SYMMETRY-FORCED FORMATION CONTROL

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Cooperative Networks
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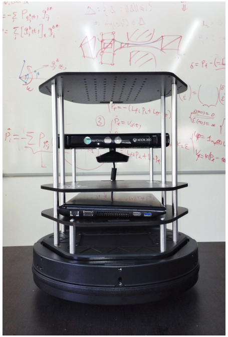
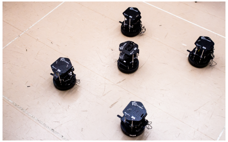
Rigidity in Action
April 7, 2024

FORMATION CONTROL OF ROBOTS

WHAT IS GOING ON?



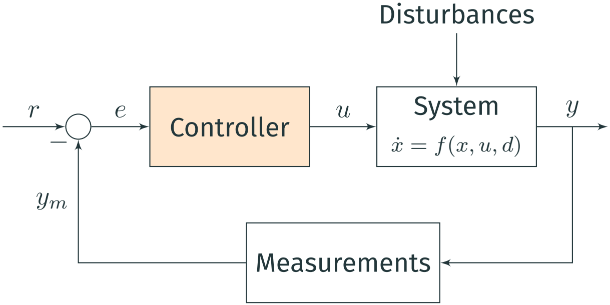


- How do I autonomously control robots?
- How do I control networks of robots?
- What should I consider?
  - who talks to whom?
  - who defines team mission?
  - how do sensors impact solutions?
  - . . .

What is the correct mathematical language for studying the coordination of teams of autonomous systems?

- Control Theory
- Graph Theory
- Optimization Theory

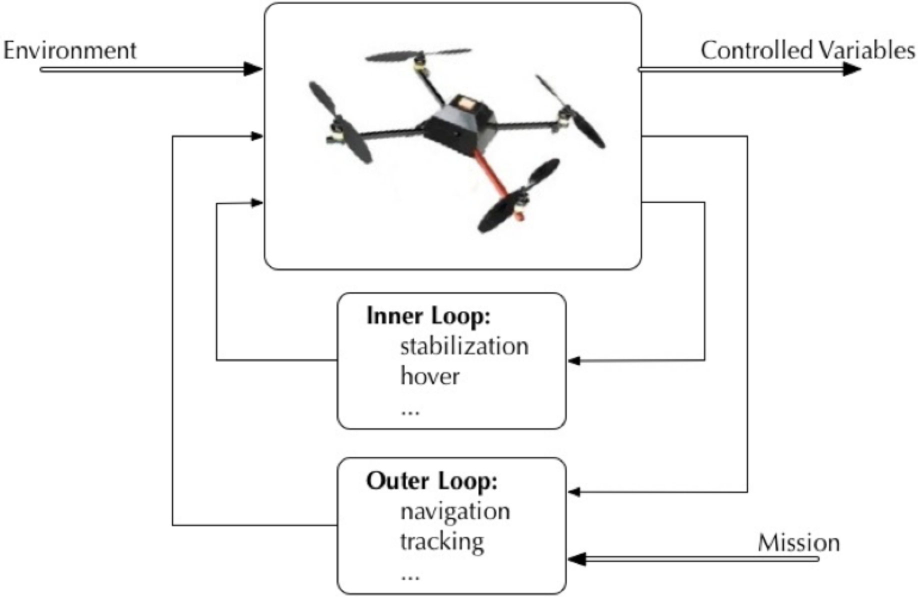
A CLASSIC CONTROL SYSTEM



A control systems engineer aims to design a controller that ensures the closed-loop system

- is stable
- satisfies some performance criteria

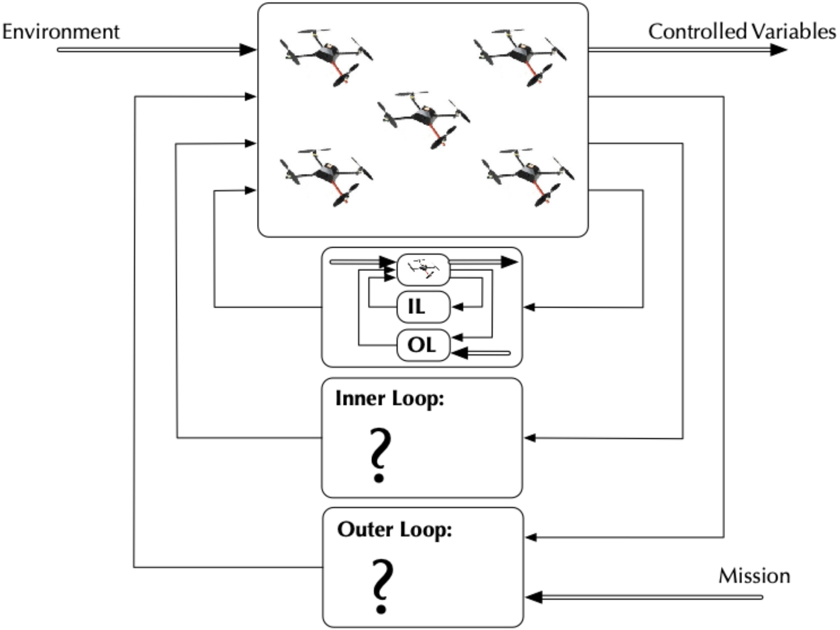
CONTROL ARCHITECTURES



Inner Loop:
stabilization
hover
...

Outer Loop:
navigation
tracking
...

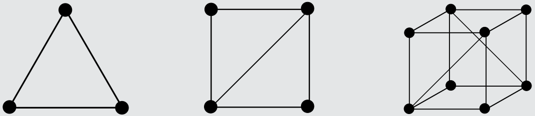
TOWARDS A MULTI-AGENT ARCHITECTURE



FORMATION CONTROL

Formation Control Objective

Given a team of robots endowed with the ability to sense/communicate with neighboring robots, design a control for each robot using only local information that moves the team into a desired spatial configuration.



FORMATION CONSTRAINTS

- The desired formation is characterized by a set of M constraints, encoded in the function $F : \mathbb{R}^{nd} \to \mathbb{R}^{M}$, and a configuration $\mathbf{p}^{\star}$ satisfying the constraints.
- The set of all feasible formations is

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \,|\, F(p) = F(\mathbf{p}^{\star})\}$$

Formation Control Objective

For an ensemble of n agents with dynamics

$$\dot{p}_i = u_i,$$

with $p_i(t) \in \mathbb{R}^d$, an information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and formation constraint function $F : \mathbb{R}^{nd} \to \mathbb{R}^{M}$, design a distributed control law for each agent $i \in \{1, \dots, n\}$ such that

$$\lim_{t \to \infty} p(t) \in \mathcal{F}(p),$$

i.e., $\mathcal{F}(p)$ is asymptotically stable.

DISTRIBUTED CONTROL

- Agents are represented by the nodes in a graph

$$i \in \mathcal{V} \mapsto x_i(t)$$

- Dynamics of each agent

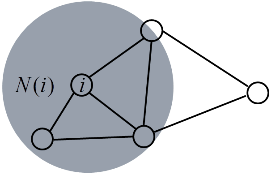
$$\dot{x}_i(t) = f(x_i(t), u_i(t))$$

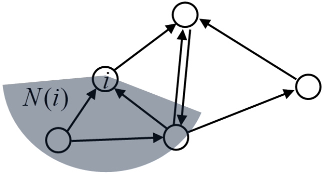
- Agent i acquires information from the set of its neighbors $\mathcal{N}_i$

$$\mathcal{I}_i(t) = \{x_j(t) \,|\, j \in \mathcal{N}_i \cup \{i\}\}$$

- control $u_i(t)$ is distributed if

$$u_i(t) \equiv u_i(\mathcal{I}_i(t))$$





A GRADIENT FLOW APPROACH

- Define a formation potential function

$$F_f(p) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^2 \right)^2$$

    - distance errors on each edge of graph

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  - distance errors on each edge of graph
- proposed control:

$$u_i = -\frac{\partial F_f(p)}{\partial p_i} = \sum_{ij\in\mathcal{E}}(\|p_i - p_j\|^2 - \mathbf{d}_{ij}^2)(p_j - p_i)$$

  - a distributed control!

RIGIDITY THEORY AND FORMATION CONTROL

Theorem - Distance Constrained Formation Control **[Krick 2009]**

Consider the potential function

$$F_f(p) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^2 \right)^2$$

and assume the desired distances $\mathbf{d}_{ij}$ correspond to a feasible formation. Then the gradient dynamical system

$$u_i = -\nabla_{p_i} F_f(p) = \sum_{ij \in \mathcal{E}} \left(\|p_i - p_j\|^2 - \mathbf{d}_{ij}^2 \right) (p_j - p_i)$$

asymptotically converges to the critical points of the potential function, i.e., $\frac{\partial F_f(p)}{\partial p} = 0$.

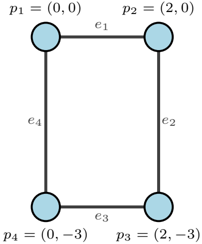
A NOTE ON FORMATION POTENTIALS AND RIGIDITY THEORY

$$F_f(p) = \frac{1}{4}\sum_{ij\in\mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^{\star})^2\right)^2$$

- formation potential can be written in terms of a rigidity function

$$\boxed{F_f(p) = \frac{1}{2}\|r_{\mathcal{G}}(p) - r_{\mathcal{G}}(\mathbf{p})\|^2}$$

  - $r_{\mathcal{G}} : p \mapsto \begin{bmatrix}\cdots & \frac{1}{2}\|p_i - p_j\|^2 & \cdots\end{bmatrix}^T$: distances between neighbors
  - $\mathbf{p}$: a configuration satisfying distance constraints (i.e., $\|\mathbf{p}_i - \mathbf{p}_j\|^2 = \mathbf{d}_{ij}^2$)



$$r_{\mathcal{G}}(p) = \begin{bmatrix}\|p_1 - p_2\|^2 \\ \|p_2 - p_3\|^2 \\ \|p_3 - p_4\|^2 \\ \|p_4 - p_1\|^2\end{bmatrix} = \begin{bmatrix}4 \\ 9 \\ 4 \\ 9\end{bmatrix}$$

A NOTE ON FORMATION POTENTIALS AND RIGIDITY THEORY

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  - $\mathbf{p}$: a configuration satisfying distance constraints (i.e., $\|\mathbf{p}_i - \mathbf{p}_j\|^2 = \mathbf{d}_{ij}^2$)
- rigidity theory looks for distance-preserving infinitesimal motions

$$\boxed{r_{\mathcal{G}}(p + \delta p) = r_{\mathcal{G}}(p) + \frac{\partial r_{\mathcal{G}}(p)}{\partial p}\delta p + \text{h.o.t}}$$

  - infinitesimal motions satisfy $\frac{\partial r_{\mathcal{G}}(p)}{\partial p}\delta p = 0$
  - the Rigidity matrix : $R(p) = \frac{\partial r_{\mathcal{G}}(p)}{\partial p} \in \mathbb{R}^{|\mathcal{E}|\times 2|\mathcal{V}|}$
  - ”rigid body” rotations and translations are always distance preserving: trivial motions
  - A framework $(\mathcal{G}, p)$ is infinitesimally rigid if the only infinitesimal motions are trivial

RIGIDITY THEORY AND FORMATION CONTROL

our formation control

$$u_i = -\nabla_{p_i} F_f(p) = \sum_{ij \in \mathcal{E}} \left(\|p_i - p_j\|^2 - \mathbf{d}_{ij}^2 \right) (p_j - p_i)$$

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can be expressed with rigidity matrix

$$\boxed{u = -R^T(p)(R(p)p - \mathbf{d}^2)}$$

RIGIDITY THEORY AND FORMATION CONTROL

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can be expressed with rigidity matrix

$$\boxed{u = -R^T(p)(R(p)p - \mathbf{d}^2)}$$

a proof sketch

- define error dynamics for distance error: $e = R(p)p - \mathbf{d}^2$

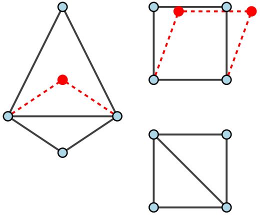
$$\dot{e} = -R(p)R^T(p)e$$

- Construct a Lyapunov function $V(e) = \frac{1}{2}\|e\|^2$
- $\frac{d}{dt}V(e) = -e^T R(p)R(p)^T e \leq 0$
    - when $R(p)R^T(p) > 0$, we have (local) exponential convergence to desired formation
    - good frameworks are i) infinitesimally rigid, and ii) full row-rank (isostatic farmeworks)

RIGIDITY THEORY AND FORMATION CONTROL

Rigidity theory helps us understand

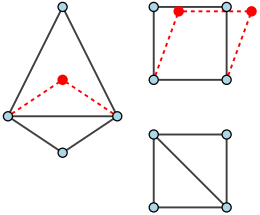
- how many constraints are required to ensure uniqueness of formation shape (modulo translations, rotations, and flip ambiguities)
- how the constraints should be distributed in the network



A widely accepted architectural requirement for distance constrained formation control is that isostatic frameworks are required. Equivalent to:

$$\operatorname{rk} R(p) = 2|\mathcal{V}| - 3 \text{ and } |\mathcal{E}| = 2|\mathcal{V}| - 3 \quad \text{(in } \mathbb{R}^2\text{)}$$

RIGIDITY THEORY AND FORMATION CONTROL



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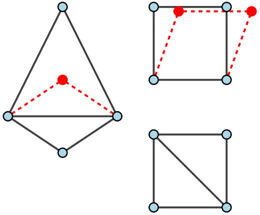

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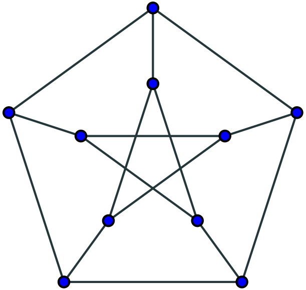
Q: is this a necessary condition? (can we solve the problem with fewer edges?)

A: Impose additional symmetry constraints without requiring more information exchange (in fact, less!)

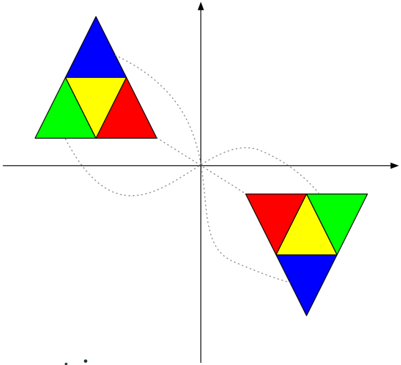
GRAPH SYMMETRIES AND POINT GROUPS

Graph Symmetries



- graph automorphisms

Point Groups



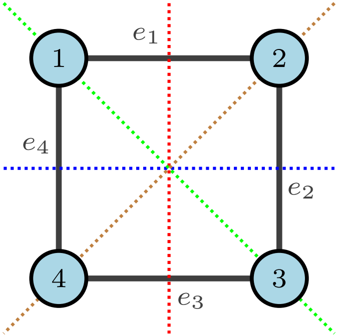
- isometries

SYMMETRY AND GRAPH AUTOMORPHISMS

Graph Automorphism

An automorphism of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of of its vertex set such that

$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$



Automorphisms encode graph symmetries

- identity: $\mathrm{Id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$
- $90°$ rotation: $\psi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
- $180°$ rotation: $\psi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
- $270°$ rotation: $\psi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

AUTOMORPHISM GROUP

Automorphisms of a graph form a *group* - $\mathrm{Aut}(\mathcal{G})$

- $\mathrm{Aut}(\mathcal{G}) = \{\mathrm{Id}, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\}$

A subgroup is a subset of a group, and also satisfies all properties of a group

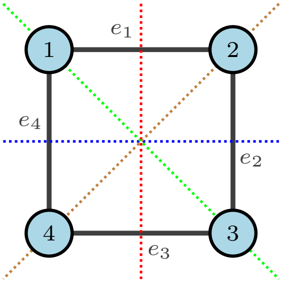
- $\{\mathrm{Id}, \psi_1, \psi_2, \psi_3\}$
- $\{\mathrm{Id}, \psi_2, \psi_4, \psi_5\}$
- $\{\mathrm{Id}, \psi_2\}$
- $\{\mathrm{Id}, \psi_6\}$
- $\{\mathrm{Id}, \psi_7\}$

- Subgroups of $\mathrm{Aut}(\mathcal{G})$ define specific symmetries in $\mathcal{G}$
- for any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$, we say that $\mathcal{G}$ is Γ-symmetric

Γ-SYMMETRIC GRAPHS

Definition

For a Γ-symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and vertex $i \in \mathcal{V}$, the set $\Gamma_i = \{\gamma(i) \,|\, \gamma \in \Gamma\}$ is called the vertex orbit of i. Similarly, for an edge $e = ij \in \mathcal{E}$, the set $\Gamma_e = \{\gamma(i)\gamma(j) \,|\, \gamma \in \Gamma\}$ is termed the edge orbit of e.



Consider $\Gamma = \{\mathrm{Id}, \psi_2\}$ (ψ_2 is the $180°$ rotation)

- **Vertex Orbit:**

  $\Gamma_1 = \Gamma_3 = \{1, 3\}, \; \Gamma_2 = \Gamma_4 = \{2, 4\}$

  vertices inside a vertex orbit are equivalent

  representative vertex set: $\mathcal{V}_0 = \{1, 2\}$

- **Edge Orbit:**

  $\Gamma_{e_1} = \Gamma_{e_3} = \{e_1, e_3\},$

  $\Gamma_{e_2} = \Gamma_{e_4} = \{e_2, e_4\}$

  representative edge set: $\mathcal{E}_0 = \{e_1, e_2\}$

$\tau(\Gamma)$-SYMMETRIC FRAMEWORK

combine notions of graph symmetries with point groups

- let $\mathcal{G}$ be a Γ-symmetric graph
- Γ also represented as a *point group*
    - a set of isometries that preserve symmetries
    - homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$
    - τ assigns an orthogonal matrix (describing an isometry of $\mathbb{R}^d$ such as a rotation or reflection) to each element of Γ

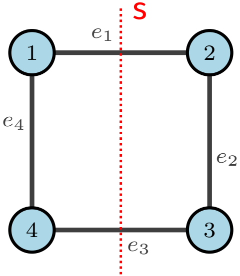
Definition

A framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called $\tau(\Gamma)$-symmetric if

$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}.$$

$\tau(\Gamma)$-SYMMETRIC FRAMEWORK

example...



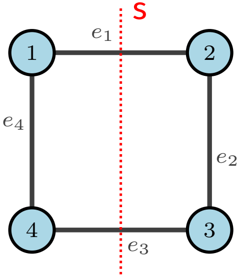
- consider $\Gamma = \{\mathrm{Id}, \psi_4\} \subseteq \mathrm{Aut}(\mathcal{G})$
- $\gamma = \psi_4 \in \Gamma$ (reflection about mirror S)
- isometry $\tau(\gamma) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\tau(\gamma)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix}$

satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$.

- note: for a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ and for every $j \in \Gamma_i$, there is a $\gamma_j \in \Gamma$ such that $\tau(\gamma_j)p_j = p_i$ for all $j \in \Gamma_i$

$\tau(\Gamma)$-SYMMETRIC FRAMEWORK

example...



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- note: for a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ and for every $j \in \Gamma_i$, there is a $\gamma_j \in \Gamma$ such that $\tau(\gamma_j)p_j = p_i$ for all $j \in \Gamma_i$

isometries of configuration p coincide with symmetries of the automorphisms of $\mathcal{G}$

- in $\tau(\Gamma)$-symmetric frameworks, the configurations p are in a special geometric position (not necessarily generic)
- symmetry can lead to unexpected infinitesimal flexibility/rigidity

SYMMETRIC RIGIDITY

Definition

An infinitesimal motion u of a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric* if

$$\tau(\gamma)(u_i) = u_{\gamma(i)} \qquad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}.$$

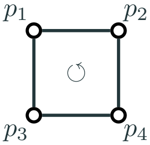
We say that $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric infinitesimally rigid* if every $\tau(\Gamma)$-symmetric infinitesimal motion is trivial.

- recall that infinitesimal motions are in the kernel of the rigidity matrix

$$R(p)\delta p = 0$$

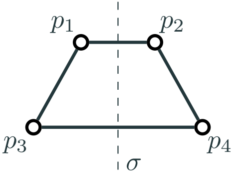
- we can find a subspace of the kernel that is isomorphic to the space of ‘fully-symmetric’ infinitesimal motions
- velocity assignments to the points of $(\mathcal{G}, p)$ that exhibit exactly the same symmetry as the configuration p

SYMMETRIC RIGIDITY



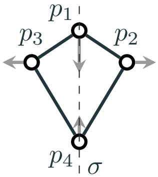
(a)

- $\mathcal{C}_{4v}$-symmetric (and hence $\tau(\Gamma)$-symmetric for any subgroup $\tau(\Gamma)$ of $\mathcal{C}_{4v}$)
- $\tau(\Gamma)$-symmetric infinitesimally rigid



(b)

- $\mathcal{C}_s$-symmetric (with respect to the reflection σ)
- $\tau(\Gamma)$-symmetric infinitesimally rigid



(c)

- $\mathcal{C}_s$-symmetric (with respect to the reflection σ) with a non-trivial $\mathcal{C}_s$-symmetric infinitesimal motion
- $\tau(\Gamma)$-symmetric infinitesimally flexible

SYMMETRIC CONFIGURATION FORMATION CONTROL

Symmetric Formation Control Objective

Consider a group of n integrator agents that interact over the Γ-symmetric sensing graph $\mathcal{G}$. Let $\mathbf{p} \in \mathbb{R}^{dn}$ be a configuration such that $(\mathcal{G}, \mathbf{p})$ is $\tau(\Gamma)$-symmetric for some desired point group $\tau(\Gamma)$, and let $\mathcal{V}_0$ be a set of representatives of the vertex orbits of $\mathcal{G}$ under Γ. Design a control $u_i(t)$ for each agent i such that

(i) $\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = \|\mathbf{p}_i - \mathbf{p}_j\| = \mathbf{d}_{ij}$ for all $ij \in \mathcal{E}$; (distance constraints)

(ii) $\lim_{t\to\infty} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\| = 0$ for all $u, v \in \Gamma_i$, $i \in \mathcal{V}_0$. (symmetry constraints)

A GRADIENT APPROACH

- the formation potential

$$F_f(p(t)) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^2 \right)^2$$

A GRADIENT APPROACH

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- the symmetry potential

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu}) p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

A GRADIENT APPROACH

- the formation potential

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- the symmetry potential

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Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the symmetric formation potential

$$F(p(t)) = F_f(p(t)) + F_s(p(t))$$

A GRADIENT APPROACH

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

A GRADIENT APPROACH

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- closed-loop dynamics

$$\dot{p}(t) = -R(p(t))^T \left(R(p(t))p(t) - \mathbf{d}^2\right) - Qp(t)$$

where Q is symmetric and a block-diagonal matrix with

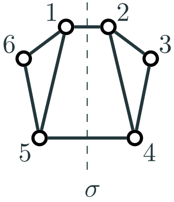
$$[Q_i]_{uv} = \begin{cases} d_{\Gamma_i}(u)I, & u = v,\, u \in \Gamma_i \\ -\tau(\gamma_{uv}), & uv \in \mathcal{E}, u, v \in \Gamma_i \\ 0, & \text{o.w.} \end{cases}.$$

- $Q_i \in \mathbb{R}^{|\Gamma_i|d \times |\Gamma_i|d}$
- $[Q]_{uv} \in O(\mathbb{R}^d)$ (orthogonal group)
- $\tau(\gamma_{uv})^{-1} = \tau(\gamma_{uv})^T$

  - Q_i has a decomposition $Q_i = E(\Gamma_i)E(\Gamma_i)^T$
  - $Q = \bar{E}(\Gamma)\bar{E}(\Gamma)^T$
  - any p in a symmetric position satisfies $Qp = 0$

“NICE” GRAPHS

- symmetric formation potential makes no assumption on relation between the graph $\mathcal{G}$ and the point group $\tau(\Gamma)$
- we restrict our study to graphs where communication required by symmetric potential use same edges as $\mathcal{G}$



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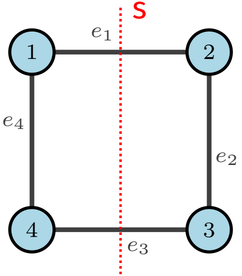


- $\Gamma = \{\mathrm{Id}, \psi_4\} \subseteq \mathrm{Aut}(\mathcal{G})$
- $\Gamma_1 = \Gamma_2 = \{1, 2\},\ \Gamma_3 = \Gamma_4 = \{3, 4\}$
- $\mathcal{V}_0 = \{1, 4\}$
- isometry $\tau(\gamma) : (a, b) \mapsto (-a, b)$

satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$ and
for each $i \in \mathcal{V}_0$ and $j \in \Gamma_i \setminus \{i\}$,
the edge ij is in $\mathcal{E}$ (i.e. $\mathcal{G}(\Gamma_i)$ is connected)

A GRADIENT APPROACH

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- closed-loop dynamics

$$\dot{p}(t) = -R(p(t))^T \left(R(p(t))p(t) - \mathbf{d}^2\right) - Qp(t)$$

- dynamics at for each agent

$$\boxed{\dot{p}_i(t) = \sum_{ij\in\mathcal{E}} (\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^2)(p_j(t) - p_i(t)) + \sum_{\substack{ij\in\mathcal{E}\\ i,j\in\Gamma_u}} (\tau(\gamma_{ij})p_j(t) - p_i(t))}$$

MAIN RESULT 1

Theorem **[Z, Shulze, Tanigawa '23]**

Consider a team of n integrator agents interacting over a Γ-symmetric graph $\mathcal{G}$ satisfying Assumption 1 that can be drawn with maximum point group symmetry $\mathcal{S}$ in $\mathbb{R}^d$, and let

$$\mathcal{F}_f = \{p \in \mathbb{R}^{dn} \,|\, \|p_i - p_j\| = \mathbf{d}_{ij}\; ij \in \mathcal{E}\}, \text{ and } \mathcal{F}_s = \{p \in \mathbb{R}^{dn} \,|\, \tau(\gamma)(p_i) = p_{\gamma(i)}\; \forall\gamma \in \Gamma,\; i \in \mathcal{V}\}.$$

Then for initial conditions $p_i(0)$ satisfying

$$\sum_{ij\in\mathcal{E}} (\|p_i(0) - p_j(0)\| - \mathbf{d}_{ij})^2 \leq \epsilon_1, \text{ and } \|p_i(0) - \tau(\gamma_{ij})p_j(0)\|^2 \leq \epsilon_2$$

for all $i, j \in \Gamma_u$ and $u \in \mathcal{V}_0$, for a sufficiently small and positive constant ϵ_1 and ϵ_2, the control

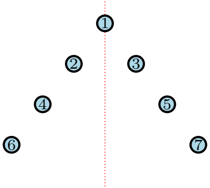
$$u = -\nabla F(p(t)),$$

renders the set $\mathcal{F}_f \cap \mathcal{F}_s$ exponentially stable, i.e.

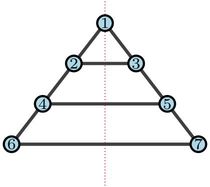
$$\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = \mathbf{d}_{ij} \text{ and } \lim_{t\to\infty} \tau(\gamma)(p_i(t)) = \lim_{t\to\infty} p_{\gamma(i)}(t) \quad \text{for all } \gamma \in \Gamma, i \in \mathcal{V}.$$

EXAMPLE: THE VIC FORMATION

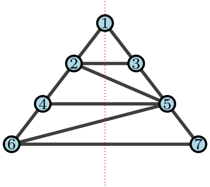
- formation flight for aircraft originated in WWI
- Vic formation used by pilots to improve visual communication and defensive advantages



Vic formation with symmetry mirror

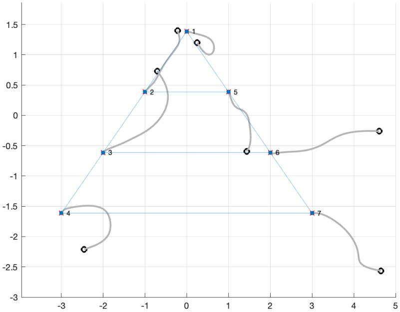


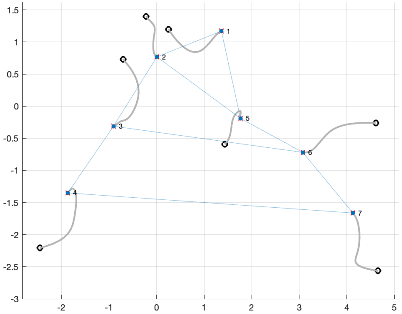
Flexible framework (9 edges; satisfies Assumption 1)



Minimally Rigid framework (11 edges)

EXAMPLE: THE VIC FORMATION





- symmetry constraints force agents to correct formation
- requires less agent communication than standard formation control with MIR requirement

- with flexible framework and only formation potential can not guarantee convergence to correct shape

EXPLOIT MORE SYMMETRY

- proposed strategy does not take advantage of the full power of symmetry

EXPLOIT MORE SYMMETRY

- proposed strategy does not take advantage of the full power of symmetry
- can we find redundant information between the symmetry constraints and the distance constraints?

Γ-SYMMETRIC FRAMEWORK

Definition

An infinitesimal motion u of a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric* if

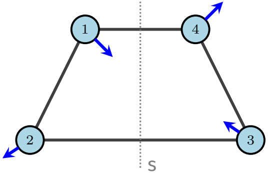
$$\tau(\gamma)(u_i) = u_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}. \tag{1}$$

We say that $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric infinitesimally rigid* if every $\tau(\Gamma)$-symmetric infinitesimal motion is trivial.

infinitesimal motions can also be studied in this framework

- $\tau(\gamma)(u_i) = u_{\gamma(i)}$
- understanding symmetry structure means we only need to find infintesimal motion for one representative vertex in each vertex orbit

EXAMPLE



$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-c, d)^T$
- $p_4 = (-a, b)^T$

Rigidity matrix

$$R(p) = \begin{bmatrix} (a-c \;\; b-d) & (c-a \;\; d-b) & (0 \;\; 0) & (0 \;\; 0) \\ (2a \;\; 0) & (0 \;\; 0) & (0 \;\; 0) & (-2a \;\; 0) \\ (0 \;\; 0) & (2c \;\; 0) & (-2c \;\; 0) & (0 \;\; 0) \\ (0 \;\; 0) & (0 \;\; 0) & (a-c \;\; d-b) & (c-a \;\; b-d) \end{bmatrix}$$

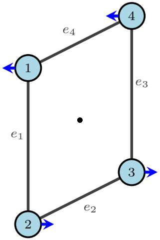
- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by

$(1 \;\; -1 \;\; -1 \;\; \frac{2(c-a)+b-d}{d-b} \;\; -1 \;\; -\frac{2(c-a)+b-d}{d-b} \;\; 1 \; 1)^T$

flex is not symmetric with respect to s

EXAMPLE



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$$R(p) = \begin{bmatrix} (a-c\ b-d) & (c-a\ d-b) & (0\ 0) & (0\ 0) \\ (a+c\ b+d) & (0\ 0) & (0\ 0) & (-a-c\ \ -b-d) \\ (0\ 0) & (0\ 0) & (c-a\ d-b) & (a-c\ b-d) \\ (0\ 0) & (a+c\ b+d) & (-a-c\ \ -b-d) & (0\ 0) \end{bmatrix}$$

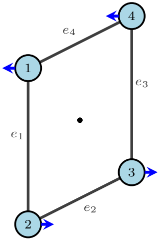
- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by

$(-1\ 0\ \frac{cd-ab}{ad-bc}\ \frac{a^2-c^2}{ad-bc}\ 1\ 0\ -\frac{cd-ab}{ad-bc}\ \ -\frac{a^2-c^2}{ad-bc})^T$

flex is symmetric with respect to $180°$ rotation ($\mathcal{C}_2$)

EXAMPLE



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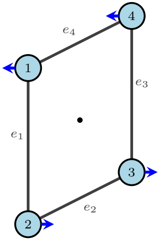
- 180° rotation of points corresponds to $\psi_2 \in \mathrm{Aut}(\mathcal{G})$
- recall: vertex orbits : $\{1, 3\}$, $\{2, 4\}$, edge orbits: $\{e_1, e_3\}$, $\{e_2, e_4\}$

EXAMPLE



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symmetries make certain rows and columns of the rigidity matrix redundant

ORBIT RIGIDITY MATRIX

symmetries make certain rows and columns of the rigidity matrix redundant

$$R(p) = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 = \psi_2(1) & 4 = \psi_2(2) \end{array} \\ \begin{array}{c} e_1 \\ e_4 \\ \psi_2(e_1) \\ \psi_2(e_4) \end{array} & \begin{pmatrix} (a-c\ \ b-d) & (c-a\ \ d-b) & (0\ \ 0) & (0\ \ 0) \\ (a+c\ \ b+d) & (0\ \ 0) & (0\ \ 0) & (-a-c\ \ -b-d) \\ (0\ \ 0) & (0\ \ 0) & (c-a\ \ d-b) & (a-c\ \ b-d) \\ (0\ \ 0) & (a+c\ \ b+c) & (-a-c\ \ -b-d) & (0\ \ 0) \end{pmatrix} \end{array}$$

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Orbit Rigidity Matrix

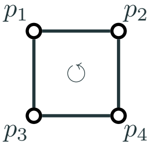
$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} e_1 \\ e_4 \end{array} & \begin{pmatrix} (p_1 - p_2)^T & (p_2 - p_1)^T \\ (p_1 - \psi_2(p_2))^T & (p_2 - \psi_2^{-1}(p_1))^T \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} (a-c,\ b-d) & (c-a,\ d-b) \\ (a+c,\ b+d)) & (c+a,\ d+b) \end{pmatrix} \end{array}$$

- 2 rows - one for each representative of edge orbits under action of ψ_2
- 4 columns - nodes p_1, p_2 each have two dof; nodes $p_3 = \psi_2(p_1)$ and $p_4 = \psi_2(p_2)$ are uniquely determined by the symmetries

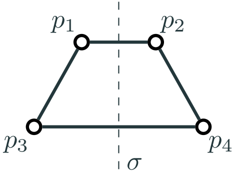
QUOTIENT GAIN GRAPHS

- relation between vertices within vertex orbits and between vertex orbits (through edge orbits) captured by quotient gain graph of a Γ-symmetric graph
  - node set is representative vertex set $\mathcal{V}_0$
  - edge set is representative edge set $\mathcal{E}_0$: choose edge of form $i\gamma(j)$ with $i, j \in \mathcal{V}_0$
    - it is ok for $i = j$
    - edges are directed with ‘edge gain’ being the group action $\gamma \in \Gamma$

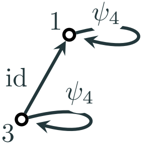
QUOTIENT GAIN GRAPHS

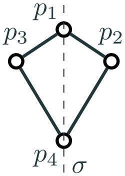


- $\Gamma = \{\mathrm{Id}, \psi_1\}$ (rotation)
- $\Gamma_i = \{1, 2, 3, 4\}$
- $\mathcal{V}_0 = \{1\}$, $\mathcal{E}_0 = \{e_1\}$

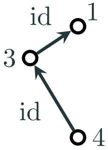


- $\Gamma = \{\mathrm{Id}, \psi_4\}$ (reflection)
- $\Gamma_{1,2} = \{1, 2\}$, $\Gamma_{3,4} = \{3, 4\}$
- $\mathcal{V}_0 = \{1, 3\}$,
  $\mathcal{E}_0 = \{12,\ 13,\ 24\}$





- $\Gamma = \{\mathrm{Id}, \psi_6\}$ (reflection)
- $\Gamma_1 = \{1\}$, $\Gamma_4 = \{4\}$,
  $\Gamma_{2,3} = \{2, 3\}$
- $\mathcal{V}_0 = \{1, 3, 4\}$, $\mathcal{E}_0 = \{13,\ 14\}$



ORBIT RIGIDITY MATRIX

Definition [Shulze 2011]

The orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, \bar{p})$ of $(\mathcal{G}, p)$ is the $|\mathcal{E}_0| \times d|\mathcal{V}_0|$ matrix defined as follows. The row corresponding to an edge $((i,j);\gamma)$, where $i \neq j$, has the form:

$$\begin{pmatrix} 0\cdots0 & (\bar{p}_i - \tau(\gamma)\bar{p}_j)^T & 0\cdots0 & (\bar{p}_j - \tau(\gamma)^{-1}\bar{p}_i)^T & 0\cdots0 \end{pmatrix},$$

with the d-dimensional entries $(\bar{p}_i - \tau(\gamma)\bar{p}_j)^T$ and $(\bar{p}_j - \tau(\gamma)^{-1}\bar{p}_i)^T$ being in the columns corresponding to vertex i and j, respectively. The row corresponding to a loop $((i,i);\gamma)$ has the form:

$$\begin{pmatrix} 0\cdots0 & (2\bar{p}_i - \tau(\gamma)\bar{p}_i - \tau(\gamma)^{-1}\bar{p}_i)^T & 0\cdots0 \end{pmatrix},$$

with the d-dimensional entry $(2\bar{p}_i - \tau(\gamma)\bar{p}_i - \tau(\gamma)^{-1}\bar{p}_i)^T$ being in the columns corresponding to vertex i.

ORBIT RIGIDITY MATRIX

Theorem [Shulze 2011]

Let $(\mathcal{G}, p)$ be a $\tau(\Gamma)$-symmetric framework with orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, \bar{p})$. Then,

(i) the kernel of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ is isomorphic to the space of $\tau(\Gamma)$-symmetric infinitesimal motions of $(\mathcal{G}, p)$, and

(ii) the cokernel of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ is isomorphic to the space of $\tau(\Gamma)$-symmetric self-stresses of $(\mathcal{G}, p)$.

- Orbit rigidity matrix can be used to identify symmetric infinitesimal flexes
- full-rank $\mathcal{O}(\mathcal{G}_0, \bar{p})$ implies none exist
- size of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ does not depend on p, but only the graph and symmetry constraints
- $\tau(\Gamma)$-isostatic frameworks have orbit rigidity matrices with full row-rank

ORBIT RIGIDITY MATRIX

key point: quotient gain graph and orbit rigidity matrix suggests a further way to exploit symmetry in formation control

- representative edges used to maintain distances
- symmetry within vertex orbits have no need for distance constraints

A MODIFIED FORMATION POTENTIAL

- the representative edge formation potential

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij \in \mathcal{E}_0} \left(\|p_i - \tau(\gamma) p_j\|^2 - \mathbf{d}^2_{i\gamma(j)} \right)^2$$

  - γ is label of edge in quotient gain graph

A MODIFIED FORMATION POTENTIAL

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  - γ is label of edge in quotient gain graph
- the symmetry potential

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu}) p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

A MODIFIED FORMATION POTENTIAL

- the representative edge formation potential

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$$F_s(p(t)) = \frac{1}{2} \sum_{i\in\mathcal{V}_0} \sum_{\substack{u,v\in\Gamma_i\\ uv\in\mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the symmetric formation potential

$$F(p(t)) = F_e(p(t)) + F_s(p(t))$$

A MODIFIED FORMATION CONTROL

- node relabeling - representative vertices first

$$\tilde{p} = Pp = \begin{bmatrix} p_o^T & p_f^T \end{bmatrix}^T$$

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

A MODIFIED FORMATION CONTROL

Then the control for each agent $i \in \mathcal{V}_0$ can be expressed as

$$u_i(t) = u_i^{(a)}(t) + u_i^{(b)}(t) + u_i^{(c)}(t),$$

where

$$\begin{aligned}
u_i^{(a)}(t) &= \sum_{\substack{i\gamma(j)\in\mathcal{E}_0 \\ j\in\mathcal{V}_0,\, i\neq j}} \big(\|p_i(t) - \tau(\gamma)p_j(t)\|^2 - \mathbf{d}_{ij}^2\big)(\tau(\gamma)p_j(t) - p_i(t)) \\
u_i^{(b)}(t) &= \sum_{i\gamma(i)\in\mathcal{E}_0} (\|(I - \tau(\gamma))p_i\|^2 - \mathbf{d}_{i\gamma(i)}^2)(2I - \tau(\gamma) - \tau(\gamma)^{-1})p_i \\
u_i^{(c)}(t) &= \sum_{ij\in\mathcal{E}(\Gamma_i)} (\tau(\gamma_{ij})p_j(t) - p_i(t)).
\end{aligned}$$

The control for the agents in $\mathcal{V} \setminus \mathcal{V}_0$ is simply

$$u_i(t) = \sum_{ij\in\mathcal{E}(\Gamma_u)} (\tau(\gamma_{ij})p_j(t) - p_i(t)),$$

for each $u \in \mathcal{V}_0$.

A MODIFIED FORMATION CONTROL

in state-space form

$$\boxed{\begin{bmatrix}\dot p_0(t)\\ \dot p_f(t)\end{bmatrix} = \begin{bmatrix}-\mathcal{O}^T(\mathcal{G}_0, p_0(t))\left(\mathcal{O}(\mathcal{G}_0, p_0(t))p_0(t) - \mathbf{d}_0^2\right)\\ 0\end{bmatrix} - PQP^T\begin{bmatrix}p_0(t)\\ p_f(t)\end{bmatrix}}$$

recall our earlier idea

$$\dot p(t) = -R(p(t))^T\left(R(p(t))p(t) - \mathbf{d}^2\right) - Qp(t)$$

MAIN RESULT

we can define an error system with

$$e = \begin{bmatrix} \sigma \\ q \end{bmatrix} = \begin{bmatrix} \mathcal{O}(\mathcal{G}_0, p_0(t))p_0(t) - \mathbf{d}_0^2 \\ \bar{E}(\Gamma)^T P^T p(t) \end{bmatrix}$$

orbit error dynamics

$$\begin{aligned} \begin{bmatrix} \dot{\bar{\sigma}}(t) \\ \dot{\bar{q}}(t) \end{bmatrix} &= -\underbrace{\begin{bmatrix} \mathcal{O}\mathcal{O}^T & \mathcal{O}\bar{E}_0(\Gamma) \\ \bar{E}_0^T(\Gamma)\mathcal{O}^T & \bar{E}^T(\Gamma)\bar{E}(\Gamma) \end{bmatrix}}_{\mathcal{M}} \underbrace{\begin{bmatrix} \bar{\sigma}(t) \\ \bar{q}(t) \end{bmatrix}}_{e(t)} \\ &= -\begin{bmatrix} \begin{bmatrix} \mathcal{O} & 0 \end{bmatrix} \\ \bar{E}^T(\Gamma)P^T \end{bmatrix} \underbrace{\begin{bmatrix} \begin{bmatrix} \mathcal{O}^T \\ 0^T \end{bmatrix} & P\bar{E}(\Gamma) \end{bmatrix} \begin{bmatrix} \bar{\sigma}(t) \\ \bar{q}(t) \end{bmatrix}}_{u(t)}. \end{aligned}$$

MAIN RESULT

Theorem

Let p be the target formation satisfying conditions (i) and (ii) of the Symmetry-Forced Formation Control Problem, and assume that $(\mathcal{G}, \mathbf{p})$ is a $\tau(\Gamma)$-symmetric isostatic framework. Then the origin is a locally exponentially stable equilibrium of the orbit error dynamics.

proof sketch

- Define Lyapunov function $V(e) = \frac{1}{2}e^Te$
- $\frac{d}{dt}V(e) = -e^T\mathcal{M}e \leq 0$
  - $\tau(\Gamma)$-symmetric isostatic framework means $\mathcal{M}$ is positive definite
  - error converges exponentially fast to origin

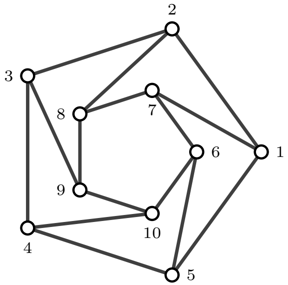
MAIN RESULT

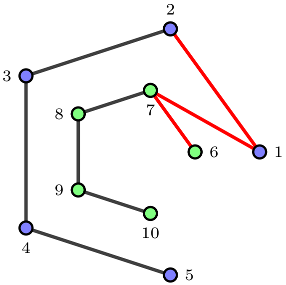
Theorem

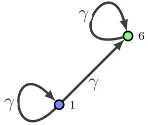
The orbit rigidity control uses at most $(1 + 1/|\Gamma|)|\mathcal{V}|$ edges.

- can be significantly less than $2|\mathcal{V}| - 3$

EXAMPLE





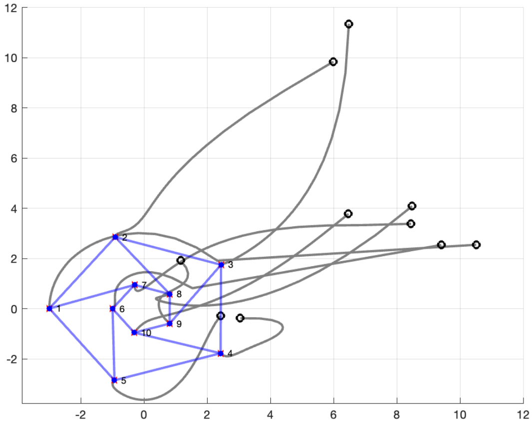


- quotient gain graph

- graph has 15 edges
- at least 17 edges required for infinitesimal rigidity
- flexible framework

- $2\pi/5$ rotational symmetry
- can use only spanning tree subgraph for each vertex orbit
- only 3 distances required

EXAMPLE



- nice...but symmetries are defined with respect to a global origin

CENTROID CONSENSUS

idea: augment a virtual consensus dynamics

$$\begin{bmatrix} \dot{p}_0(t) \\ \dot{p}_f(t) \end{bmatrix} = \begin{bmatrix} -\mathcal{O}^T(\mathcal{G}_0, c_0(t))\left(\mathcal{O}(\mathcal{G}_0, c_0(t))c_0(t) - \mathbf{d}_0^2\right) \\ 0 \end{bmatrix} - PQP^T \begin{bmatrix} c_0(t) \\ c_f(t) \end{bmatrix}$$

$$\dot{r} = -L(\mathcal{G})r$$

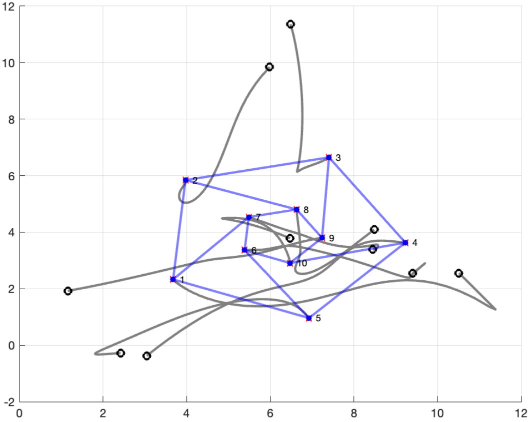
with $c(t) = p(t) - r(t)$

- Laplacian flow

$$r(t) \mapsto e^{-L(\mathcal{G})t} r(0)$$

    - when $\mathcal{G}$ is connected, $r(t) \mapsto \frac{1}{n}(\mathbb{1}^T r(0))\mathbb{1}$
- cascade structure
- same analysis idea

CENTROID CONSENSUS



CONCLUDING REMARKS

Summary

- $\tau(\Gamma)$-symmetric graphs captures symmetry of configurations and graphs
- symmetric formation potential used to design distributed control law with less edges compared to "traditional" formation control strategies
- opportunities for more sophisticated motion coordination

Zelazo, Tanigawa and Shulze, *Forced Symmetric Formation Control*, arXiv 2024.

Future Work

- formation maneuvering requires time-varying point group symmetries
- is it possible to distributedly decide on certain symmetries?
- can we eliminate need for requiring self-state in protocol?
- more?

Questions?