Bearing-only Cyclic Pursuit in 2-D for Capture of Moving Target

Dwaipayan Mukherjee*, Minh Hoang Trinhˆ, Daniel Zelazo*, and Hyo-Sung Ahnˆ
*Aerospace Engineering
Technion- Israel institute of Technology, Haifa-32000
Israel
ˆMechanical Engineering
Gwangju Institute of Science and Technology, Gwangju-500712
Republic of Korea

Background

- Projection matrices used to generate control laws for multi-agent systems sensing over undirected graphs.
- Control makes use of bearing information only.
- Led to rigidity theory, etc.
- Problem of bearing-only formation control over digraph $-$ an open problem!
- Emphasis of current work: Cycle digraph.
- Paradigm closely related to classical cyclic pursuit.
- Control modified to capture moving target.

The Projection Matrix

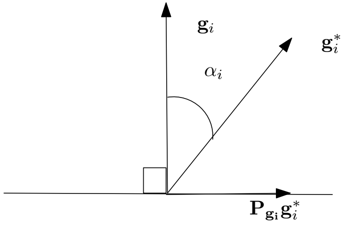


Figure: Effect of Projection Matrix on vectors in $\mathbb{R}^2$

Symmetric, Idempotent, Positive Semi-definite

$\mathbf{P}_{\mathbf{g_i}} = I_2 - \mathbf{g}_i\mathbf{g}_i^T \; : \mathbf{g}_i \in \mathbb{R}^2$ and $\mathbf{g}_i^T\mathbf{g}_i = 1$

$\mathbf{P}_{\mathbf{g_i}} = {\mathbf{P}_{\mathbf{g_i}}}^T = {\mathbf{P}_{\mathbf{g_i}}}^2 = {\mathbf{P}_{\mathbf{g_i}}}^T \geq 0$

$\mathcal{N}(\mathbf{P}_{\mathbf{g_i}}) = span\{\mathbf{g}_i\}$

How is this useful?

- Suppose target is fixed.
- Specific bearing w.r.t. target desired.
- How should an agent move?

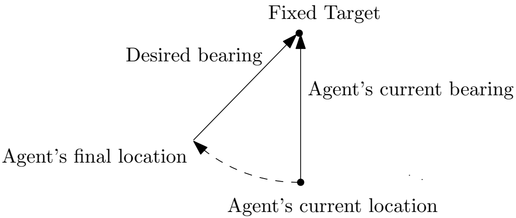


Figure: Movement along circular arc leads to desired bearing

How can Projection Matrices help?

- Choose $\mathbf{g}_i^*$: desired bearing unit vector
  $\mathbf{g}_i$: current bearing unit vector.
- Observation 1: $\mathbf{P}_{\mathbf{g_i}}\mathbf{g}_i^*$ is orthogonal to $\mathbf{g}_i$.
- Observation 2: Tangent is orthogonal to radius at point of intersection.
- $-\mathbf{P}_{\mathbf{g_i}}\mathbf{g}_i^*$: direction of motion.
- Agent will achieve desired bearing.

Bearing-only Cyclic Pursuit

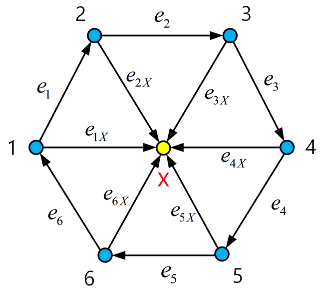


Figure: A group consisting of six agents and a target **X**.

Single Integrator agents

$$\dot{\mathbf{p}}_i = \mathbf{u}_i$$

Aim of present paper

Each agent i must achieve a bearing $\mathbf{g}_i^*$ with respect to $i+1$ and $\mathbf{g}_{iX}^*$ with respect to the target **X**. Both these desired bearings are known to each agent.

Problem

Design $\mathbf{u}_i$ to meet the goal of the paper.

Assumptions and Definitions

Assumption 1

Every agent has access to a global reference frame in $\mathbb{R}^2$. The positions of the agents, $\mathbf{p}_i \in \mathbb{R}^2$, are initially non-collocated, i.e., $\mathbf{p}_i(0) \neq \mathbf{p}_j(0)$, for all $1 \leq i \neq j \leq n$.

Assumption 2

Each agent i senses the bearing vectors with respect to agent $i+1$ and the target **X**. Thus the sensing topology of the agents is a directed cycle graph with n nodes and an additional node whose information is sensed by all other nodes. Additionally, the target's velocity is available to every agent in the group.

Assumptions and Definitions contd.

Unit Bearing Vectors

$$\mathbf{g}_i = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{\|\mathbf{p}_{i+1} - \mathbf{p}_i\|} = \frac{\mathbf{z}_i}{\|\mathbf{z}_i\|}, \ \mathbf{g}_{iX} = \frac{\mathbf{p}_X - \mathbf{p}_i}{\|\mathbf{p}_X - \mathbf{p}_i\|} = \frac{\mathbf{z}_{iX}}{\|\mathbf{z}_{iX}\|},$$

$$\mathbf{g} = [\mathbf{g}_1^T, \ldots, \mathbf{g}_n^T, \mathbf{g}_{1X}^T, \ldots, \mathbf{g}_{nX}^T]^T \in \mathbb{R}^{4n}$$

Feasible Desired Bearing Set

The set $\mathcal{B}_n = \{\mathbf{g}_i^*, \mathbf{g}_{iX}^*\}_{i=1,\ldots,n}$ is called a feasible bearing vector set if and only if the following conditions hold $\forall i$:

(a) $\mathbf{g}_i^* \neq \pm\mathbf{g}_{i+1}^*$, $\mathbf{g}_i^* \neq \pm\mathbf{g}_{iX}^*$, $\mathbf{g}_{i-1}^* \neq \pm\mathbf{g}_{iX}^*$, and there exist positive scalars d_i^* such that $\sum_{i=1}^{n} d_i^* \mathbf{g}_i^* = \mathbf{0}$, and

(b) positive scalars d_{iX}^* exist such that $d_i^* \mathbf{g}_i^* - d_{iX}^* \mathbf{g}_{1X}^* + d_{i+1,X}^* \mathbf{g}_{i+1,X}^* = \mathbf{0}$

More Definitions & a Result

Bearing Equivalency

Frameworks $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ are *bearing equivalent* if $\mathbf{P}_{(\mathbf{p}_i - \mathbf{p}_j)}(\mathbf{p}_i' - \mathbf{p}_j') = \mathbf{0}$ for all $(i,j) \in \mathcal{E}$.

Bearing Congruency

Frameworks $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ are *bearing congruent* if $\mathbf{P}_{(\mathbf{p}_i - \mathbf{p}_j)}(\mathbf{p}_i' - \mathbf{p}_j') = \mathbf{0}$ for all $i, j \in \mathcal{V}$.

Lemma

Under the two assumptions made, given two formations $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ with the graph as described in Assumption 2, if $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ are bearing equivalent, they are also bearing congruent. Moreover, $d_{ij}/d'_{ij} = \eta \in \mathbb{R}$, for all $i,j \in \mathcal{V}$, $i \neq j$.

Idea of the proof

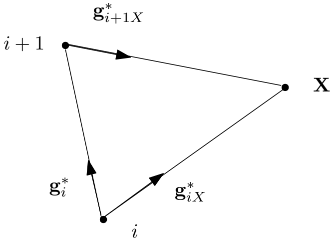


Figure: Similar triangles completely specified by three angles

- Similar triangles differ by scaling factor.
- Total formation: combination of triangles.

Main Results: Observations

Proposed Control Law

$$\mathbf{u}_i = -\mathbf{P}_{\mathbf{g}_i}\mathbf{g}_i^* - \mathbf{P}_{\mathbf{g}_{iX}}\mathbf{g}_{iX}^* + \mathbf{v}_T$$

Some Observations

- In $\mathbb{R}^2$, $\mathbf{P}_{\mathbf{g}_i} = \mathbf{g}_i^\perp(\mathbf{g}_i)^{\perp T}$, where

$$\mathbf{g}_i^\perp = \mathbf{J}\mathbf{g}_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{g}_i$$

  is a unit vector orthogonal to $\mathbf{g}_i$. Similarly, $\mathbf{P}_{\mathbf{g}_{iX}} = \mathbf{g}_{iX}^\perp(\mathbf{g}_{iX})^{\perp T}$
- $\mathbf{z}_i = \mathbf{p}_{i+1} - \mathbf{p}_i$, $\mathbf{z}_{iX} = \mathbf{p}_X - \mathbf{p}_i$, $d_i = \|\mathbf{z}_i\|$, $d_{iX} = \|\mathbf{z}_{iX}\|$, for $i = 1, \ldots, n$.
- $\mathbf{P}_{\mathbf{g}_i}\mathbf{z}_i = d_i\mathbf{P}_{\mathbf{g}_i}\mathbf{g}_i = \mathbf{0}$; $\mathbf{P}_{\mathbf{g}_{iX}}\mathbf{z}_{iX} = d_{iX}\mathbf{P}_{\mathbf{g}_{iX}}\mathbf{g}_{iX} = \mathbf{0}$.

Main Results: Equilibria

Lemma

The cyclic pursuit system with the proposed control law has two types of equilibria which are symmetric about the target's position: the desired equilibrium $\mathbf{p}^*$ corresponding to $\mathbf{g} = \mathbf{g}^*$ and the undesired equilibrium corresponding to $\mathbf{g} = -\mathbf{g}^*$.

Idea of proof:

- At equilibrium:
  $-\mathbf{P}_{\mathbf{g}_i}\mathbf{g}_i^* - \mathbf{P}_{\mathbf{g}_{iX}}\mathbf{g}_{iX}^* = \mathbf{0}$
  $\Rightarrow \mathbf{g}_i^T\mathbf{P}_{\mathbf{g}_{iX}}\mathbf{g}_{iX}^* = 0.$
- Either $\mathbf{g}_i = \pm\mathbf{g}_{iX}$, or $\mathbf{g}_{iX} = \pm\mathbf{g}_{iX}^*$
- Three possibilities arise...
- Two ruled out by contradiction!
- Investigate third

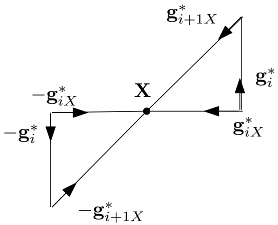


Figure: Triplet of $\mathbf{g}_i^*$, $-\mathbf{g}_{iX}^*$ and $\mathbf{g}_{i+1X}^*$ forming a triangle

Main Results: Equilibria

Sets of Equilibria

$\mathcal{Q} := \{\mathbf{p} \in \mathbb{R}^{2n} | \mathbf{g}_i = \pm\mathbf{g}_i^* \text{ and } \mathbf{g}_{iX} = \pm\mathbf{g}_{iX}^*,\ i = 1, \ldots, n\},$
$\mathcal{D} := \{\mathbf{p} \in \mathbb{R}^{2n} | \mathbf{g}_i = \mathbf{g}_i^* \text{ and } \mathbf{g}_{iX} = \mathbf{g}_{iX}^*,\ i = 1, \ldots, n\},$
$\mathcal{U} := \{\mathbf{p} \in \mathbb{R}^{2n} | \mathbf{g}_i = -\mathbf{g}_i^* \text{ and } \mathbf{g}_{iX} = -\mathbf{g}_{iX}^*,\ i = 1, \ldots, n\}.$

Technique: All variables transformed in terms of angles.

Main Results: System Dynamics

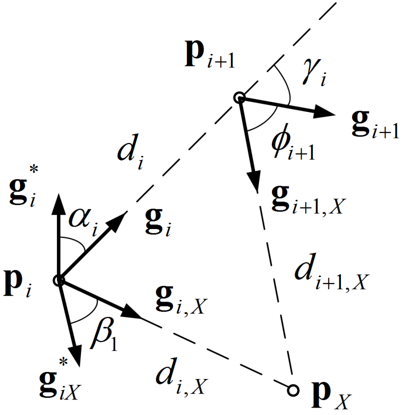


Figure: Various angles defined

β_i-dynamics

$$\cos\beta_i = (\mathbf{g}_{iX}^*)^T \mathbf{g}_{iX}$$
$$\Rightarrow \sin\beta_i \dot{\beta}_i = -(\mathbf{g}_{iX}^*)^T \dot{\mathbf{g}}_{iX}$$
$$= -(\mathbf{g}_{iX}^*)^T \frac{\mathbf{P}_{\mathbf{g}_{iX}}}{d_{iX}} (\dot{\mathbf{p}}_X - \dot{\mathbf{p}}_i)$$
$$= -(\mathbf{g}_{iX}^*)^T \frac{\mathbf{P}_{\mathbf{g}_{iX}}}{d_{iX}} (\mathbf{v}_T - \dot{\mathbf{p}}_i)$$
$$= -(\mathbf{g}_{iX}^*)^T \frac{\mathbf{P}_{\mathbf{g}_{iX}}}{d_{iX}} (-\dot{\tilde{\mathbf{p}}}_i)$$

Use $\mathbf{P}_{\mathbf{g}_{iX}} = \mathbf{g}_{iX}^{\perp}(\mathbf{g}_{iX}^{\perp})^T$

$$d_{iX} \sin\beta_i \dot{\beta}_i =$$
$$-(\mathbf{g}_{iX}^*)^T \mathbf{g}_{iX}^{\perp}(\mathbf{g}_{iX}^{\perp})^T \mathbf{g}_{iX}^{\perp}(\mathbf{g}_{iX}^{\perp})^T \mathbf{g}_{iX}^*$$
$$-(\mathbf{g}_{iX}^*)^T \mathbf{g}_{iX}^{\perp}(\mathbf{g}_{iX}^{\perp})^T \mathbf{g}_{i}^{\perp}(\mathbf{g}_{i}^{\perp})^T \mathbf{g}_{i}^*$$
$$= -\sin^2\beta_i +$$
$$(\pm\sin\beta_i)(\cos\phi_i)(\pm\sin\alpha_i)$$

Main Results: System Dynamics

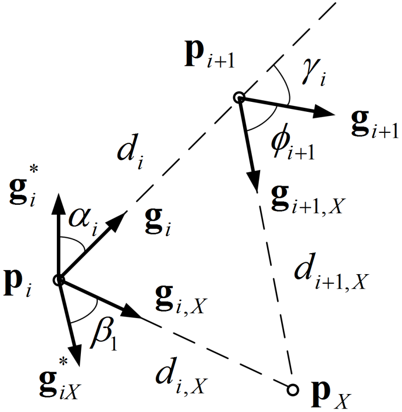


Figure: Various angles defined

α_i-dynamics

$$\cos\alpha_i = (\mathbf{g}_i^*)^T\mathbf{g}_i$$
$$\Rightarrow d_i \sin\alpha_i\dot{\alpha}_i = -\sin^2\alpha_i \pm \sin\alpha_i \sin\alpha_{i+1}\cos\gamma_i \pm \sin\alpha_i \sin\beta_{i+1}\cos(\gamma_i \pm \phi_i) \pm \sin\alpha_i \sin\beta_i \cos\phi_i$$

Main Results: System Dynamics

System equations in terms of angles

For each agent i,

$$\dot{\beta}_i = -\frac{\sin\beta_i}{d_{iX}} \pm \frac{\sin\alpha_i\cos\phi_i}{d_{iX}}.$$

$$\dot{\alpha}_i = -\frac{\sin\alpha_i}{d_i} \pm \frac{\sin\alpha_{i+1}\cos\gamma_i}{d_i} \pm \frac{\sin\beta_{i+1}\cos(\gamma_i \pm \phi_i)}{d_i} \pm \frac{\sin\beta_i\cos\phi_i}{d_i}.$$

Define $\mathbf{\Theta} = [\alpha_1 \dots \alpha_{n-1}\ \beta_1 \dots \beta_{n-1}] \in \mathbb{R}^{2(n-1)}$

Main Results: Stability Analysis

Theorem

In $\mathbb{R}^2$, the equilibria corresponding to $\mathcal{D}$ are locally asymptotically stable, while those corresponding to $\mathcal{U}$ are unstable.

- Linearize the system. Remove redundant states.
- At desired equilibria

$$\Delta\dot{\mathbf{\Theta}} = \mathbf{A}\Delta\mathbf{\Theta} = \begin{bmatrix} P & Q \\ D_1 & D_2 \end{bmatrix}\Delta\mathbf{\Theta},\ D_1 = \text{diag}(\pm\frac{\cos\phi_1^*}{d_{1X}}, \ldots, \pm\frac{\cos\phi_{n-1}^*}{d_{n-1X}})$$

$$D_2 = \text{diag}(-\tfrac{1}{d_{1X}}, \ldots, -\tfrac{1}{d_{n-1X}})$$

Main Results: Stability Analysis

$$P = \begin{bmatrix} -\frac{1}{d_1^*} & \pm\frac{\cos\gamma_1^*}{d_1^*} & 0 & \cdots & 0 \\ 0 & -\frac{1}{d_2^*} & \pm\frac{\cos\gamma_2^*}{d_2^*} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \pm\frac{\cos\gamma_{n-2}^*}{d_{n-2}^*} \\ 0 & 0 & 0 & \cdots & -\frac{1}{d_{n-1}^*} \end{bmatrix}$$

$$Q = \begin{bmatrix} -\frac{\cos\phi_1^*}{d_1^*} & \pm\frac{\cos\left(\gamma_1^* \pm \phi_1^*\right)}{d_1^*} & 0 & \cdots & 0 \\ 0 & -\frac{\cos\phi_1^*}{d_2^*} & \pm\frac{\cos\left(\gamma_2^* \pm \phi_2^*\right)}{d_2^*} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -\frac{\cos\phi_{n-2}^*}{d_{n-2}^*} & \pm\frac{\cos\left(\gamma_{n-2}^* \pm \phi_{n-2}^*\right)}{d_{n-2}^*} \\ 0 & 0 & \cdots & 0 & -\frac{\cos\phi_{n-1}^*}{d_{n-1}^*} \end{bmatrix}$$

Main Results: Stability Analysis

- Recall: determinant of block matrices when two lowrr blocks commute

$$\begin{aligned} det(\lambda I_{2n-2} - M) &= det(\begin{bmatrix} \lambda I_{n-1} - A & -B \\ -C & \lambda I_{n-1} - D \end{bmatrix} \\ &= det((\lambda I_{n-1} - A)(\lambda I_{n-1} - D) - BC) \\ &= \prod_{i=1}^{n-1} \left(\left(\lambda + \frac{1}{d_i^*}\right) \left(\lambda + \frac{1}{d_{iX}^*}\right) \pm \frac{(\cos\phi_1^*)^2}{d_1^* d_{1X}^*} \right) \\ &= \prod_{i=1}^{n-1} \left(\lambda^2 + \left(\frac{1}{d_1^*} + \frac{1}{d_{iX}^*}\right)\lambda + \frac{1 \pm (\cos\phi_1^*)^2}{d_i^* d_{iX}^*} \right) \end{aligned}$$

- $\lambda^2 + \left(\frac{1}{d_i^*} + \frac{1}{d_{iX}^*}\right)\lambda + \frac{1 \pm \left(\cos\phi_i^*\right)^2}{d_i^* d_{iX}^*} = 0$ has two roots in the open left half plane.

Main Results: Stability Analysis

- Similarly, at undesired equilibria:

$$\Delta\dot{\mathbf{\Theta}} = \tilde{\mathbf{A}}\Delta\mathbf{\Theta} = \begin{bmatrix} \tilde{P} & \tilde{Q} \\ \tilde{D_1} & \tilde{D_2} \end{bmatrix} \Delta\mathbf{\Theta},\ \tilde{D_1} = \text{diag}(\pm\frac{\cos\phi_1^*}{d_{1X}}, \ldots, \pm\frac{\cos\phi_{n-1}^*}{d_{n-1X}})$$

$$\tilde{D_2} = \text{diag}(\tfrac{1}{d_{1X}}, \ldots, \tfrac{1}{d_{n-1X}})$$

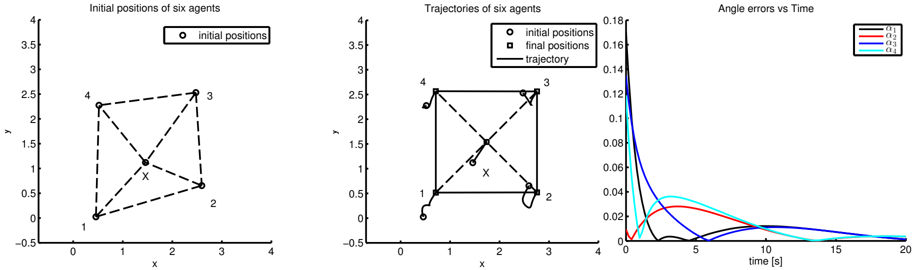
Main Results: Stability Analysis

$$\tilde{P} = \begin{bmatrix} \frac{1}{d_1^*} & \pm\frac{\cos\gamma_1^*}{d_1^*} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2^*} & \pm\frac{\cos\gamma_2^*}{d_2^*} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \pm\frac{\cos\gamma_{n-2}^*}{d_{n-2}^*} \\ 0 & 0 & 0 & \cdots & \frac{1}{d_{n-1}^*} \end{bmatrix}$$

$$\tilde{Q} = \begin{bmatrix} -\frac{\cos\phi_1^*}{d_1^*} & \pm\frac{\cos(\gamma_1^* \pm \phi_1^*)}{d_1^*} & 0 & \cdots & 0 \\ 0 & -\frac{\cos\phi_1^*}{d_2^*} & \pm\frac{\cos(\gamma_2^* \pm \phi_2^*)}{d_2^*} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -\frac{\cos\phi_{n-2}^*}{d_{n-2}^*} & \pm\frac{\cos\left(\gamma_{n-2}^* \pm \phi_{n-2}^*\right)}{d_{n-2}^*} \\ 0 & 0 & \cdots & 0 & -\frac{\cos\phi_{n-1}^*}{d_{n-1}^*} \end{bmatrix}$$

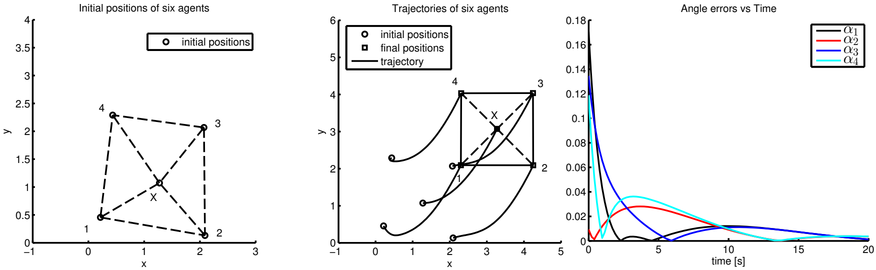
Conclusion: roots in rhp, hence unstable!

Simulations



(a) The initial formation. (b) Trajectories and the final formation. (c) Angle errors vs. time.

Figure: Capturing a target moving along a line.



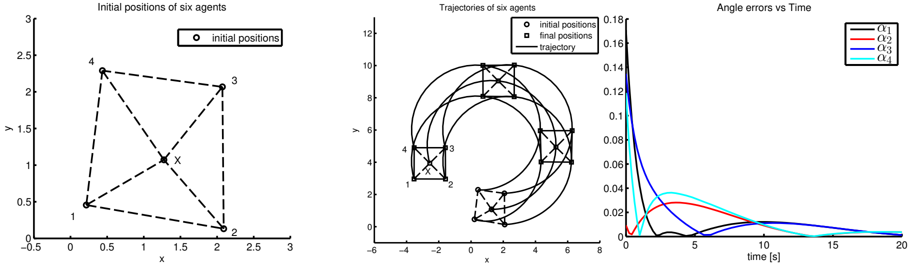
(a) The initial formation.

(b) Trajectories and the final formation.

(c) Angle errors vs. time.

Figure: Capturing a target moving along a parabola.

Simulations



(a) The initial formation. (b) Trajectories and the final formation. (c) Angle errors vs. time.

Figure: Capturing a target moving along a circle.

Conclusions

- Bearing-only cyclic pursuit strategy for capturing a moving target proposed.
- Target velocity known to all agents.
- Desired formation shape achieved up to a scaling factor.
- Desired formation: locally asymptotically stable.
- Undesired formation: unstable.

Potential Future Work

- Can control law be modified to fix the scale of formation?
- Extension to higher dimensions.
- Capture when target velocity known to some agents.
- Estimate region of attraction for desired equilibria.

Acknowledgements

The work of D. Mukherjee and D. Zelazo was supported in part at the Technion by a fellowship of the Israel Council for Higher Education and the Israel Science Foundation (grant No. 1490/1).

The work of M. H. Trinh and H.-S. Ahn was supported in part by Ministry of Culture, Sports and Tourism (MCST) and Korea Creative Content Agency (KOCCA) in the Culture Technology (CT) Research & Development Program 2014 and the National Research Foundation of Korea under Grant NRF-2015M2A8A4049953.

Thank you!

Questions?