvectors, generically denoted by x for now, can be chosen to be orthogonal to $\mathbf{1}$

$$L(K_n)x = (nI - \mathbf{1}\mathbf{1}^T)x = \lambda x$$

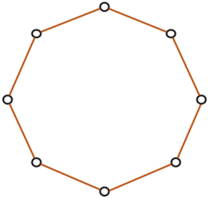
Hence for all these eigenvectors

$$nx = \lambda x$$

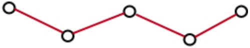
The spectrum of $L(K_n)$ is thus

$$0, n, n, \ldots n; \quad \text{check that} \quad \textbf{trace}\,\{L(K_n)\} = n(n-1)$$

SPECTRA OF SOME OTHER CLASSES OF GRAPHS



$$2\,(1-\cos 2k\pi/n), \quad k=0,1,\ldots n-1$$



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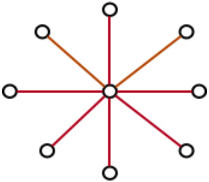

$n-2$ eigenvalues of 1, one eigenvalue of zero (as always) and last one is $2(n-1)-(n-2)=n$

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Introduction to Networked Dynamic Systems

Basic Graph Theory

Protocols on Graphs

Structural Stability of Linear Time-Invariant Systems

Graphs and Input-Output Properties of Network Systems

Unexplored Opportunities

We now what to see how this machinery (graphs and linear algebra, spectra vs. structure) helps us understand dynamics on networks

Our Action Plan

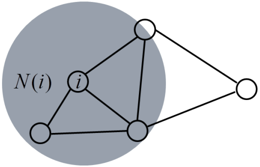
1. we start with a baseline dynamics/distributed algorithm called consensus
2. we relate consensus behavior to structure of the graph

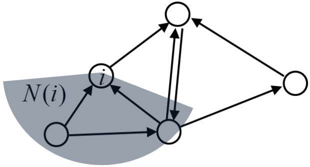
We then move on to show that this distributed algorithm can be used in many different context to do very useful distributed tasks.

However, it is important to note that the same line of research could have been pursued with a different baseline/distributed protocol or viewed completely from the perspective of patterned matrices independent of particular protocol!

NETWORK IN THE DYNAMICS – GENERAL SETUP

- Graph $\mathcal{G}$ is composed of physical nodes $\mathcal{V}$ and coupling edges $\mathcal{E}$
- Node i acquires information from the set of its neighbors $\mathcal{N}(i)$





- Node i has a state $x_i(t)$ and neighbor information $I_i(t) = \{x_j(t) | j \in \mathcal{N}(i)\}$
- Provides a naturally distributed dynamics over $\mathcal{G}$

$$\dot{x}_i(t) = f_i(x_i(t), I_i(t))$$

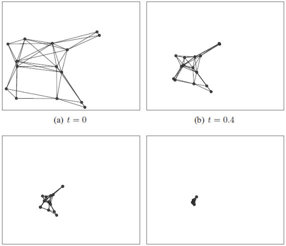
- some of the earlier works in distributed decision-making include: DeGroot ('74), Borkar and Varaiya ('82), Tsitsiklis ('84) ...

Consensus Model

$$\dot{x}_i(t) = -\sum_{j \in N(i)} w_{ij}\left(x_i(t) - x_j(t)\right)$$

$$\rightsquigarrow \dot{x}(t) = -L\left(\mathcal{G}\right)x(t)$$

where $L(\mathcal{G})$ is the (weighted) Laplacian matrix.



(a) $t = 0$ (b) $t = 0.4$ (c) $t = 0.8$ (d) $t = 1.8$

- appears in: flocking, formation control, opinion dynamics, energy systems, synchronization, distributed estimation, distributed optimization, among many others!

Let us examine the convergence of the algorithm a bit more ... in terms of the graph structure. We will assume that $w_{ij} = 1$ for this purpose, although our observations generalize seamlessly to weighted graphs

Let us consider consensus on undirected networks ... spectral factorization of the Laplacian is of the form

$$L(\mathcal{G}) = U\Lambda U^\top$$

where

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

as such,

$$\begin{aligned} x(t) &= e^{-L(\mathcal{G})t}x(0) = Ue^{-t\Lambda}U^T x(0) \\ &= u_1^\top x(0)\, u_1 + e^{-\lambda_2 t}u_2^\top x(0)\, u_2 + \ldots + e^{-\lambda_n t}u_n^\top x(0)\, u_n \end{aligned}$$

so if the graph is connected (noting that $u_1 = \mathbf{1}/\sqrt{n}$)

$$x(t) \to \frac{\mathbf{1}^T x(0)}{n}\,\mathbf{1} \quad \text{at a rate proportional to } \lambda_2(\mathcal{G})!$$

in fact,

$$\begin{aligned}\|x(t) - \frac{\mathbf{1}^T x(0)}{n}\| &= \|\sum_{i=2}^{n} e^{-\lambda_i t} \underbrace{u_i^\top x(0)}_{\alpha_i} u_i\| \\ &= \sum_{i=2}^{n} e^{-\lambda_i t} |\alpha_i| \leq (n-1) \underbrace{\beta}_{\max_i |\alpha_i|} e^{-\lambda_2 t}\end{aligned}$$

so if we want $\|x(t) - \frac{\mathbf{1}^T x(0)}{n}\| \leq \epsilon$ for some $\epsilon > 0$, then we need

$$t \geq \{\ln \frac{\beta(n-1)}{\epsilon}\}/\lambda_2(\mathcal{G}) \propto \frac{1}{\lambda_2(\mathcal{G})}$$

higher algebraic connectivity directly translates to faster convergence (in a linear way)!

some observations:

- Recall that $\lambda_2(P_n) = 2(1 - \cos k\pi/n)$, $\lambda_2(C_n) = 2(1 - \cos 2k\pi/n)$, $\lambda_2(S_n) = 1$, and $\lambda_2(K_n) = n$
- what this means is that as $n \to \infty$, the rate of convergence for P_n and C_n goes to zero!
- in the meantime, the rate of convergence for K_n grows linearly with n
- however, the number of edges for P_n, C_n grow linearly with n but for K_n the number of edges is $O(n^2)$!

this thread of thought leads to the area of graph synthesis

HOW BASELINE CONSENSUS CAN BE USED FOR MORE ELABORATE DISTRIBUTED ALGORITHMS

- as a distributed subroutine for mixing
- including the right inputs to consensus (not just driven by initial conditions)
- consensus with nonlinear and/or state-dependent weights (used in preserving connectivity in distributed robotics)
- consensus with negative, complex-valued, and matrix weights
- consensus across scales
- consensus with security and privacy considerations

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WHICH STRUCTURED LTI SYSTEMS CAN SUSTAIN STABLE DYNAMICS?

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ a_{31} & 0 & a_{32} & 0 \\ 0 & a_{42}0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{bmatrix} u$$

- Does there exist values of the a_{ij}'s that yield asymptotically stable dynamics? If so, we call the system structurally stable.

WHICH STRUCTURED LTI SYSTEMS CAN SUSTAIN STABLE DYNAMICS?

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- Does there exist values of the a_{ij}'s that yield asymptotically stable dynamics? If so, we call the system structurally stable.
- Does there exist values of the a_{ij}'s and b_i's that yield controllable dynamics? If so, we call the system structurally controllable.

WHICH STRUCTURED LTI SYSTEMS CAN SUSTAIN STABLE DYNAMICS?

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ a_{31} & 0 & a_{32} & 0 \\ 0 & a_{42}0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{bmatrix} u$$

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- Recall: LTI dynamics are asymptotically stable iff the eigenvalues of the system matrix have strictly negative real parts.

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- Does there exist values of the a_{ij}'s and b_i's that yield controllable dynamics? If so, we call the system structurally controllable.
- Recall: LTI dynamics are asymptotically stable iff the eigenvalues of the system matrix have strictly negative real parts.
- Graph theory is the natural framework to study structural stability.

$$A = \begin{bmatrix} 0 & * & * & 0 & * \\ * & * & 0 & * & * \\ 0 & * & 0 & * & 0 \\ 0 & * & 0 & * & * \\ * & 0 & 0 & * & 0 \end{bmatrix}$$

$*$ entries are arbitrary real

0 entries are fixed to zero

Definition (Zero-pattern (ZP))

Set E_{ij} to be the $n \times n$ matrix with all entries 0 except for the ijth one, which is 1. We call a *zero pattern* a vector space $\mathcal{Z}$ of matrices

$$A = \sum_{(i,j)\in\mathcal{N}} a_{ij} E_{ij}.$$

- Does the ZP contain stable (Hurwitz) matrices?
- We call a ZP that contains Hurwitz matrices stable

- Think of a ZP as an adjacency matrix with

$$\begin{aligned} 0 &\longrightarrow \quad 0 \\ * &\longrightarrow \quad 1 \end{aligned}$$

HURWITZ DIGRAPHS AND ZERO-PATTERNS

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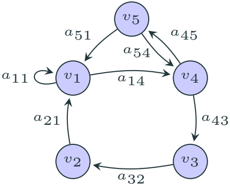

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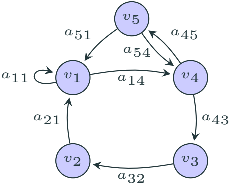

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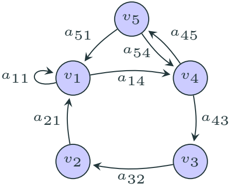
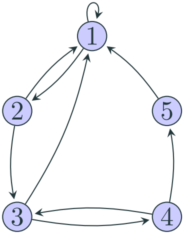
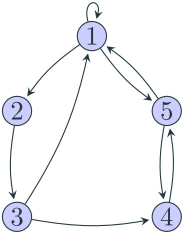


- We call a graph Hurwitz or stable if the corresponding ZP is stable.

How to determine if a graph is Hurwitz? How to create Hurwitz graphs?

WHICH GRAPH IS STABLE?





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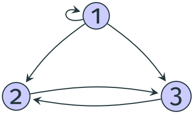
Which graph is stable?

Lemma

A digraph G is stable only if every strongly connected component has a node with a self-loop



Not stable: the strongly connected component $\{2,3\}$ has no nodes with a self-loop.

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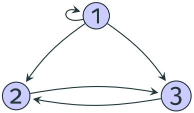

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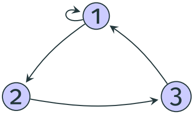
This is not the end of the story...

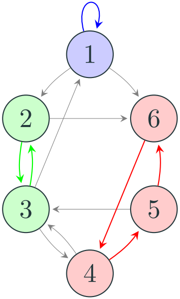


The graph is strongly connected and has a self-loop, yet not stable.

$\longrightarrow$ need to find the graphical structure that enables stability

K-DECOMPOSITIONS

- k-cycle in G: a sequence of k distinct nodes connected by edges.



1-cycle = (1)
2-cycle: (23)
3-cycle: (456)
3-decomp.: $(1)(23)$ or (456)
4-decomp.: $(1)(456)$
5-decomp.: $(23)(456)$

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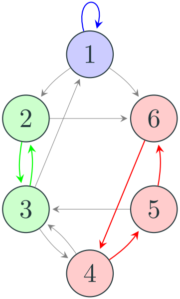

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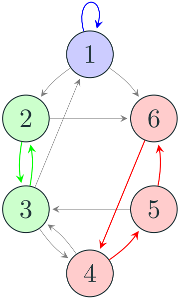
  A k-decomposition is given by cycles $S_1, \dots, S_l$ if the S_i are disjoint and $|S_1| + \cdots + |S_l| = k$.



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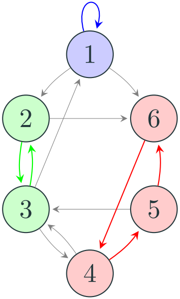
- Hamiltonian cycle (resp. decomposition): n-cycle (resp. decomposition).



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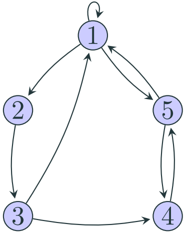
A NECESSARY CONDITION FOR STABILITY

Theorem[1]

A digraph G is stable only if it contains a k-decomposition for each $k = 1, 2, \ldots, n$



$$\begin{bmatrix} * & * & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & * & 0 \end{bmatrix}$$

1-decomp.: (1), 2-decomp.: (15), 3-decomp.:$(1)(45)$ but no 4-decomp. $\longrightarrow$ not stable.

[1] B. “Sparse Stable Systems”, Systems and Control Letters, 2013

A NECESSARY CONDITION FOR STABILITY: SKETCH OF PROOF

- S_k: symmetric group on k characters.

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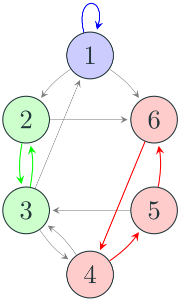
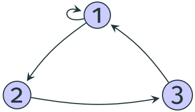
- It is known that A is Hurwitz only if all coefficients of its characteristic polynomial are non-zero.
- Characteristic polynomial of A is given by

$$\det(I\lambda - A) = \sum_{k=0}^{n-1} (-1)^k \lambda^k \sum_{\sigma \in S_{n-k}} (-1)^\sigma \prod_{i=1}^{n-k} a_{i,\sigma(i)}$$

- Each term $\prod_{i=1}^{k} a_{i,\sigma(i)}$ corresponds to a k-decomposition.

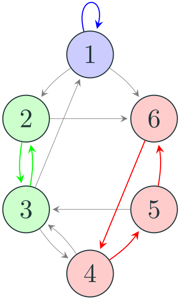


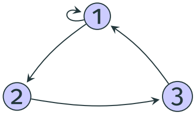


$$p(s) = s^3 - a_{11}s^2 + 0s - a_{12}a_{23}a_{31}.$$

A NECESSARY CONDITION FOR STABILITY: SKETCH OF PROOF (II)

- Each term $\prod_{i=1}^{k} a_{i,\sigma(i)}$ corresponds to a k-decomposition.
- Said otherwise: each permutation in S_k corresponds to a k-decomposition: e.g. permutation in S_3 that sends $\{4,5,6\}$ to $\{5,6,4\}$ is depicted in red. permutation in S_3 that sends $\{1,2,3\}$ to $\{1,3,2\}$ is depicted in blue+green.





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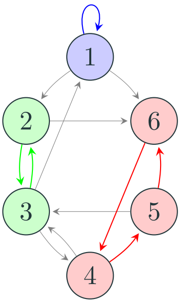
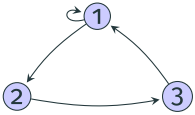
- Conclusion: no k-decompositions $\Longrightarrow$ degree $n-k$ term in characteristic polynomial of any matrix in $\mathcal{Z}$ is zero $\Longrightarrow$ graph and ZP are not stable



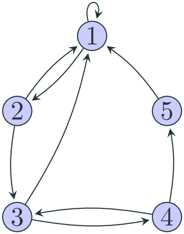


$$p(s) = s^3 - a_{11}s^2 + 0s - a_{12}a_{23}a_{31}.$$

Theorem[2]

A digraph G is stable if it contains a sequence of *nested* k-decomposition for each $k = 1, 2, \dots, n$.

We say that a k-decomposition K_1 is nested in K_2 if the node set of K_1 is included in the one of K_2



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1-decomp.: (1), 2-decomp.: (12), 3-decomp.:(123), 4-decomp.:$(12(34)$, 5-decomp.:(12345).

[2] B. "Sparse Stable Systems", Systems and Control Letters, 2013

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- Stability is not generic. The proportion of stable matrices in a ZP can be *very small*.
- Hence simulations studies are “hard”: one needs to sample many matrices in a SMS to conclude non-stability. Very unlike structural controllability: almost all systems in a zero-pattern are controllable. Sample one system: with probability one, it is controllable if the zero pattern is.

Observation: adding an edge to a stable graph yields another stable graph.

Graph stability is monotone with respect to edge addition.

$$\begin{bmatrix} * & * & 0 \\ * & 0 & * \\ * & 0 & 0 \end{bmatrix} \subset \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix} \subset \begin{bmatrix} * & * & 0 \\ * & * & * \\ * & 0 & * \end{bmatrix}$$

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stable $\longrightarrow$

MINIMAL STABLE GRAPHS AND NOTIONS OF ROBUSTNESS

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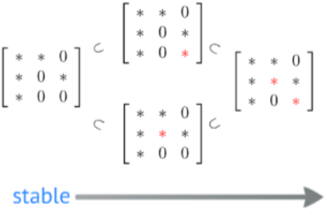
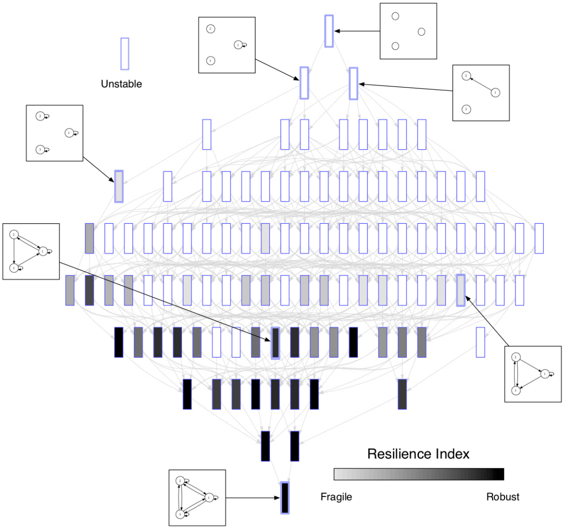


- Minimal stable graphs: stable graphs for which removing *any* edge yields an *unstable* graph. All stable graphs are “descendants” of minimal stable graphs. We can think of them as “prime” graphs.
- Robustly stable graphs: stable graphs for which removing *any* edge yields a *stable* graph.



Box $\to$ graph on three nodes

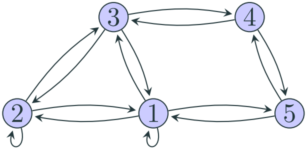
Same # edges $\to$ same row

Edge between box denotes inclusion

Shade: $\dfrac{\text{\# stable ancestors}}{\text{\# ancestors}}$

Minimal stable: lightest shade. There are 7.

- It is often the case that information exchange is *bilateral*: $i \leftrightarrow j$.
- We call a graph reciprocal or symmetric of to every edge $(i,j) \in E$ there is an edge $(j,i) \in E$.
- The corresponding ZP is symmetric:



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- Two cases: either the matrices in the ZP are symmetric (*strongly symmetric ZP*) or not necessarily symmetric (*weakly symmetric ZP*).

Definition[3]

A ZP is *weakly symmetric* if to a free variable in position ij corresponds a free variable in position ji. A ZP is *strongly symmetric* if it only contains symmetric matrices.

[3] A. Kirkoryan and B. “Symmetric Sparse Systems”, CDC 2014.

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Theorem[3]

A weakly symmetric ZP is stable *if and only if* its graph is so that

1. Every node is strongly connected to a self-loop
2. The graph contains a Hamiltonian decomposition.

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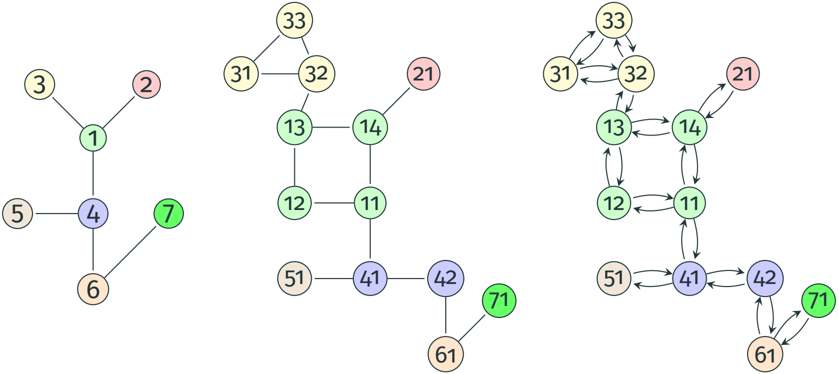
The proof of the last theorem is graphical in nature.
A tree graph is a graph without cycles.



- Tree graph → Nodes can be cycles → Edges are symmetric → fat tree

- Proof idea: Given a symmetric graph G, show that if

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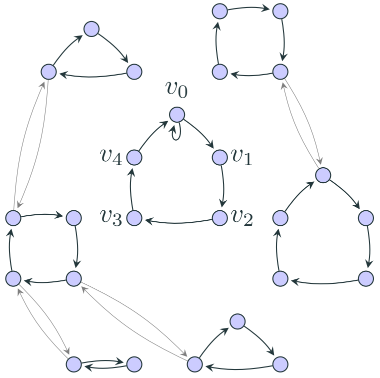
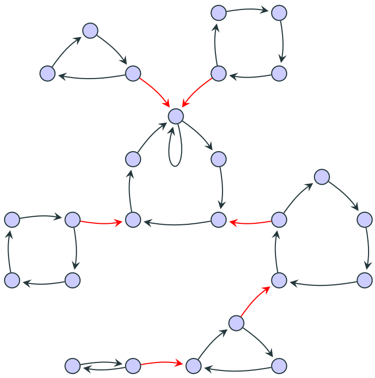
- Proof technique: find a fat tree in G. Fat trees provide a natural ordering of nodes. Use the ordering to exhibit nested k-decompositions:
  We label (order) the nodes so that $\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, \ldots, n\}$ all have k-decompositions. By construction, they are nested.

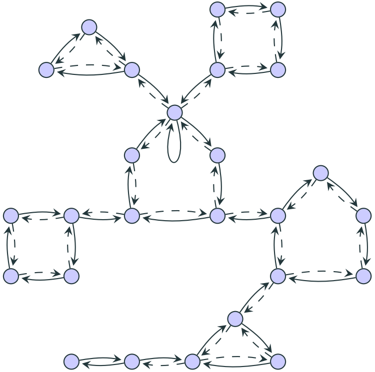


Draw the cycles of a Hamiltonian decomposition of G. This is a subgraph of G.

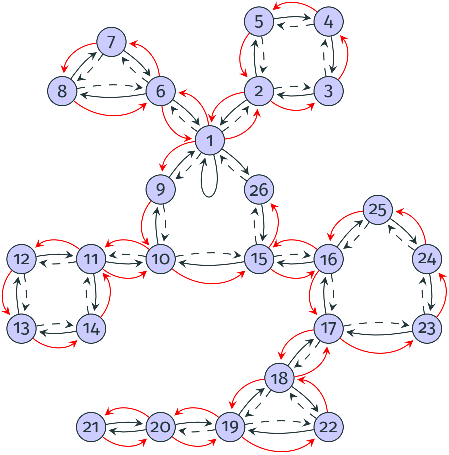


Connect every cycle to the cycle with the self-loop. We can do so by assumption 1.

STABILITY OF SYMMETRIC GRAPHS (III)



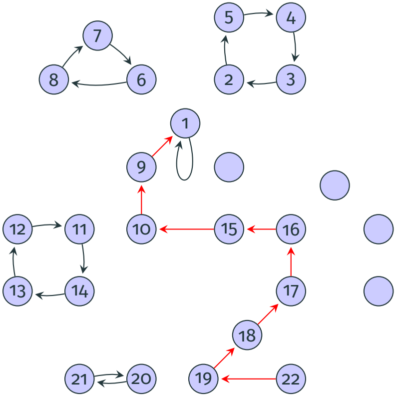
Add reciprocal edges. The resulting graph is a planar subgraph of G by construction.



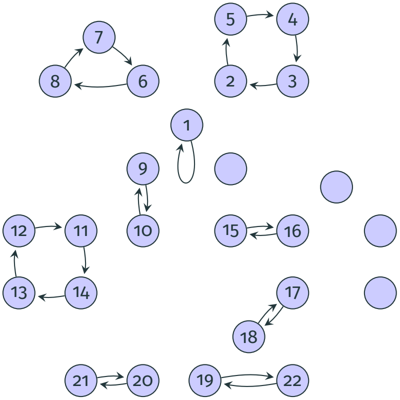
Ordering: Set v_0 at 1. Order nodes counter-clockwise. Skip already numbered nodes. By construction, no node lies inside $\rightarrow$ complete ordering. Call this graph P.

STABILITY OF SYMMETRIC GRAPHS: (IV)

The last graph shown is a subgraph of G. We show that is satisfies the hypothesis of Theorem 2.



- There is a unique path from any node k to 1 using the plain edges of P only.
- Key observation: by construction, the subgraph induced by the node set $\{1, 2, \ldots, k\}$ is the union of the path joining 1 to k and l-cycles.



A $n = 22$-decomposition

- The subgraph induced by nodes $\{1, \ldots, k\}$ admits a Hamiltonian decomposition, which is thus a k-decomposition of G.
- Depending on whether the path joining 1 to k has an even or odd number of nodes, the decomposition is in 2-cycles (even) or self=loop+2 cycles (odd).
- Repeating the procedure for each node $k = 1, \ldots, n$, we obtain nested k-decompositions.

3.36pt

Introduction to Networked Dynamic Systems

Basic Graph Theory

Protocols on Graphs

Structural Stability of Linear Time-Invariant Systems

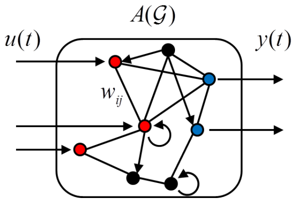
Graphs and Input-Output Properties of Network Systems

Unexplored Opportunities

Symmetry and Controllability

- Model

$$\dot{x}_i(t) = -w_{ii}x_i(t) + \sum_{i\sim P} w_{iP}x_P(t) + u_i(t)$$

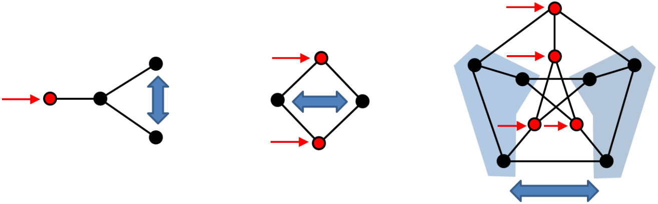


that in general assumes the form:

$$\dot{x}(t) = A(\mathcal{G})x(t) + B(S)u(t)$$

Controllability/observability: stabilization via feedback, observer design, disturbance/noise rejection, optimal control, and pole placement

For the LTI plant $(A(\mathcal{G}, S), B(S))$ what are the structural conditions for controllability? One approach is to link uncontrollability to symmetry



For today, we will use the edge leader follower dynamics

$$\dot{x} = A(\mathcal{G}, S)x + B(S)u = -(L(\mathcal{G}) + B(S)B(S)^T)x + B(S)u.$$

(These results can be extended to the leader follower dynamics $\dot{x} = A(\mathcal{G}, \mathcal{R})x + B(\mathcal{R})u$ and controlled consensus dynamics $\dot{x} = -L(\mathcal{G})x + B(S)u$)

First, what do we mean by symmetry...

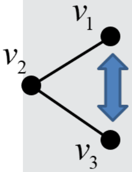
Definition

An automorphism of the graph is a mapping $\pi : \mathcal{V}(\mathcal{G}) \to \mathcal{V}(\mathcal{G})$ such that if $\{i, p\} \in \mathcal{E}(\mathcal{G}) \iff \{\pi(i), \pi(p)\} \in \mathcal{E}(\mathcal{G})$

Represented as $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, $\pi(i) = p$

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \pi(1) & \pi(2) & \pi(3) & & \pi(n) \end{array}$$

Example



$1 \to 3$, $2 \to 2$, $3 \to 1$

Mapping $\pi : \mathcal{V}(\mathcal{G}) \to \mathcal{V}(\mathcal{G})$

$\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 1$

The edges $\{\pi(i), \pi(p)\}$

$\{1, 2\} \to \{3, 2\} \in \mathcal{E}$, $\{2, 3\} \to \{2, 1\} \in \mathcal{E} \implies \pi$ is an automorphism

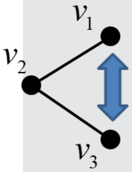
We need an algebraic representation of the automorphism π.

Definition

A permutation matrix is a $\{0,1\}$ square matrix with one "1" and one "zero" in each row and column.

$\pi \rightarrow$permutation matrix P such that $PA(\mathcal{G}) = A(\mathcal{G})P$

Example



$$
\begin{aligned}
PA(\mathcal{G}) &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
A(\mathcal{G})P &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

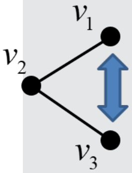
We also need a link between the automorphism and the inputs.

Definition

A system is input symmetric with respect to the input nodes if there exists a nonidentity automorphism with input nodes invariant under its action.

Input symmetry (permutation P) w.r.t. to the input nodes $\iff$ $P \neq I$, $A(\mathcal{G})P = PA(\mathcal{G})$ and $PB(S) = B(S)$.

Example



$$PB(\{v_2\}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = B(\{v_2\})$$

$\rightarrow$ Input symmetric

$$PB(\{v_3\}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq B(\{v_3\})$$

Some more preliminary work before showing our controllability conditions

For an automorphism π of $\mathcal{G}$ with permutation matrix P

$$A(\mathcal{G})P = PA(\mathcal{G}) \implies \deg(v) = \deg(\pi(v)) \implies \Delta(\mathcal{G})P = P\Delta(\mathcal{G})$$

then as $L(\mathcal{G}) = A(\mathcal{G}) - \Delta(\mathcal{G})$ we have

$$L(\mathcal{G})P = PL(\mathcal{G}).$$

For input symmetry $PB(S) = B(S)$ then

$$PB(S) = B(S) \implies \pi(\{s\}) = \{s\} \text{ for all } s \in S$$

Finally,

$$\begin{aligned} A(\mathcal{G}, S)P &= -(L(\mathcal{G}) + B(S)B(S)^T)P \\ &= -P(L(\mathcal{G}) + B(S)B(S)^T) \\ &= PA(\mathcal{G}, S). \end{aligned}$$

Theorem

Input symmetry implies uncontrollability.

Proof.

For $P \neq I$, $A(\mathcal{G})P = PA(\mathcal{G})$ and $PB(S) = B(S) \implies A(\mathcal{G}, S)P = PA(\mathcal{G}, S)$

Let v be an eigenvector of $A(\mathcal{G}, S) := A$ then

$$APv = PAv \quad = \quad P(\lambda v) = \lambda Pv$$

So Pv is also an eigenvector.

As $A(\mathcal{G}, S)$ is symmetric with a spanning set of eigenvectors then for some v, $Pv \neq v$.

Then $v - Pv$ is an eigenvector and $(v - Pv)^T B(S) = v^T B(S) - v^T P^T B(S)$; hence

$$(v - Pv)^T B(S) = v^T B(S) - v^T B(S) = 0$$

Theorem

Suppose that the network dynamics assumes the form

$$\dot{x} = \mathbf{A}(\mathcal{G})x + \mathbf{B}(\mathcal{G})u$$

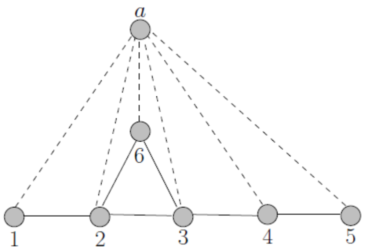
is such at there exists some $P \in$ **AUT**$(\mathcal{G})$ that commutes with the dynamics and leaves the input invariant under its action, i.e.,

$$P\mathbf{A}(\mathcal{G}) = \mathbf{A}(\mathcal{G})P \quad P\mathbf{B}(\mathcal{G}) = \mathbf{B}(\mathcal{G});$$

if $\mathbf{A}(\mathcal{G})$ is non-defective, then $(A(\mathcal{G}), B(\mathcal{G}))$ is not controllable.

No!

Consider the smallest asymmetric graph $\mathcal{G}$ controlled through a



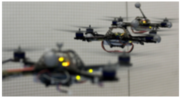
Then $A(\mathcal{G},\mathcal{R}) = L(\mathcal{G}) + I$ and $B(\mathcal{R}) = -\mathbf{1}$; $A(\mathcal{G},\mathcal{R})$ has $\mathbf{1}$ as an eigenvector:

$$A(\mathcal{G},\mathcal{R})\mathbf{1} = L(\mathcal{G})\mathbf{1} + \mathbf{1} = \mathbf{1}$$

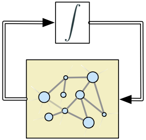
All other eigenvectors of $A(\mathcal{G},\mathcal{R})$ are orthogonal to $\mathbf{1}$; now invoke PBH!

Performance of Networks

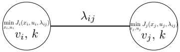
The consensus protocol is a canonical model
for studying complex networked systems



formation control



system theory over graphs

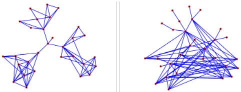


distributed optimization

Are certain information structures more favorable than others?

Can system performance be characterized using properties of the graph?

How do we synthesize good information structures?



$$\begin{matrix} \mathcal{H}_2 & & \text{cycle lengths} \\ \mathcal{H}_\infty & \propto & \text{node degree} \\ \vdots & & \vdots \end{matrix}$$

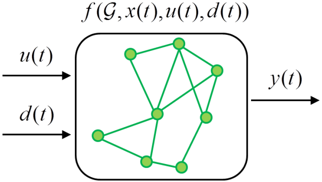
$$\min_{\mathcal{G}\in\mathbb{G}} \|\Sigma(\mathcal{G})\|$$

Networks may be influenced by

- selected leaders
- exogenous inputs (disturbances or noises)
- malicious agents

General Dynamics

$$\dot{x}(t) = f(\mathcal{G}, x(t), u(t), d(t))$$
$$y(t) = g(\mathcal{G}, x(t), u(t), d(t))$$



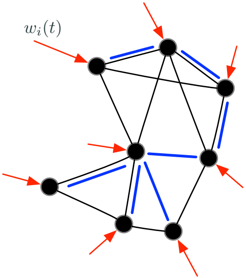
Analysis draws upon:

- Control theory:
  Input-output dynamics
- Graph theory:
  Design and reasoning on $\mathcal{G}$
- Large-scale Optimization:
  For large # nodes n
- Machine-learning:
  For uncertain dynamics and inputs

Dynamics

$$\dot{x}(t) = -L(\mathcal{G})x(t) + w(t)$$
$$y(t) = E(\mathcal{H})^T x(t)$$

- Each node corrupted by zero-mean white Gaussian noise.
- $\mathcal{H}$ models the performance network (i.e., $\mathcal{H} \subseteq \mathcal{G}$ or $\mathcal{H} = \mathcal{K}_n$)

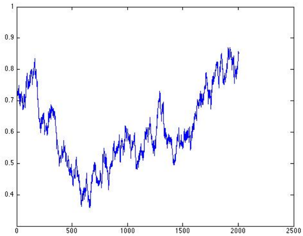


consensus state (average) is driven by noise

$$\frac{d}{dt}\text{avg}(x(t)) = \frac{1}{n}\mathbf{1}^\top w(t)$$

covariance exhibits a random walk

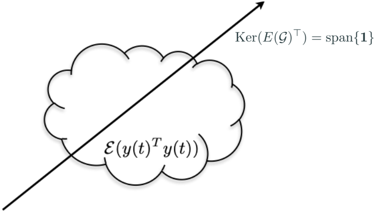
$$\mathcal{E}(\text{avg}(x(t)^2)) = \frac{\sigma_w}{n}t$$



When driven by noise, it is meaningful to examine how noises effect the stead-state covariance of the relative states

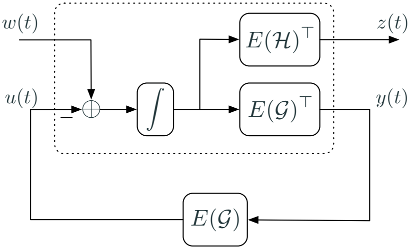
Idea

Characterized by the $\mathcal{H}_2$ performance



A two-port model

$$\dot{x}(t) = -L(\mathcal{G})x(t) + {\color{red} w(t)}$$
$$z(t) = {\color{blue} E(\mathcal{H})^T x(t)}$$

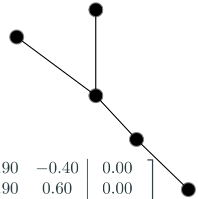


Note the system is *not* minimal (unobservable) and also has unbounded $\mathcal{H}_2$ norm (eigenvalue at 0)

$\Rightarrow$ Find a stable minimal realization!

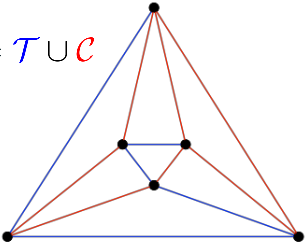
$$S = \begin{bmatrix} P & \frac{1}{\sqrt{n}}\mathbf{1} \end{bmatrix} \qquad \mathbf{1}^\top P = 0$$
$$\tilde{x}(t) = S^{-1}x(t)$$

$\mathcal{G}$



$$S^{-1}L(\mathcal{G})S = \left[\begin{array}{cccc|c} 2.90 & 0.90 & 0.90 & -0.40 & 0.00 \\ 0.90 & 1.90 & 0.90 & 0.60 & 0.00 \\ 0.90 & 0.90 & 1.90 & 0.60 & -0.00 \\ -0.40 & 0.60 & 0.60 & 1.29 & -0.00 \\ \hline 0.00 & 0.00 & 0.00 & 0.00 & -0.00 \end{array}\right]$$

A connected graph can be decomposed into a spanning tree and the edges that complete cycles (co-tree)



Cycles can be expressed as a "linear combination" of edges in the tree

$$E(\mathcal{C}) = E(\mathcal{T})R$$
$$E(\mathcal{G}) = E(\mathcal{T}) \begin{bmatrix} I & R \end{bmatrix}$$

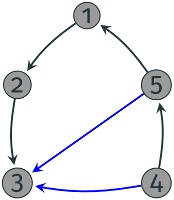
R is referred to as the *Tucker representation* of $\mathcal{G}$ with spanning tree $\mathcal{T}$

Theorem [Godsil and Royle, 2001]

The cycle space of $\mathcal{G}$ is spanned by the fundamental cycles of $\mathcal{G}$.

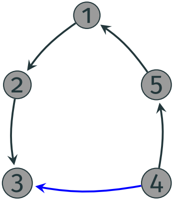
$$\mathrm{Ker}[E(\mathcal{G})] = \mathrm{Im} \begin{bmatrix} -R \\ I \end{bmatrix}$$

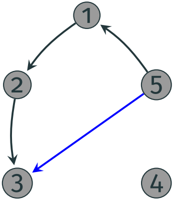
$$R = \underbrace{(E(\mathcal{T})^\top E(\mathcal{T}))^{-1} E(\mathcal{T})^\top}_{E_{\mathcal{T}}^L} E(\mathcal{C})$$



$$E(\mathcal{T}) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

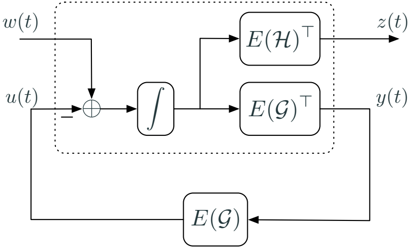
$$R = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$





A two-port model

$$\dot{x}(t) = -L(\mathcal{G})x(t) + w(t)$$
$$z(t) = E(\mathcal{H})^T x(t)$$



$\Rightarrow$ Find a stable minimal realization!

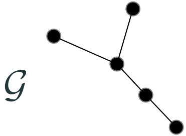
$$S^{-1} = \begin{bmatrix} E(\mathcal{T})^\top \\ \frac{1}{\sqrt{n}}\mathbf{1}^\top \end{bmatrix}$$

$$\tilde{x}(t) = \begin{bmatrix} x_\tau(t) \\ \text{avg}(x(t)) \end{bmatrix} = S^{-1}x(t)$$

$$S^{-1}L(\mathcal{G})S = \begin{bmatrix} L_{ess}(\mathcal{G}) & \mathbf{0}^\top \\ \mathbf{0} & 0 \end{bmatrix}$$

The Essential Edge Laplacian

$$L_{ess}(\mathcal{G}) := (E(\mathcal{T})^\top E(\mathcal{T}))(I + RR^\top)$$



$$S^{-1}L(\mathcal{G})S = \begin{bmatrix} E(\mathcal{T})^\top E(\mathcal{T})(I + RR^\top) & \mathbf{0}^\top \\ \mathbf{0} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = L_e(\mathcal{T})$$

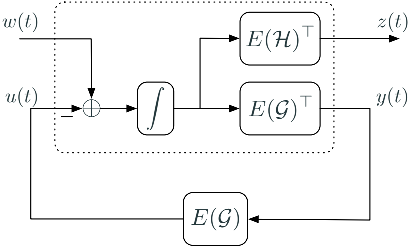
Edge Laplacian

$$L_e(\mathcal{G}) = E(\mathcal{G})^\top E(\mathcal{G}) \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{E}|}$$

- shares the same non-zero eigenvalues of $L(\mathcal{G})$
- $L_e(\mathcal{T})$ is positive definite
- indexed by the edges in the graph
- $[L_e(\mathcal{G})]_{ij} = \pm 1$ when *edge* i is adjacent to *edge* j
- $\mathrm{Ker}[L_e(\mathcal{G})]$ is spanned by fundamental cycles in $\mathcal{G}$

$\langle_2$ PERFORMANCE OF CONSENSUS

$$\dot{x}(t) = -L(\mathcal{G})x(t) + {\color{red}w(t)}$$
$$z(t) = {\color{blue}E(\mathcal{H})^T x(t)}$$



Theorem [Zelazo and Mesbahi, TAC2011]

The $\mathcal{H}_2$ performance of the consensus protocol is

$$\|\Sigma(\mathcal{G})\|_2^2 = \mathrm{Tr}[E(\mathcal{H})^\top E_{\mathcal{T}}^{L^\top} X E_{\mathcal{T}}^{L} E(\mathcal{H})],$$

where

$$X = \frac{1}{2}\left(I + RR^\top\right)^{-1}$$

is the positive definite solution to the Lyapunov equation

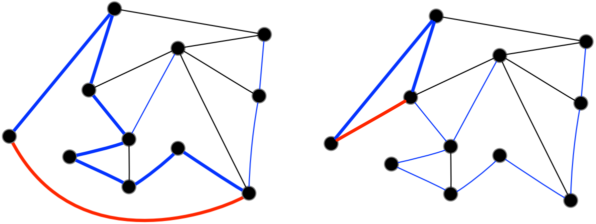
$$\mathcal{L}(X) = -L_{ess}(\mathcal{G})X - XL_{ess}(\mathcal{G})^\top + E(\mathcal{T})^\top E(\mathcal{T}) = 0.$$

Theorem [Zelazo et al., Systems & Controls Letters, 2013]

Consider the consensus protocol with $\mathcal{G} = \mathcal{H} = \mathcal{T}$ and an edge $e \notin \mathcal{G}$. Then

$$\|\Sigma(\mathcal{T} \cup e)\|_2^2 = \|\Sigma_e(\mathcal{T})\|_2^2 - \frac{\ell(c) - 1}{2\ell(c)},$$

where $\ell(c)$ is the length of the fundamental cycle created by adding the edge e.



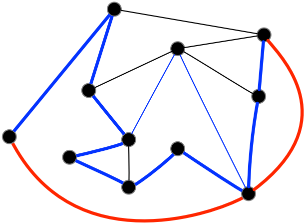
- *long* cycles are better than *short ones*

Corollary [Zelazo et al., Systems & Controls Letters, 2013]

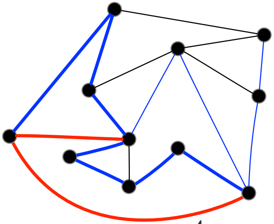
Consider the consensus protocol with $\mathcal{G} = \mathcal{H} = \mathcal{T}$ and an edges $e_1, e_2 \notin \mathcal{G}$. Then

$$\|\Sigma(\mathcal{T} \cup \{e_1, e_2\})\|_2^2 = \|\Sigma_e(\mathcal{T})\|_2^2 - \left(1 - \frac{\ell(c_1) + \ell(c_2)}{2(\ell(c_1)\ell(c_2) - s_{12}^2)}\right),$$

where s_{ij} is the edge correlation number for cycles c_i and c_j.



$s_{12} = 0$

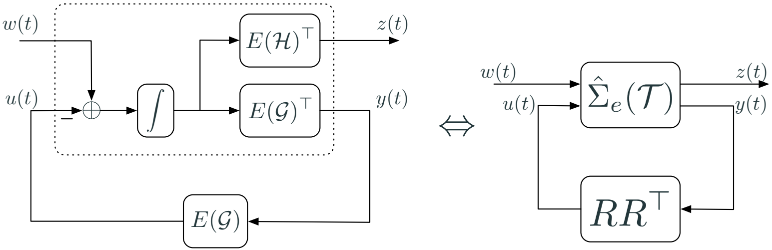


$s_{12} = 4$

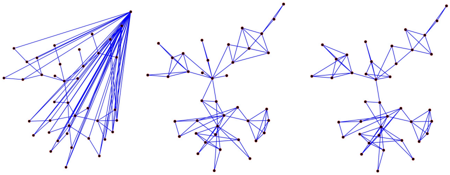
- *edge disjoint* cycles are better

A network design problem

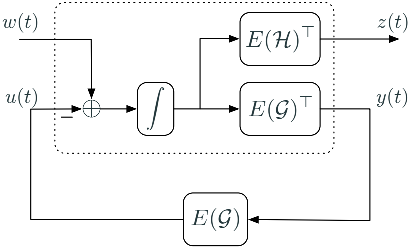
Given a graph $\mathcal{G}$ with spanning tree $\mathcal{T}$, add k edges that optimizes $\|\Sigma(\mathcal{G})\|_2^2$.



- Cycles interpreted as a *feedback* system
- Can be formulated as a *mixed-integer SDP*
- re-weighted ℓ_1 optimization; ADMM

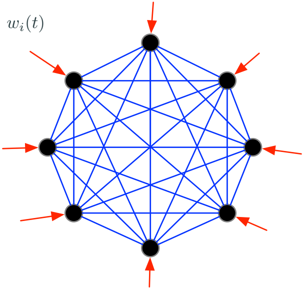


$$\dot{x}(t) = -L(\mathcal{G})x(t) + w(t)$$
$$z(t) = E(\mathcal{H})^T x(t)$$



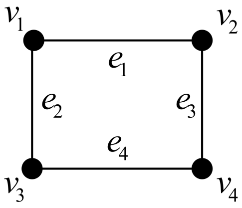
What is the performance when monitoring all relative state pairs?

$$\mathcal{H} = \mathcal{K}_n$$



Linear Consensus as an RC-Circuit

$$\dot{x}(t) = -L(\mathcal{G})x(t) + w(t)$$
$$y(t) = E(\mathcal{H})^T x(t)$$



$\mathcal{G}$

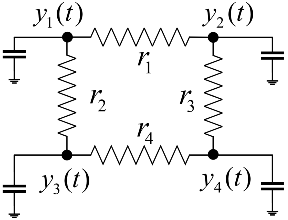
Capacitors $\Leftrightarrow$ Node Dynamics (integrators)

Resistors $\Leftrightarrow$ Edge Dynamics (linear gain)

- edge weights model the *admittance* of the resistor

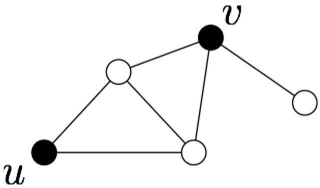
$$r_i = \frac{1}{\mathrm{w}_i}$$

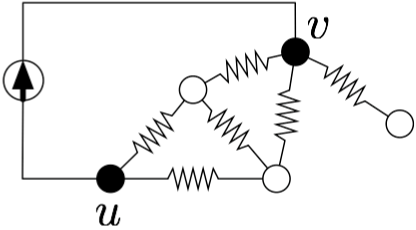
- in steady-state, network corresponds to a *resistive circuit*



RC circuit

The effective resistance between two nodes u and v is the electrical resistance measured across the nodes when the graph represents a resistive circuit.





Effective Resistance Calculation [Klein and Randić 1993]

$$\mathcal{R}_{uv}(\mathcal{G}) = [L^\dagger(\mathcal{G})]_{uu} + 2[L^\dagger(\mathcal{G})]_{uv} + [L^\dagger(\mathcal{G})]_{vv}$$

The *total effective resistance* of a graph is the sum over all pairs of nodes of $\mathcal{R}_{uv}(\mathcal{G})$,

Proposition

Consider a graph $\mathcal{G}$ with spanning tree $\mathcal{T}$ and Tucker matrix R. Let R_{uv} satisfy $(\mathbf{e}_u - \mathbf{e}_v) = E(\mathcal{T})R_{uv}$. Then the effective resistance between nodes u and v can be computed as

$$\mathcal{R}_{uv}(\mathcal{G}) = R_{uv}^\top (I + RR^\top)^{-1} R_{uv}.$$

This can be extended to derive an expression for the total effective resistance. Let $R_{\mathcal{K}_n}$ satisfy $E(\mathcal{K}_n) = E(\mathcal{T})R_{\mathcal{K}_n}$, representing the Tucker matrix for all possible edges, then

$$\mathcal{R}_{tot}(\mathcal{G}) = \mathrm{Tr}[R_{\mathcal{K}_n}^\top (I + RR^\top)^{-1} R_{\mathcal{K}_n}].$$

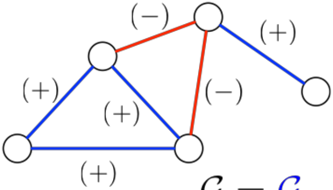
Performance when $\mathcal{H} = \mathcal{K}_n$

$$\|\Sigma(\mathcal{G})\|_2^2 = \frac{1}{2}\mathcal{R}_{tot}(\mathcal{G})$$

a **signed graph** is a graph with positive and negative edge weights

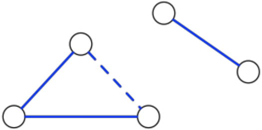
$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

$$\mathcal{W} : \mathcal{E} \to \mathbb{R}$$

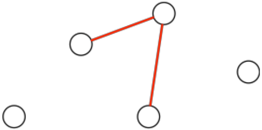


$$\mathcal{G} = \mathcal{G}_+ \cup \mathcal{G}$$

$$\mathcal{E}_+ = \{e \in \mathcal{E} \ : \ \mathcal{W}(e) > 0\} \qquad \mathcal{E}_- = \{e \in \mathcal{E} \ : \ \mathcal{W}(e) < 0\}$$



$E(\mathcal{G}_+) = E_+$



$E(\mathcal{G}_-) = E_-$

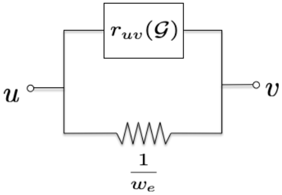
$$L(\mathcal{G}) = E(\mathcal{G}_+)W_+E(\mathcal{G}_+)^T - E(\mathcal{G}_-)|W_-|E(\mathcal{G}_-)^T$$

Theorem [Zelazo and Bürger, TCNS2017]

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$ be a strictly positive network with edge functions $\mu_k = w_k \zeta_k$ (i.e., $w_k > 0$ for all $k \in \mathcal{E}$) and let $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}_> \cup e)$ where $e = (u, v)$ is a negative edge with weight $w_e < 0$. Then the signed consensus network reaches agreement if and only if

$$|w_e| \leq r_{uv}^{-1},$$

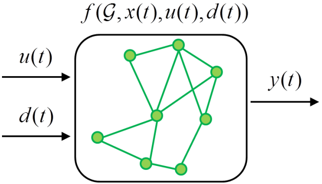
where r_{uv} is the effective resistance in $\mathcal{G}$ between nodes u and v.



The negative edge weights effectively creates an open circuit

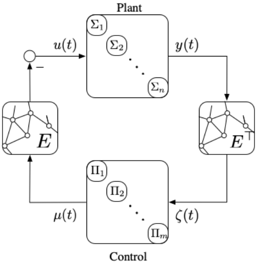
General Dynamics

$$\dot{x}(t) = f(\mathcal{G}, x(t), u(t), d(t))$$
$$y(t) = g(\mathcal{G}, x(t), u(t), d(t))$$



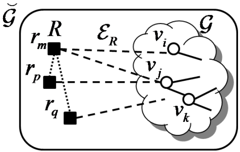
- network structure influences the performance of network systems
- in linear consensus, $\mathcal{H}_2$ performance can be understood in terms of fundamental structural properties of the graph: trees and co-trees
- effective resistance is a powerful concept for analyzing performance and robustness of linear consensus
- design of networks leverages combinatorial understanding of performance with modern optimization methods

Explore graph-theoretic interpretations for more general networked systems structures



Leader-follower networks

$$\dot{x}(t) = A(\mathcal{G}, \mathcal{R})x(t) + B(\mathcal{R})u(t)$$



- leader selection and $\mathcal{H}_2$ performance
- effective resistance interpretations
- network design using online

3.36pt

Introduction to Networked Dynamic Systems

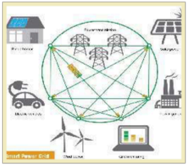
Basic Graph Theory

Protocols on Graphs

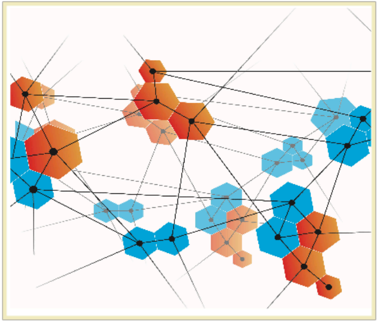
Structural Stability of Linear Time-Invariant Systems

Graphs and Input-Output Properties of Network Systems

Unexplored Opportunities



NETWORKS OF DYNAMICAL SYSTEMS ARE ONE OF THE ENABLING TECHNOLOGIES OF THE FUTURE

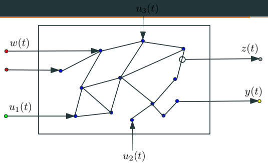


Why do we need this tutorial?

Network analysis and control	MoA03.5, MoA05.6, MoA07.2, MoA09.6, MoA12.2, MoA12.6, MoB03.3, MoB05.3, MoB10.6, MoB12.1, MoC03.3, MoC03.4, MoC06.1, MoC06.2, MoC13.6, MoC14.3, TuA03.6, TuA04.3, TuB09.1, TuB09.2, TuB12.1, TuB12.2, TuB12.3, TuB12.4, TuB12.5, TuB12.6, TuB16.3, TuB16.4, TuC04.3, TuC04.6, TuC05.1, TuC09.1, TuC09.3, TuC10.5, TuC12.1, TuC12.2, TuC12.3, TuC12.4, TuC12.5, TuC12.6, TuC18.5, TuC18.6, WeA09.2, WeA09.3, WeA09.4, WeA09.5, WeA09.6, WeA12.1, WeA12.2, WeA12.3, WeA12.4, WeA12.5, WeA12.6, WeB03.6, WeB05.1, WeB05.4, WeB12.5, WeB12.6, WeB13.1, WeB13.2, WeB13.3, WeB13.4, WeB14.1, WeB14.6, WeB18.6, WeC01.6, WeC05.1, WeC05.5, WeC12.1, WeC12.3
Networked control systems	MoA01.3, MoA03.4, MoA03.5, MoA04.2, MoA04.3, MoA04.4, MoA04.5, MoA04.6, MoA05.1, MoA05.3, MoA10.5, MoA12.1, MoA12.2, MoA12.3, MoA12.4, MoA12.5, MoA12.6, MoB03.2, MoB04.3, MoB04.4, MoB04.6, MoB09.4, MoB12.1, MoB12.2, MoB12.3, MoB12.4, MoB12.5, MoB12.6, MoB14.4, MoC03.4, MoC04.2, MoC04.3, MoC04.5, MoC09.4, MoC10.5, MoC12.1, MoC12.2, MoC12.3, MoC12.4, MoC12.5, MoC12.6, MoC13.1, MoC13.2, MoC13.4, MoC13.5, MoC18.4, MoC19.4, TuA01.6, TuA05.2, TuA10.1, TuA12.1, TuA12.2, TuA12.3, TuA12.4, TuA12.5, TuA12.6, TuA15.5, TuA21.3, TuB02.4, TuB04.4, TuB07.2, TuB12.4, TuB12.5, TuB14.3, TuB14.4, TuB19.3, TuC02.4, TuC03.2, TuC05.3, TuC05.5, TuC09.1, TuC11.6, TuC13.1, TuC16.4, WeA03.1, WeA03.4, WeA03.5, WeA05.6, WeA09.3, WeA09.5, WeA12.4, WeA12.5, WeA12.6, WeA14.4, WeA16.1, WeB01.3, WeB04.3, WeB04.4, WeB04.5, WeB08.1, WeB09.5, WeB10.3, WeB10.4, WeB12.3, WeB12.4, WeB13.1, WeB18.1, WeB19.3, WeB19.4, WeC07.2, WeC15.6, WeC17.3, WeC20.4, WeC21.1, WeC21.4

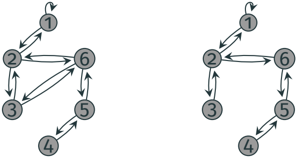
Control system architecture	MoA17.2, MoC07.6, TuA04.5, TuB06.3, TuB12.6, WeA06.2, WeB14.3, WeC05.4 See also Large-scale Systems
Cooperative control	MoA03.3, MoA03.4, MoA03.6, MoA11.6, MoA14.1, MoA14.2, MoA14.3, MoA14.4, MoA14.5, MoB03.1, MoB03.4, MoB05.5, MoB12.6, MoB14.1, MoB14.2, MoB14.3, MoB14.4, MoB14.5, MoB16.5, MoB17.2, MoB17.4, MoC03.6, MoC12.2, MoC14.1, MoC14.2, MoC14.3, MoC14.4, MoC14.5, MoC14.6, MoC17.2, MoSP1.1, TuA03.2, TuA03.3, TuA05.2, TuA09.6, TuA10.6, TuA11.1, TuA12.1, TuA14.4, TuA16.5, TuB04.1, TuB14.5, TuB17.6, TuC05.1, TuC09.2, TuC11.5, TuC11.6, TuC14.2, TuC14.3, WeA03.5, WeA05.2, WeA05.4, WeA05.5, WeA05.6, WeA14.2, WeA14.3, WeA14.5, WeB04.4, WeB12.3, WeB13.1, WeB13.2, WeB14.2, WeB14.3, WeB14.4, WeB14.5, WeC14.1, WeC20.4

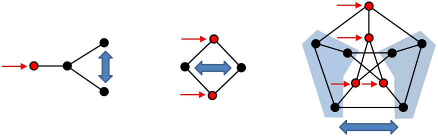
The network approach to systems is here to stay. This tutorial aims to bring to the forefront the role of graphs in these systems.



So far in this tutorial...

- graphs and modelling of network systems
- stability of network systems
- input-output properties of network systems





We are interested in morphisms between

$$\text{(networks/operations)} \quad \Longleftrightarrow \quad \text{(systems/properties)}$$

Our thesis is that for control theoretic methods to have an impact in the growing field of networks, our techniques should be modular, scalable, and offer flexibility in their use.

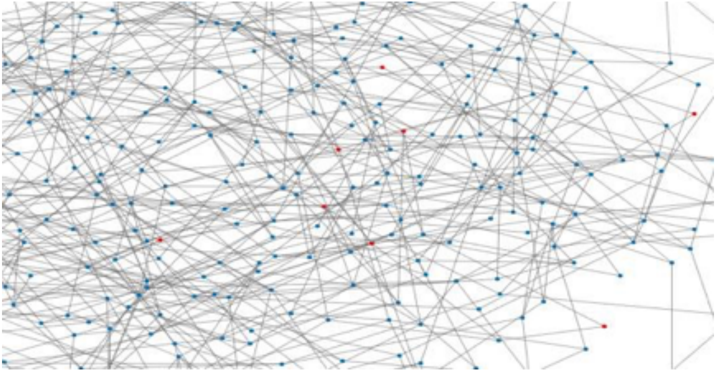
Some areas that have been explored in this direction include:

- structural considerations
- compositional perspective/motifs
- approximations
- randomness

We believe this area is highly unexplored!

Extremal Graphs

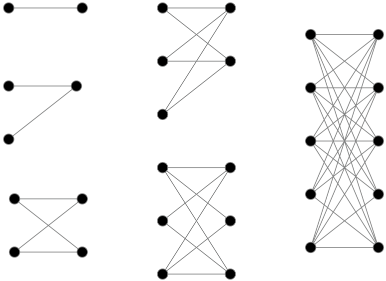
How do we analyze networks that are too large to model?



- fault detection and isolation
- power distribution networks
- transportation networks
- internet-of-things
- cyber-pysical systems
- social networks

Mantel's Theorem (1907)

If a graph $\mathcal{G}$ on n vertices contains no triangles, then it contains at most $\frac{n^2}{4}$ edges.



The complete bipartite graphs are extremal

Extremal graph theory studies how global properties of a graph (i.e., number of edges) relate to local substructures (i.e., a triangle subgraph)

Forbidden Subgraph Problem

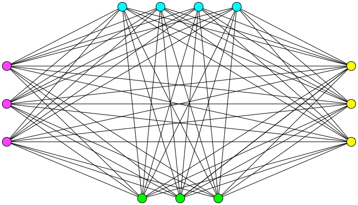
Given a set $\mathbb{H}$ of *forbidden graphs*, what is the maximum number of edges in a graph $\mathcal{G}$ on n nodes (denoted $e(\mathcal{G})$) such that $\mathcal{H} \not\subseteq \mathcal{G}$ for any $\mathcal{H} \in \mathbb{H}$?

Extremal Number $\quad ex(n, \mathcal{G}) = \max_{\mathcal{H} \not\subseteq \mathcal{G}} e(\mathcal{G})$

Generalize Mantel's Theorem for $\mathcal{K}_r$

Túran Graphs $T(n, r)$ - complete r-partite graphs with n vertices

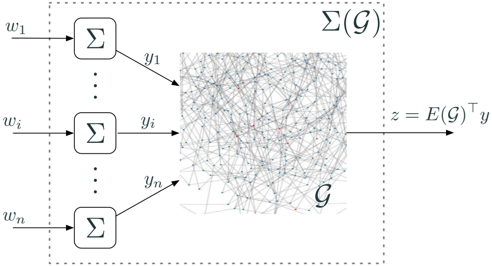
$$e(n, \mathcal{K}_r) \leq \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right)$$



$T(13, 4)$

- avoiding paths of length k
- avoiding Hamiltonian cycles
- avoiding even length cycles
- avoiding edge disjoint cycles

A simple example...



A relative sensing network

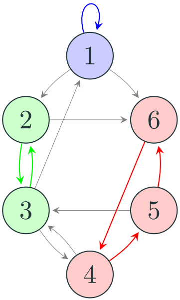
$$\|\Sigma(\mathcal{G})\|_2^2 = 2|\mathcal{E}|\|\Sigma\|_2^2$$

Proposition

Let $\Sigma(\mathcal{G})$ be a relative sensing network with n agents such that $\mathcal{G}$ is K_{r+1}-free. Then the $\mathcal{H}_2$ performance of $\Sigma(\mathcal{G})$ is at most $n^2 \frac{r1}{r} \|\Sigma\|_2^2$.

RECALL: K-DECOMPOSITIONS

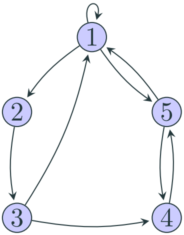
- k-cycle in $\mathcal{G}$: a sequence of k distinct nodes connected by edges.
- Two cycles are disjoint if they have no nodes in common.
- k-decomposition in $\mathcal{G}$: union of *disjoint cycles* covering k nodes.
  A k-decomposition is given by cycles $S_1, \dots, S_l$ if the S_i are disjoint and $|S_1| + \cdots + |S_l| = k$.
- Hamiltonian cycle (resp. decomposition): n-cycle (resp. decomposition).



1-cycle = (1)
2-cycle: (23)
3-cycle: (456)
3-decomp.: $(1)(23)$ or (456)
4-decomp.: $(1)(456)$
5-decomp.: $(23)(456)$

Theorem[4]

A digraph $\mathcal{G}$ is stable only if it contains a k-decomposition for each $k = 1, 2, \dots, n$



$$\begin{bmatrix} * & * & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & * & 0 \end{bmatrix}$$

An extremal question

What is the maximum number of edges in a graph $\mathcal{G}$ on n nodes before a k-decomposition appears?

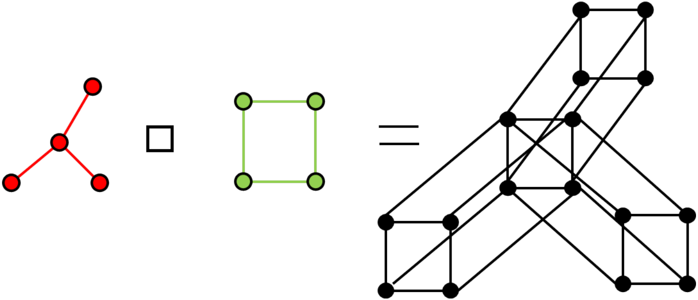
[4]B. "Sparse Stable Systems", Systems and Control Letters, 2013

Composite Networks

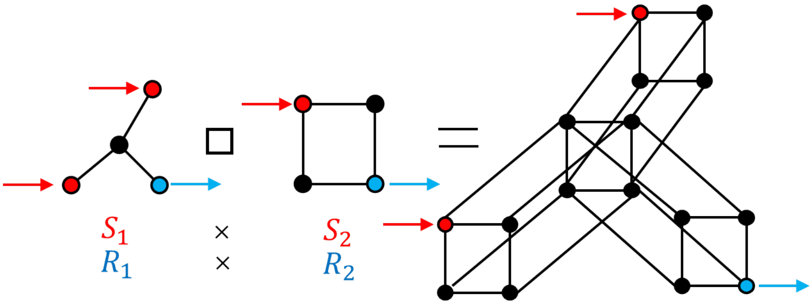
- let $\mathcal{P}$ be a system theoretic property, $\mathbf{G}$ be a class of graphs, and consider $\mathcal{P}(\mathbf{G})$
- consider a subset of $\mathbf{G}$ and examine how $\mathcal{P}$ varies over this subset
- impose algebraic operations on $\mathbf{G}$ and examine how $\mathcal{P}$ behaves with respect to this algebra
- make $\mathbf{G}$ a semi-lattice and examine how the ordering on $\mathbf{G}$ is reflected on $\mathcal{P}$

CASE IN POINT: COMPOSITE NETWORKS

Controllability of the product networks?



INPUT AND OUTPUT SET PRODUCT



Theorem 1: Product Controllability

The dynamics

$$\dot{x}(t) = -A(\prod_{\square} \mathcal{G}_i)x(t) + B(\prod_{\times} S_i)u(t)$$
$$y(t) = C(\prod_{\times} R_i)x(t)$$

where $A(\prod_{\square}\mathcal{G}_i)$ has simple eigenvalues is controllable/observable if and only if

$$\dot{x}_i(t) = -A(\mathcal{G}_i)x_i(t) + B(S_i)u_i(t)$$
$$y_i(t) = C(R_i)x_i(t)$$

is controllable/observable for all i.

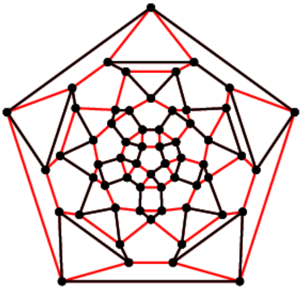
CONCLUSIONS

Graph Theory

- Algebraic graph theory
- Geometric graph theory
- Extremal graph theory
- Probabilistic graph theory
- Topological graph theory

Systems Theory

- Stability
- Performance
- Input-Output Properties
- Control Synthesis
- Control Architectures



Thank you!