Clustering, Robustness, and Effective Resistance in Linear Consensus

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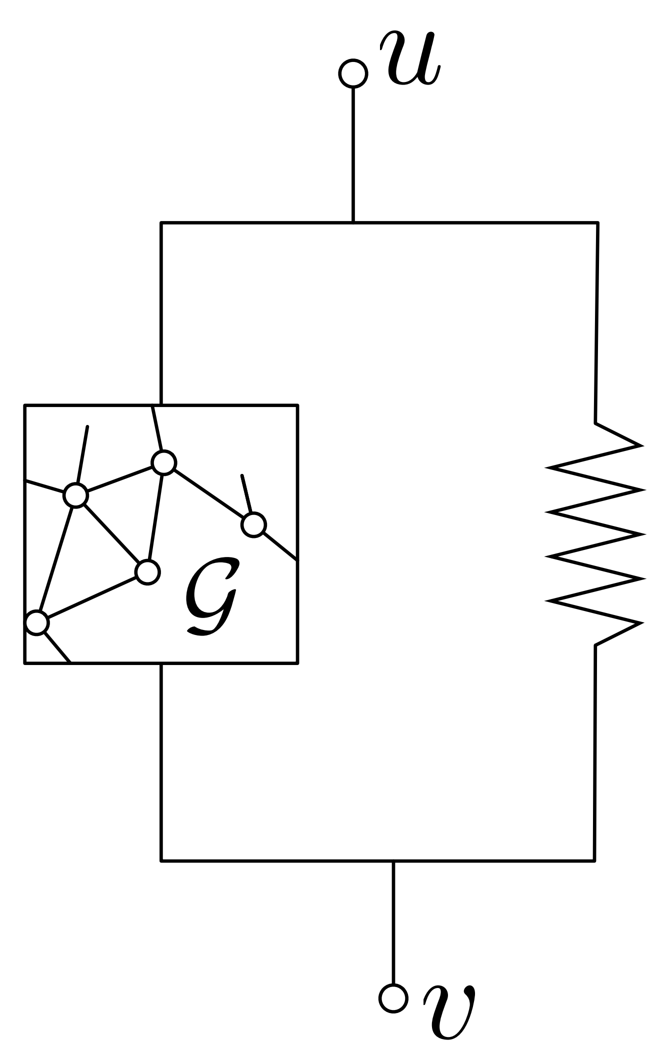


University of Minnesota
February 10, 2014

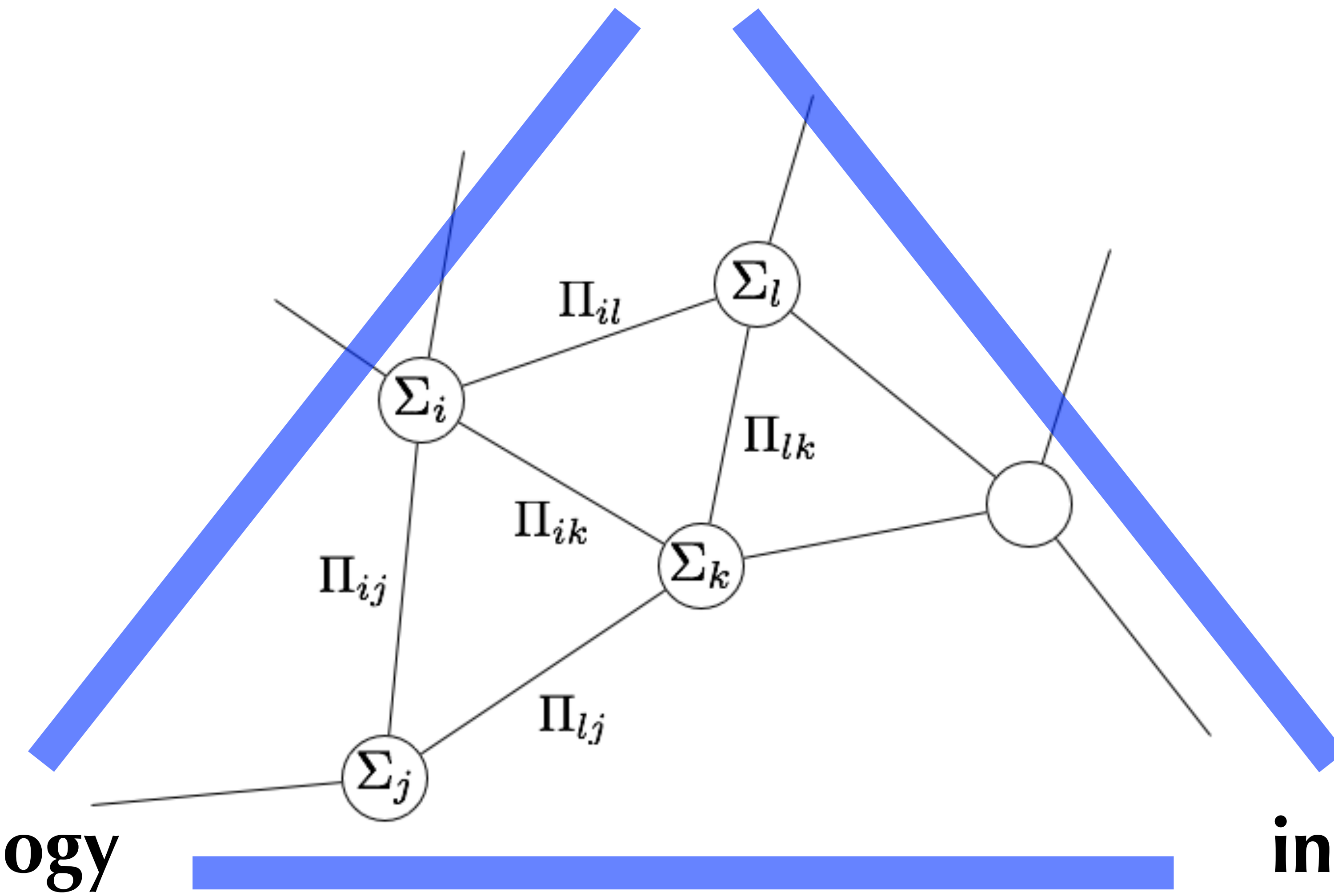
Networked Dynamic Systems

networks of dynamical systems are one of *the* enabling technologies of the future

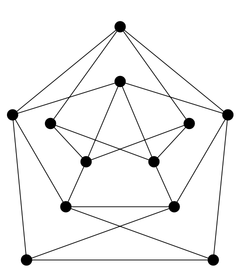
Networked Dynamic Systems

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t))$$

dynamics



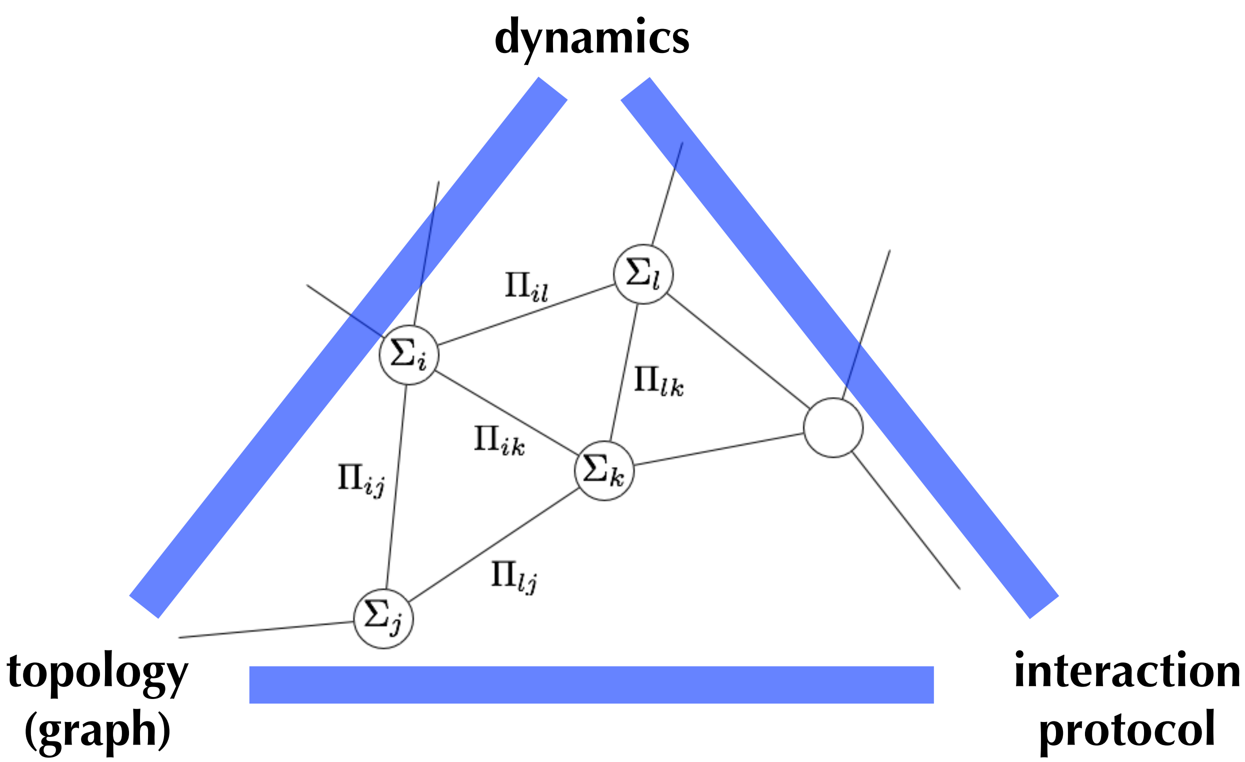
topology (graph)



interaction protocol

$$u_i(t) = \Pi_i(x(t), \mathcal{G})$$

Networked Dynamic Systems



Analysis

- steady-state behavior
- interplay between dynamics and graph
- equilibrium configurations

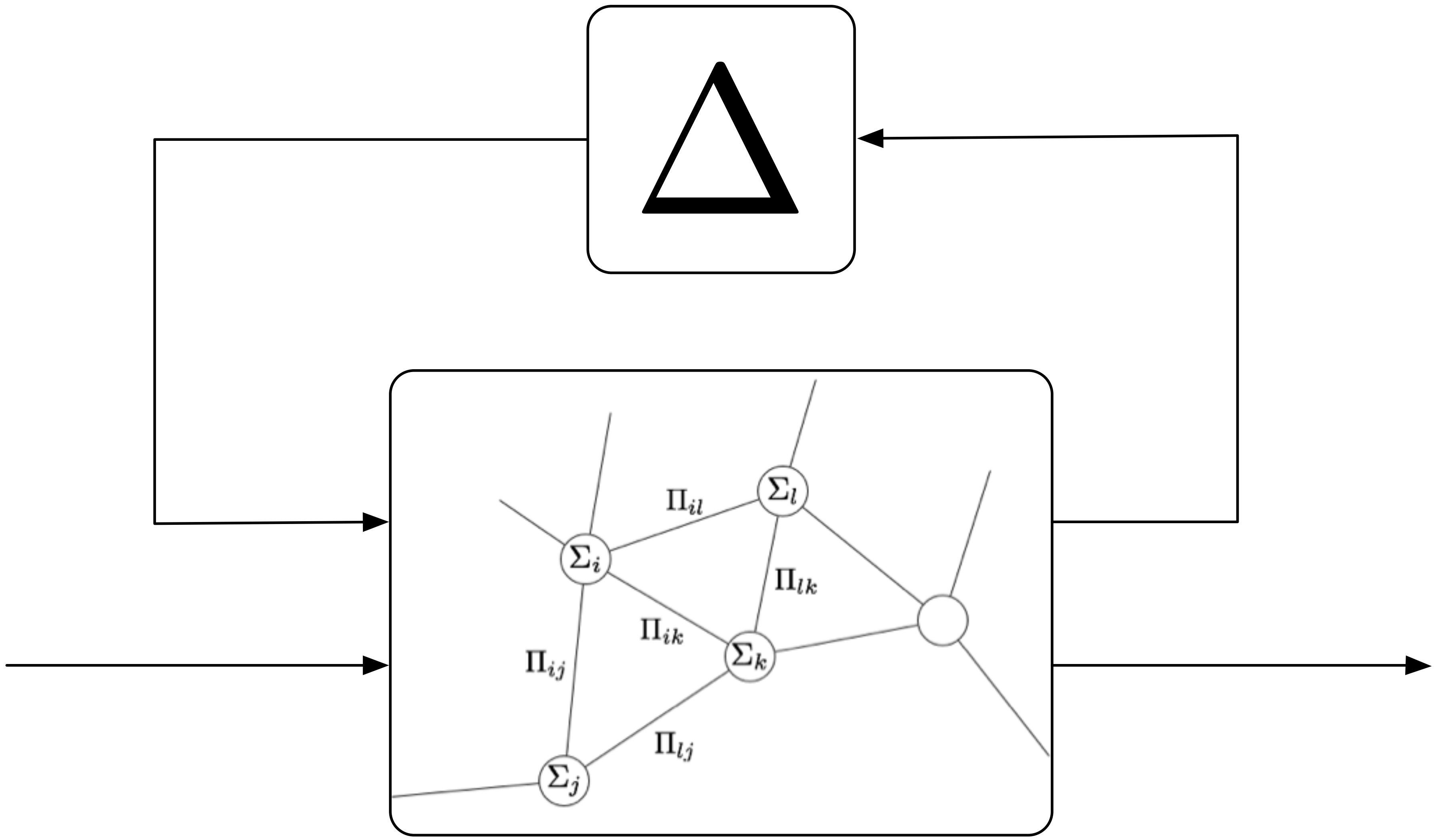
Synthesis

- design of distributed protocols
- design of “good” network structures
- good performance

can we reveal *deep* results describing the underlying behavior of these systems?

Networked Dynamic Systems

What about robustness?



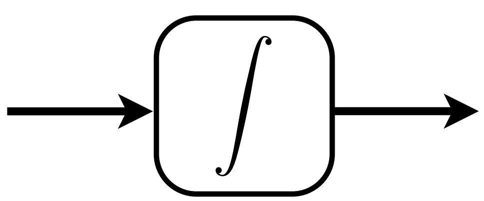
what is the right way to approach *robustness* of networked dynamic systems?

The Consensus Protocol

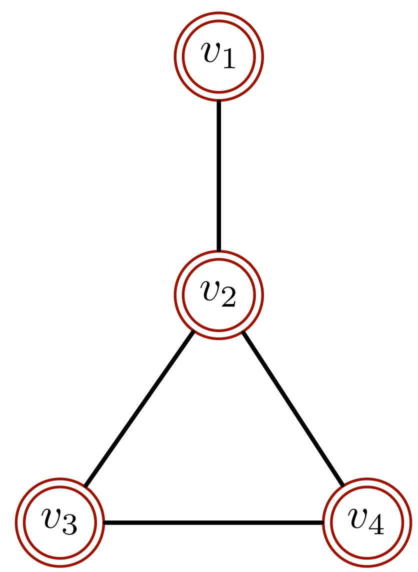
The consensus protocol is a *distributed and dynamic protocol* used for computing the average of a set of numbers.

Agent Dynamics

$$\dot{x}_i(t) = u_i(t)$$



Information Exchange Network



$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

$$\mathcal{W} : \mathcal{E} \to \mathbb{R}$$

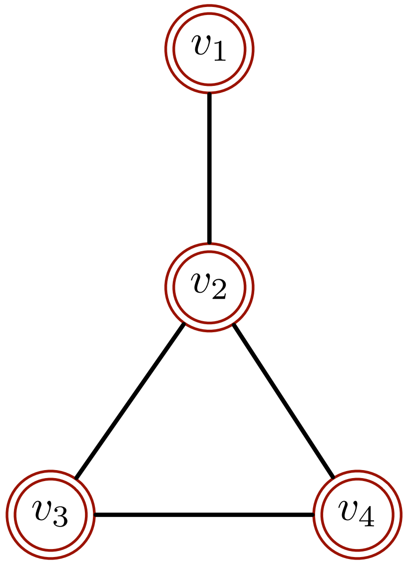
Incidence Matrix

$$E(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$$

$$E(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The Consensus Protocol

The consensus protocol is a *distributed and dynamic protocol* used for computing the average of a set of numbers.



Consensus Protocol

$$u_i(t) = \sum_{i \sim j} w_{ij}(x_j(t) - x_i(t))$$

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

Laplacian Matrix

- $L(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$
- $L(\mathcal{G}) = E(\mathcal{G})WE(\mathcal{G})^T$
- $L(\mathcal{G})\mathbb{1} = 0$

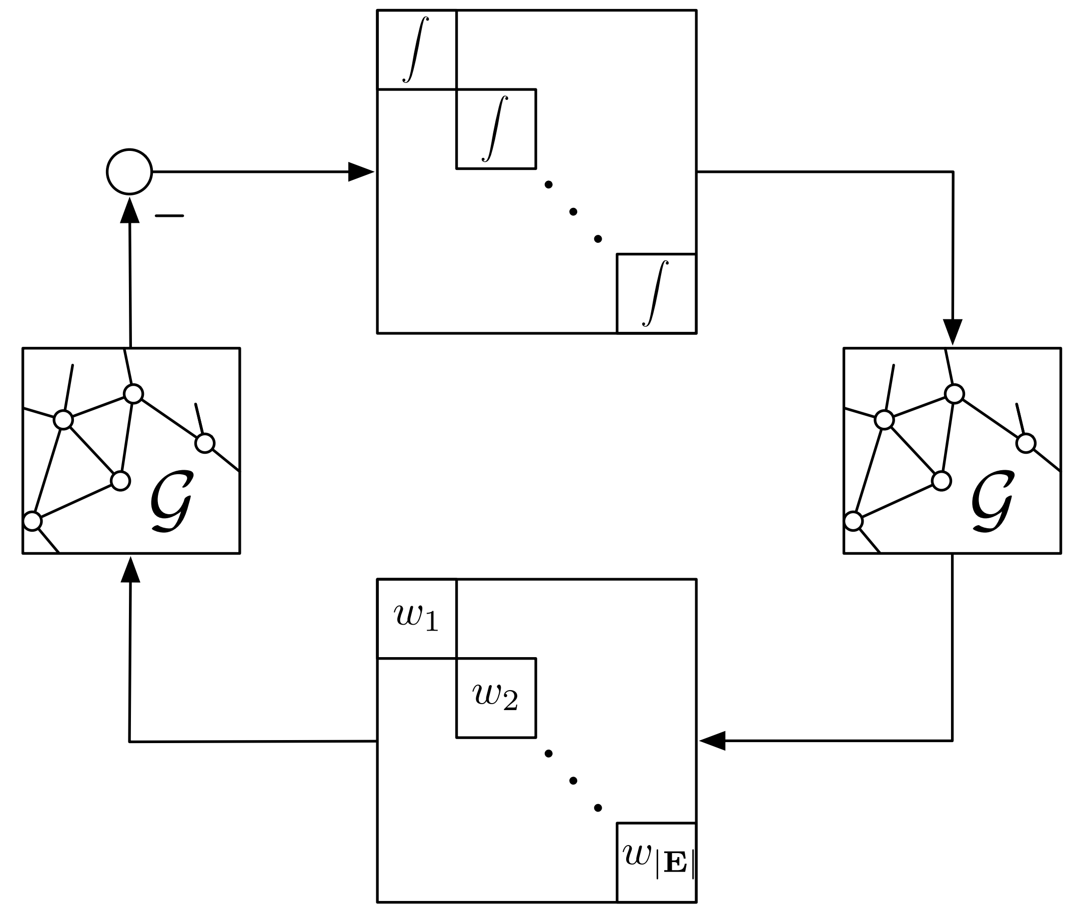
$$e = (v_i, v_j) \in \mathcal{E}$$

$$\mathcal{W}(e) = w_{ij} = [W]_{ee}$$

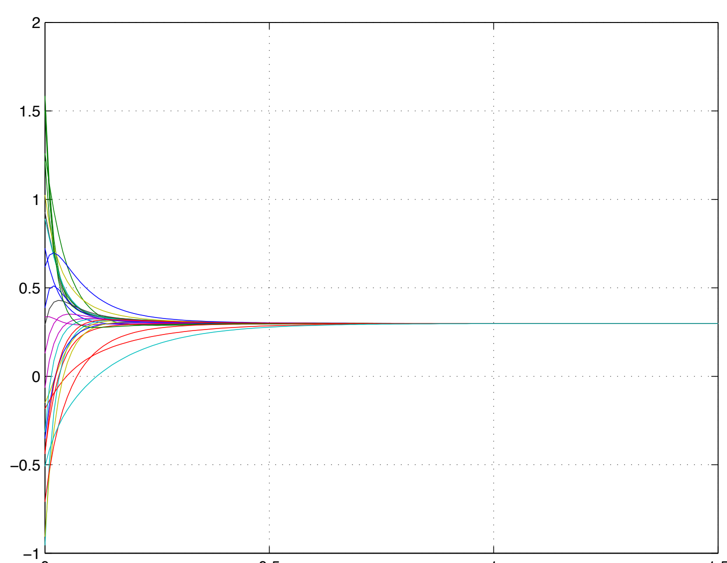
Robustness in Consensus Networks

The Linear Weighted Consensus Protocol

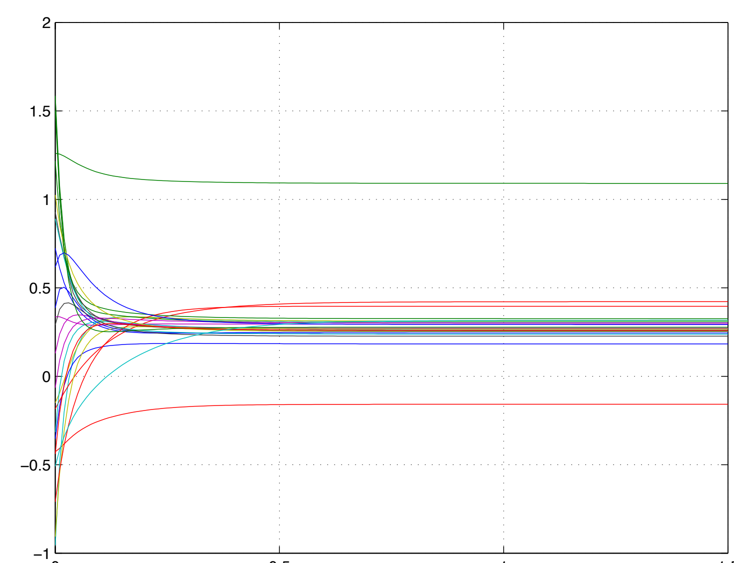
$$\dot{x}_i(t) = \sum_{i \sim j} w_{ij}(x_j(t) - x_i(t))$$



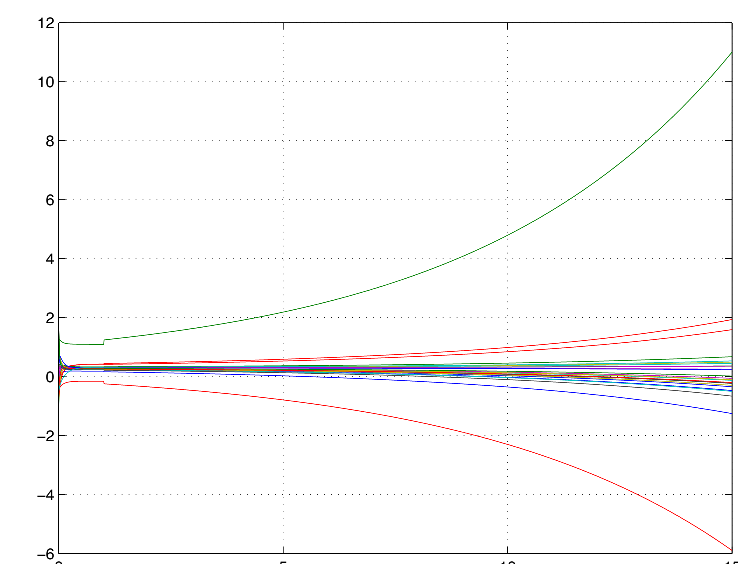
$\mathcal{G}$ 25 nodes
98 edges



$w^* > -10.1911$

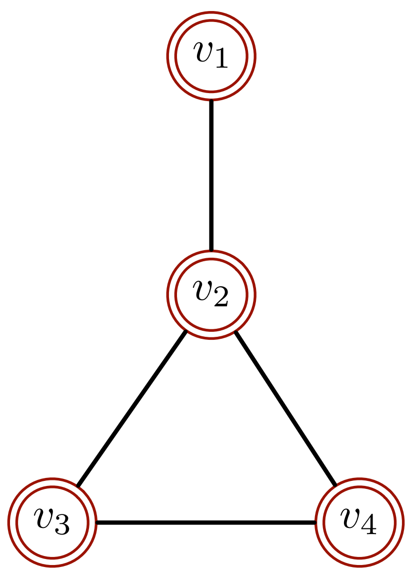


$w^* = -10.1911$



$w^* < -10.1911$

The Consensus Protocol



Consensus Protocol

$$u_i(t) = \sum_{i \sim j} w_{ij}(x_j(t) - x_i(t))$$

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

Laplacian Matrix

- $L(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$
- $L(\mathcal{G}) = E(\mathcal{G})WE(\mathcal{G})^T$
- $L(\mathcal{G})\mathbb{1} = 0$

$$e = (v_i, v_j) \in \mathcal{E}$$

$$\mathcal{W}(e) = w_{ij} = [W]_{ee}$$

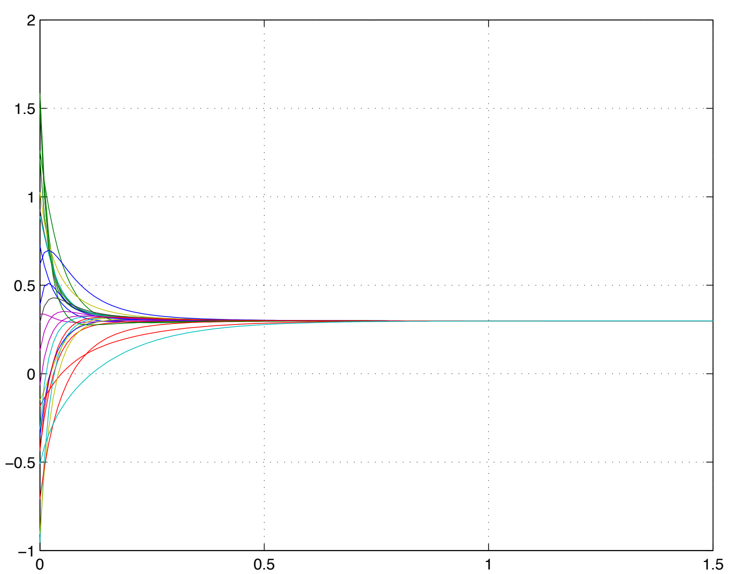
Theorem 1 *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a weighted and connected graph with positive edge weights $\mathcal{W}(k) > 0$ for $k = 1, \ldots, |\mathcal{E}|$. Then the consensus dynamics synchronizes; i.e., $\lim_{t \to \infty} x_i(t) = \beta$ for $i = 1, \ldots, |\mathcal{V}|$.*

Mesbahi & Egerstedt, Olfati-Saber, Ren

Synchronization and the Laplacian

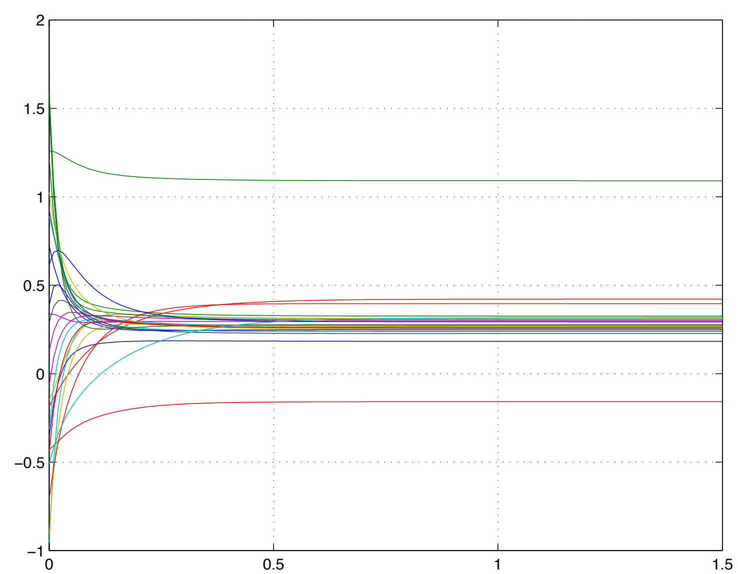
$$x(t) = e^{-L(\mathcal{G})t}x_0$$

$\lim_{t\to\infty} x(t) = \beta\mathbb{1} \Leftrightarrow L(\mathcal{G})$ has only **one** eigenvalue at the origin



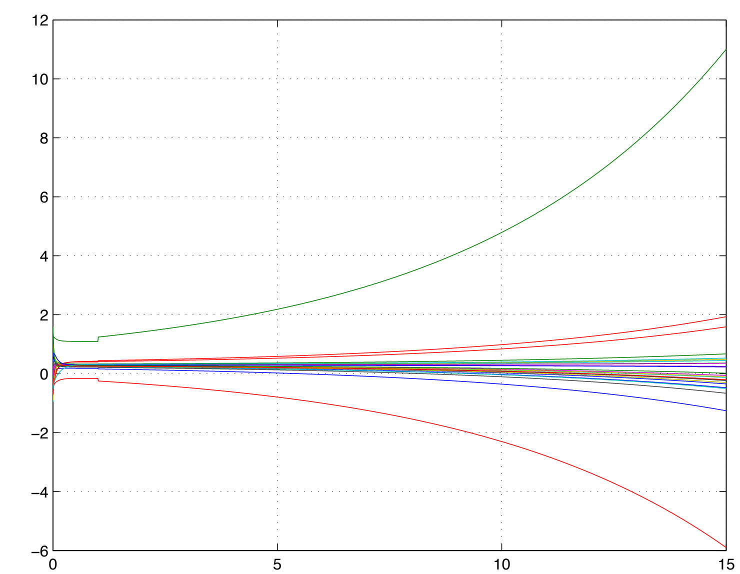
$L(\mathcal{G}) \geq 0$

has only **one** eigenvalue at the zero



$L(\mathcal{G}) \geq 0$

has **more than one** eigenvalue at the zero



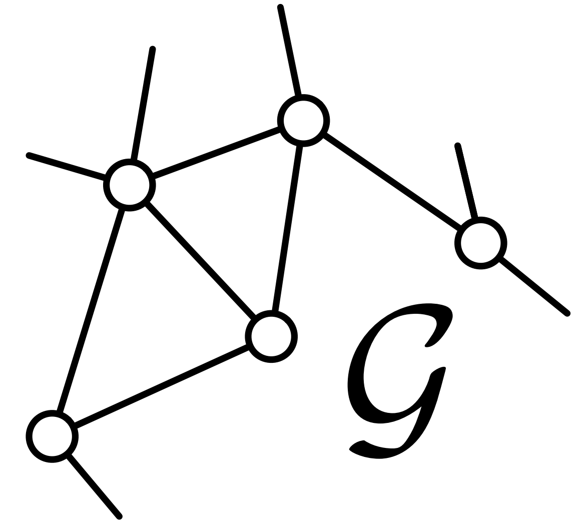
$L(\mathcal{G})$

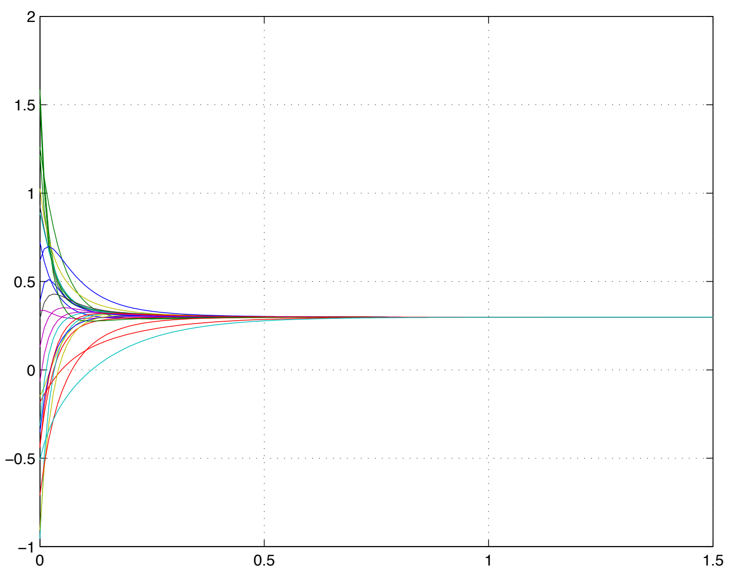
has **at least one** negative eigenvalue (indefinite)

Synchronization and the Laplacian

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

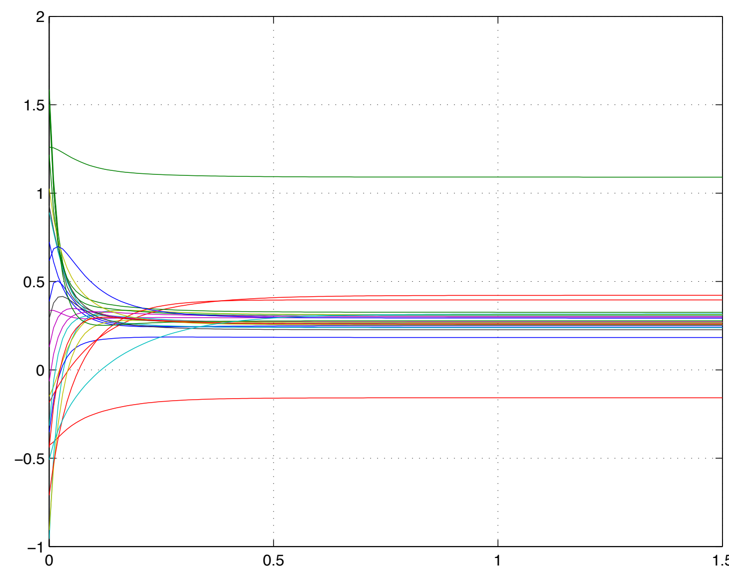
system behavior depends on the spectral properties of the graph Laplacian





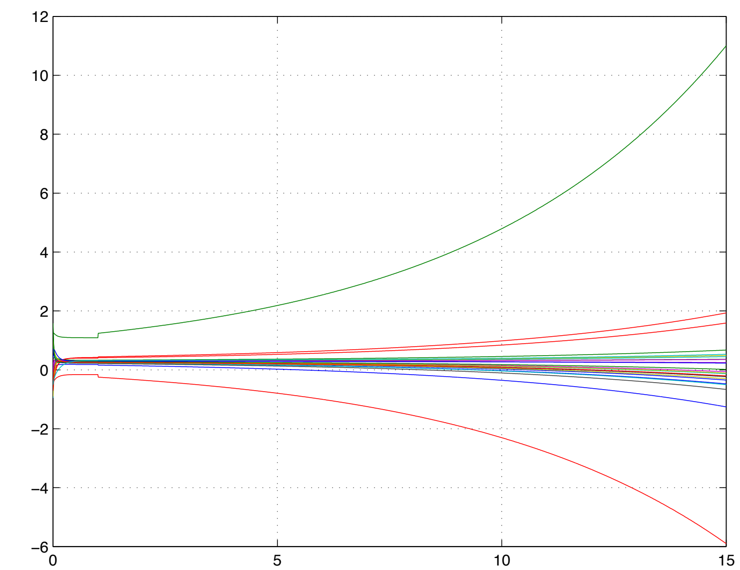
$L(\mathcal{G}) \geq 0$

has only **one** eigenvalue at the zero



$L(\mathcal{G}) \geq 0$

has **more than one** eigenvalue at the zero



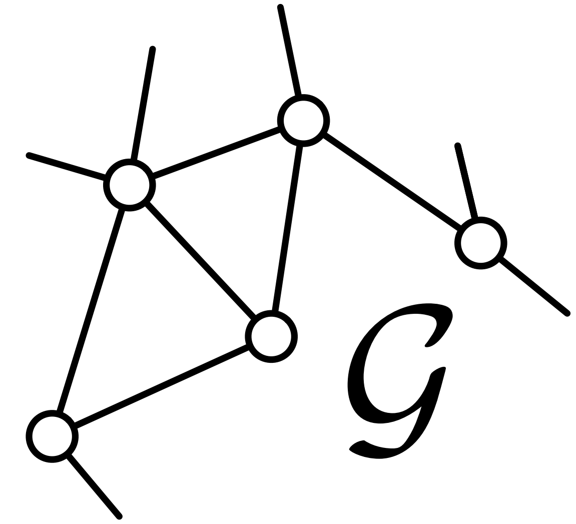
$L(\mathcal{G})$

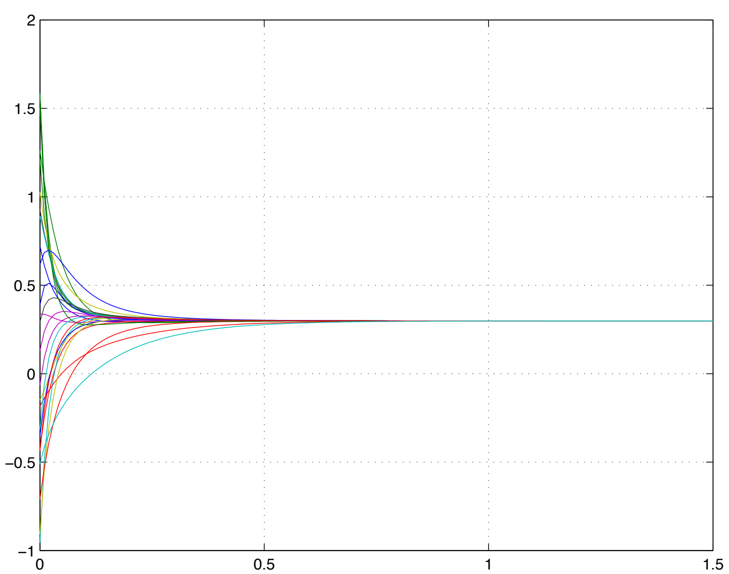
has **at least one** negative eigenvalue (indefinite)

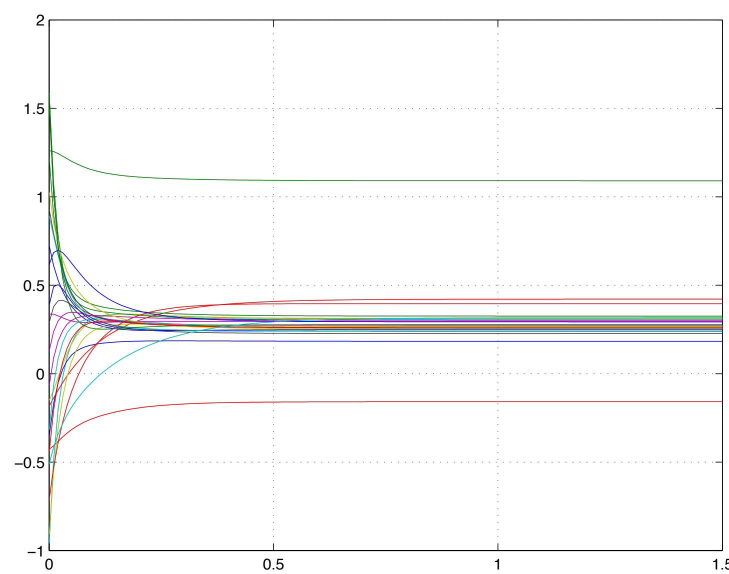
Synchronization and the Laplacian

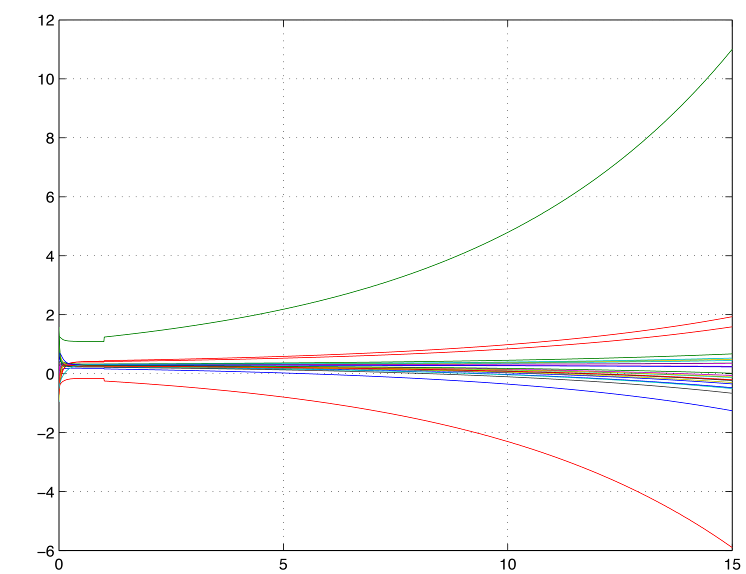
$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

can we understand spectral properties of the Laplacian from the structure of the graph?









$L(\mathcal{G}) \geq 0$ has only **one** eigenvalue at the zero

$L(\mathcal{G}) \geq 0$ has **more than one** eigenvalue at the zero

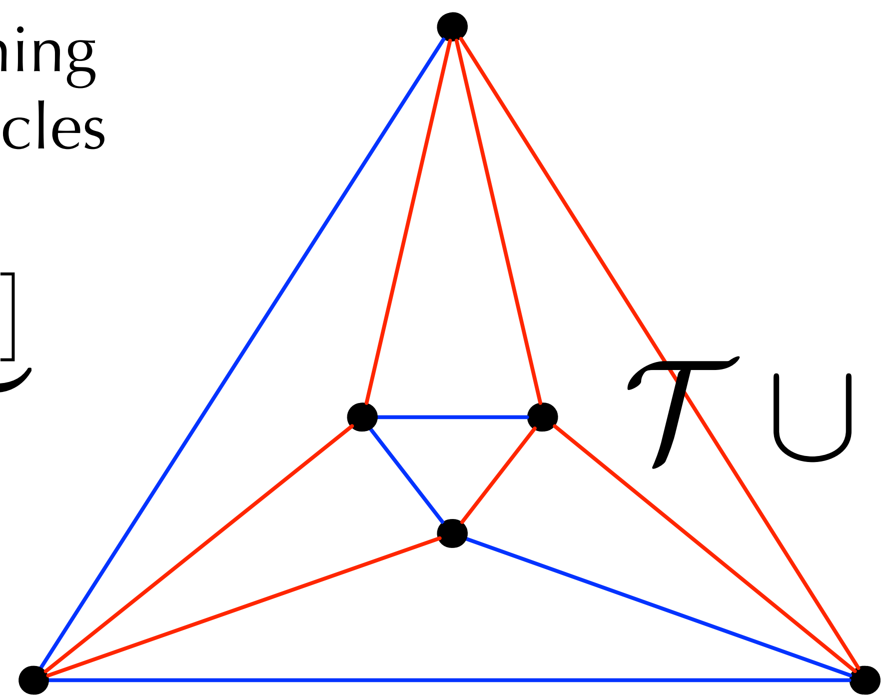
$L(\mathcal{G})$ has **at least one** negative eigenvalue (indefinite)

Spanning Trees and Cycles

A graph as the union of a spanning tree and edges that complete cycles

$$E(\mathcal{G}) = E(\mathcal{T}) \underbrace{\left[\begin{array}{cc} I & T_{(\mathcal{T},\mathcal{C})} \end{array} \right]}_{\mathcal{R}_{(\mathcal{T},\mathcal{C})}}$$

$$T_{(\mathcal{T},\mathcal{C})} = \underbrace{(E_{\mathcal{T}}^T E_{\mathcal{T}})^{-1} E_{\mathcal{T}}^T}_{E_{\mathcal{T}}^L} E(\mathcal{C})$$



$\mathcal{T} \cup \mathcal{C}$

a spanning tree

remaining edges "complete cycles"

Weighted Edge Laplacian

$$L_e(\mathcal{G}) = W^{\frac{1}{2}} E(\mathcal{G})^T E(\mathcal{G}) W^{\frac{1}{2}}$$

Essential Edge Laplacian

$$L_e(\mathcal{T}) R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T$$

similarity between edge and graph Laplacians

$L(\mathcal{G})$ ⇄ $L_e(\mathcal{G})$

$\mathcal{R}_{(\mathcal{T},\mathcal{C})}$ rows form a basis for the *cut space* of the graph

Some Properties of $L_e(\mathcal{G})$

Proposition 1 *The matrix* $L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T$ *has the same inertia as* $R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T$*. Similarly, the matrix* $(L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T)^{-1}$ *has the same inertia as* $(R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T)^{-1}$*.*

Recall: The *inertia* of a matrix is the number of negative, 0, and positive eigenvalues

Proof:

$$L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T \sim L_e(\mathcal{T})^{\frac{1}{2}}R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T L_e(\mathcal{T})^{\frac{1}{2}}$$

$L_e(\mathcal{T})^{\frac{1}{2}}R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T L_e(\mathcal{T})^{\frac{1}{2}}$ is congruent to $R_{(\mathcal{T},\mathcal{C})}WR_{(\mathcal{T},\mathcal{C})}^T$

congruent matrices have the same inertia

Some Properties of $L_e(\mathcal{G})$

Proposition 1

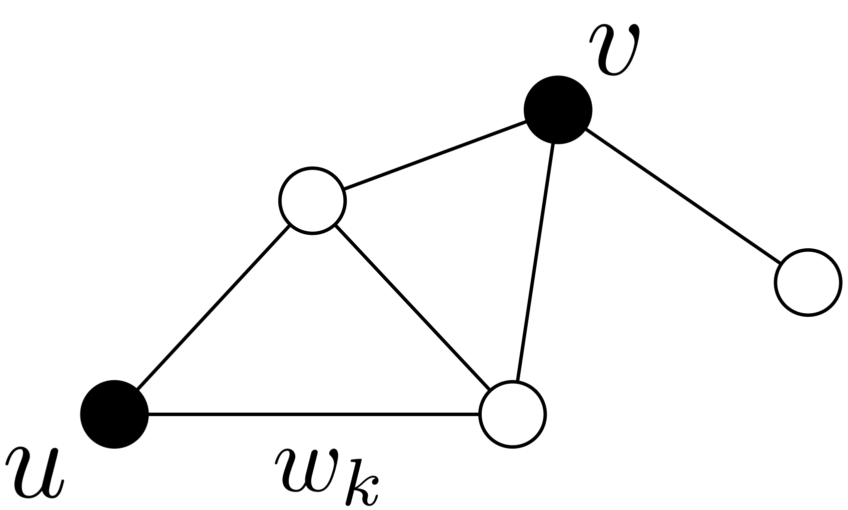
$$L(\mathcal{G}) \geq 0 \Leftrightarrow R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T \geq 0$$

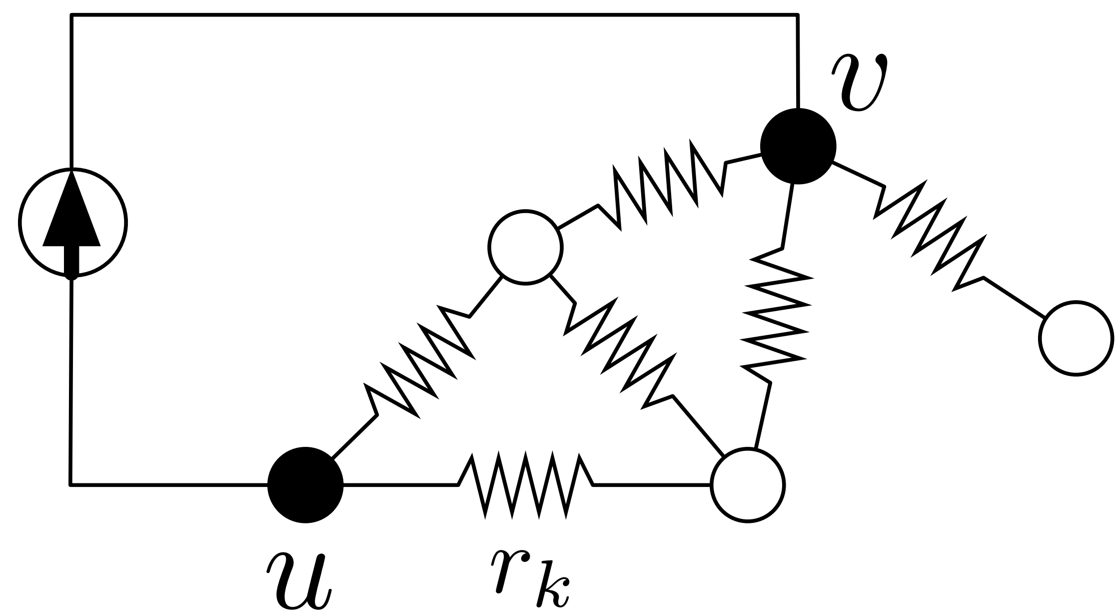
The definiteness of the graph Laplacian can be studied through another matrix!

$$R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T$$

Effective Resistance of a Graph

The **effective resistance** between two nodes *u* and *v* is the electrical resistance measured across the nodes when the graph represents an electrical circuit with each edge a resistor





$$r_k = \frac{1}{w_k}$$ edge weights are the *conductance* of *each resistor*

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T L^\dagger(\mathcal{G})(\mathbf{e}_u - \mathbf{e}_v)$$
$$= \left[L^\dagger(\mathcal{G})\right]_{uu} - 2\left[L^\dagger(\mathcal{G})\right]_{uv} + \left[L^\dagger(\mathcal{G})\right]_{vv}$$

Klein and Randić 1993

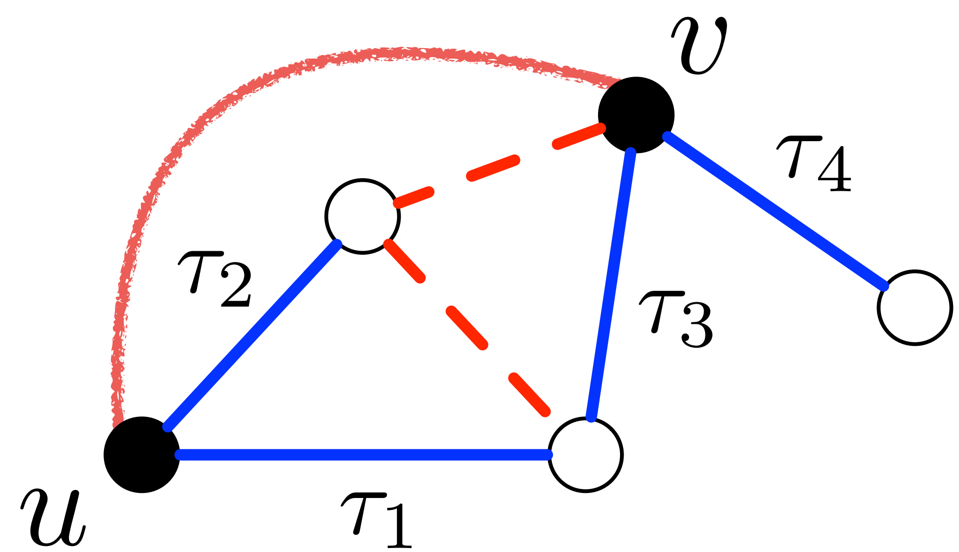
Effective Resistance of a Graph

Proposition 1

$$L^{\dagger}(\mathcal{G}) = (E_{\mathcal{T}}^{L})^{T} \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^{T}\right)^{-1} E_{\mathcal{T}}^{L}$$

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T L^{\dagger}(\mathcal{G})(\mathbf{e}_u - \mathbf{e}_v)$$

$$E_{\mathcal{T}}^{L}(\mathbf{e}_u - \mathbf{e}_v) = \begin{bmatrix} \pm 1 \\ 0 \\ \pm 1 \\ 0 \end{bmatrix} \begin{matrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{matrix}$$



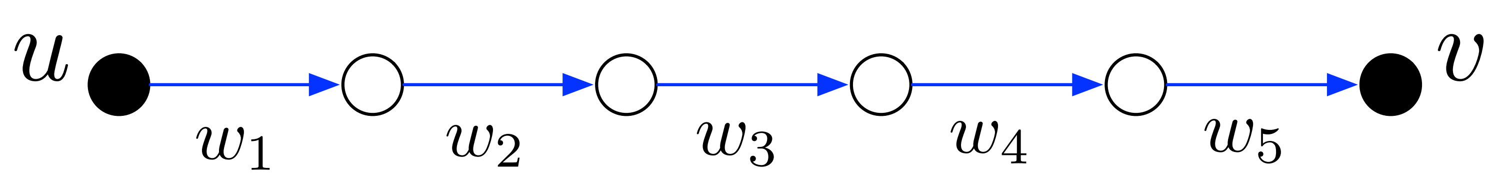
$$\mathcal{G} = \mathcal{T} \cup \mathcal{C}$$

indicates a path from node u to v using only edges in the spanning tree

$$T_{(\mathcal{T},\mathcal{C})} = \underbrace{(E_{\mathcal{T}}^{T} E_{\mathcal{T}})^{-1} E_{\mathcal{T}}^{T}}_{E_{\mathcal{T}}^{L}} E(\mathcal{C})$$

Effective Resistance of a Graph

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T (E_{\mathcal{T}}^L)^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v)$$



$$R_{(\mathcal{T},\mathcal{C})} = I$$

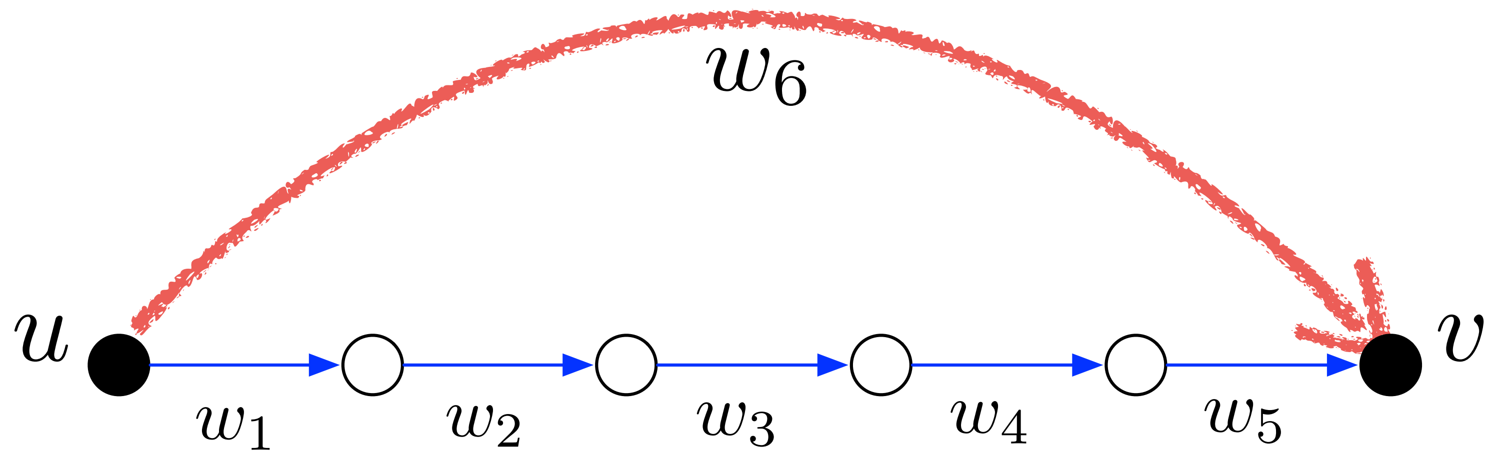
$$E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v) = \mathbb{1}$$

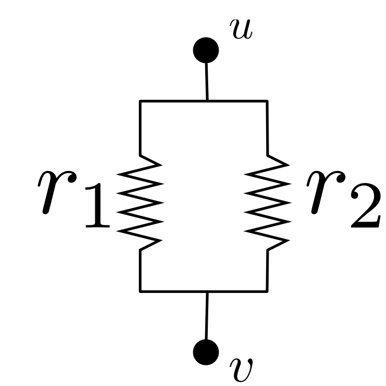
$$r_{uv} = \mathbb{1}^T W^{-1} \mathbb{1} = \sum_{i=1}^{5} \frac{1}{w_i}$$

$$r_k = \frac{1}{w_k}$$

Effective Resistance of a Graph

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T (E_{\mathcal{T}}^L)^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v)$$





$$r_{uv} = \frac{r_1 r_2}{r_1 + r_2}$$

$$R_{(\mathcal{T},\mathcal{C})} = \begin{bmatrix} I & \mathbb{1} \end{bmatrix}$$

$$E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v) = \mathbb{1}$$

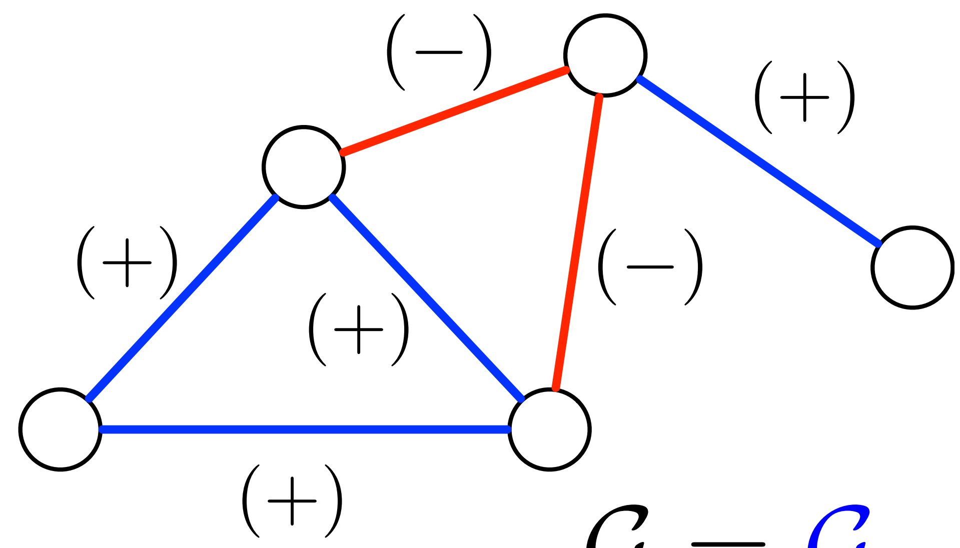
$$r_k = \frac{1}{w_k}$$

$$W_{\mathcal{T}} = \text{diag}\{w_1, \ldots, w_5\}$$

$$\begin{aligned} r_{uv} &= \mathbb{1}^T \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} \mathbb{1} \\ &= \mathbb{1}^T \left(W_{\mathcal{T}} + w_6 \mathbb{1}\mathbb{1}^T\right)^{-1} \mathbb{1} \\ &= \frac{\left(\mathbb{1}^T W_{\mathcal{T}}^{-1} \mathbb{1}\right) w_6^{-1}}{\mathbb{1}^T W_{\mathcal{T}}^{-1} \mathbb{1} + w_6^{-1}} \end{aligned}$$

Signed Graphs

a **signed graph** is a graph with positive and negative edge weights

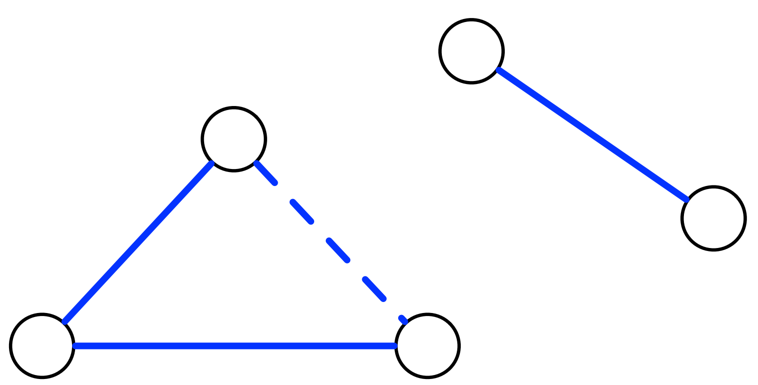


$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

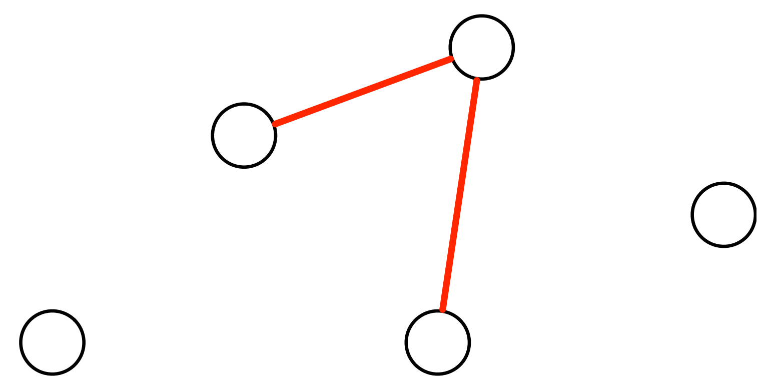
$$\mathcal{W} : \mathcal{E} \rightarrow \mathbb{R}$$

$$\mathcal{G} = \mathcal{G}_+ \cup \mathcal{G}_-$$

$$\mathcal{E}_+ = \{e \in \mathcal{E} \,:\, \mathcal{W}(e) > 0\} \qquad \mathcal{E}_- = \{e \in \mathcal{E} \,:\, \mathcal{W}(e) < 0\}$$



$$E(\mathcal{G}_+) = E_+ = E_{\mathcal{F}_+} R_{(\mathcal{F}_+, \mathcal{C}_+)}$$



$$E(\mathcal{G}_-) = E_-$$

$$L(\mathcal{G}) = E(\mathcal{G}_+) W_+ E(\mathcal{G}_+)^T - E(\mathcal{G}_-) |W_-| E(\mathcal{G}_-)^T$$

Spectral Properties of Signed Graphs

Proposition 1

$$L(\mathcal{G}) \geq 0 \Leftrightarrow \begin{bmatrix} |W_-|^{-1} & E_-^T \\ E_- & E_+W_+E_+^T \end{bmatrix} \geq 0$$

Proof:

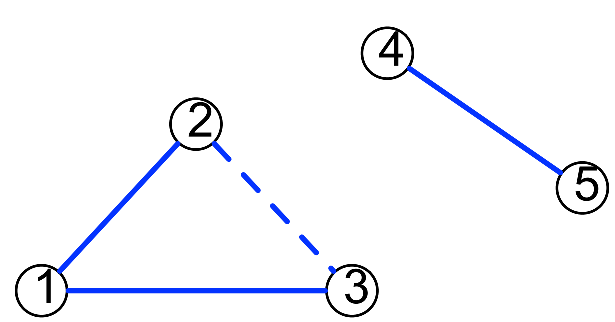
Schur Complement

$$L(\mathcal{G}) = E(\mathcal{G}_+)W_+E(\mathcal{G}_+)^T - E(\mathcal{G}_-)|W_-|E(\mathcal{G}_-)^T$$

$$E(\mathcal{G}_+) = E_+ = E_{\mathcal{F}_+}R_{(\mathcal{F}_+,\mathcal{C}_+)}$$

$$\text{IM}[N_{\mathcal{F}_+}] = \text{span}[\mathcal{N}(E_{\mathcal{F}_+}^T)]$$

Identifies how the positive weight graph is partitioned



$$N_{\mathcal{F}_+} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Spectral Properties of Signed Graphs

Proposition 1

$$L(\mathcal{G}) \geq 0 \Leftrightarrow \begin{bmatrix} |W_-|^{-1} & E_-^T (E_{\mathcal{F}_+}^L)^T & E_-^T N_{\mathcal{F}_+} \\ E_{\mathcal{F}_+}^L E_- & R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T & 0 \\ N_{\mathcal{F}_+}^T E_- & 0 & \mathbf{0} \end{bmatrix} \geq 0$$

Proof:

Congruent Transformation $S = \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} (E_{\mathcal{F}_+}^L)^T & N_{\mathcal{F}_+} \end{bmatrix} \end{bmatrix}$

If the positive portion weighted graph is connected...

$$N_{\mathcal{F}_+} = \mathbb{1}$$

$$L(\mathcal{G}) \geq 0 \Leftrightarrow \begin{bmatrix} |W_-|^{-1} & E_-^T (E_{\mathcal{F}_+}^L)^T \\ E_{\mathcal{F}_+}^L E_- & R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T \end{bmatrix} \geq 0$$

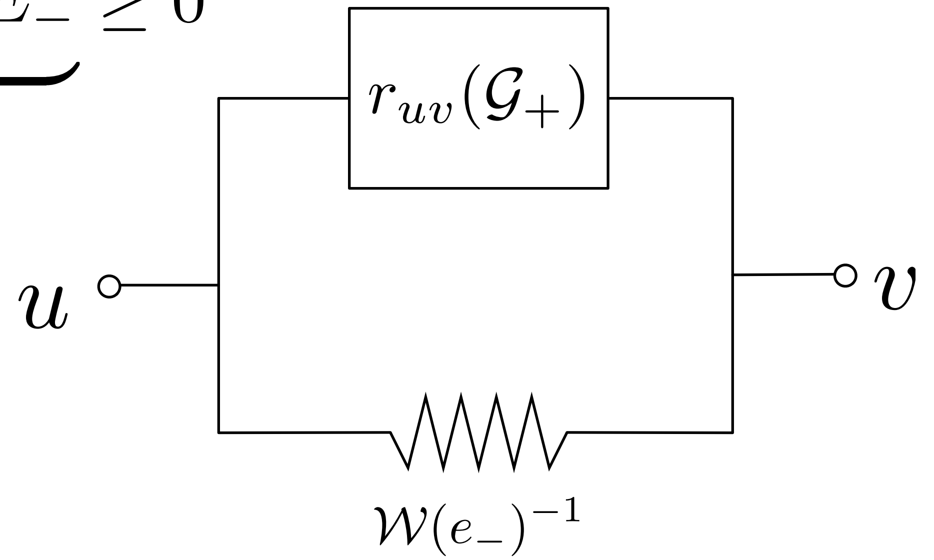
Spectral Properties of Signed Graphs

Theorem 1 *Assume that* $\mathcal{G}_+$ *is connected and* $|\mathcal{E}_-| = 1$ *and let* $\mathcal{E}_- = \{e_- = (u,v)\}$. *Let* r_{uv} *denote the effective resistance between nodes* $u, v \in \mathcal{V}$ *over the graph* $\mathcal{G}_+$. *Then*

$$L(\mathcal{G}) \geq 0 \Leftrightarrow |\mathcal{W}(e_-)| \leq r_{uv}^{-1}$$

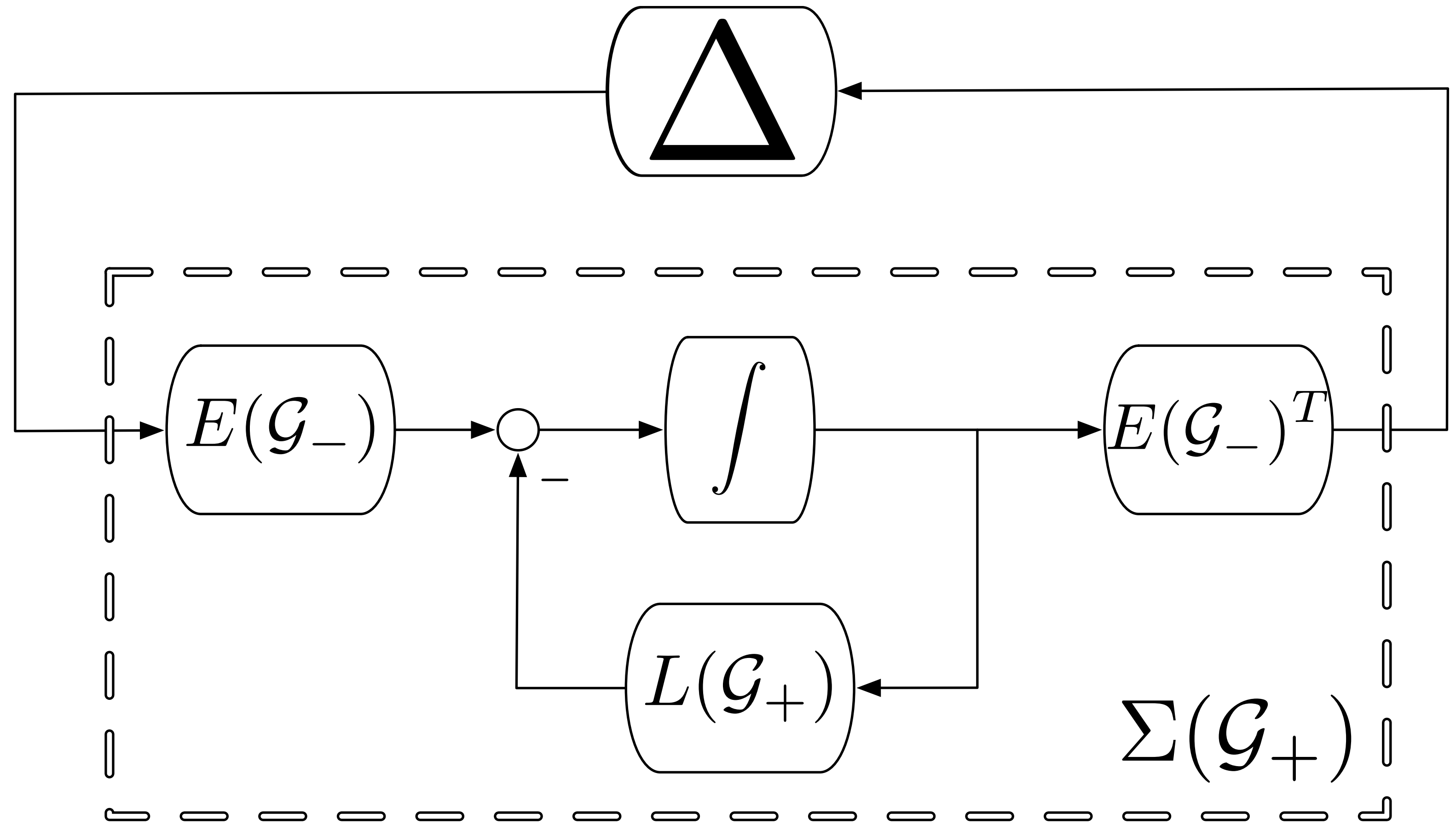
Proof:

$$|W_-|^{-1} - \underbrace{E_-^T (E_{\mathcal{F}_+}^L)^T (R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T)^{-1} E_{\mathcal{F}_+}^L E_-}_{r_{uv}(\mathcal{G}_+)} \geq 0$$



any single edge can destabilize a consensus network with a "negative enough" edge weight

A Small-Gain Interpretation



$\mathcal{F}_+ = \mathcal{T}_+$

Theorem 1 (Zelazo '11)

$$\|\Sigma(\mathcal{G}_+)\|_\infty^2 = \overline{\sigma}\underbrace{\left[E_-^T (E_{\mathcal{F}_+}^L)^T \left(R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T\right)^{-1} E_{\mathcal{F}_+}^L E_-\right]}_{r_{uv}(\mathcal{G}_+)}$$

Spectral Properties of Signed Graphs

Corollary 1 *Assume that both $\mathcal{E}_+$ and $\mathcal{E}_-$ are not empty. If $\mathcal{G}_+$ is not connected, then $L(\mathcal{G})$ is indefinite for any choice of negative weights.*

Proof:

$$\begin{bmatrix} |W_-|^{-1} & E_-^T (E_{\mathcal{F}_+}^L)^T & E_-^T N_{\mathcal{F}_+} \\ E_{\mathcal{F}_+}^L E_- & R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T & 0 \\ N_{\mathcal{F}_+}^T E_- & 0 & \mathbf{0} \end{bmatrix}$$

permutation

$$\left[\begin{array}{cc|c} |W_-|^{-1} & E_-^T N_{\mathcal{F}_+} & E_-^T (E_{\mathcal{F}_+}^L)^T \\ N_{\mathcal{F}_+}^T E_- & \mathbf{0} & 0 \\ \hline E_{\mathcal{F}_+}^L E_- & 0 & R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T \end{array}\right]$$

$[E_-^T N_{\mathcal{F}_+}]_{ik} = \pm 1$ if and only if edge k separates node *u* and *v* $e_k = (u, v)$

Spectral Properties of Signed Graphs

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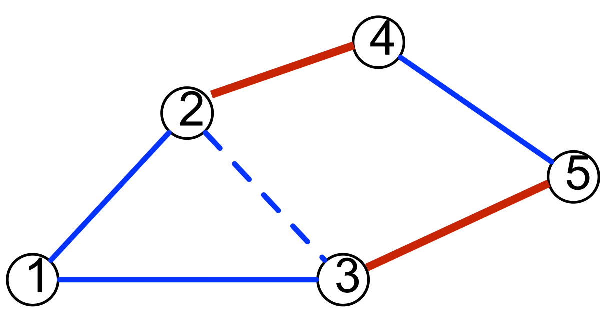
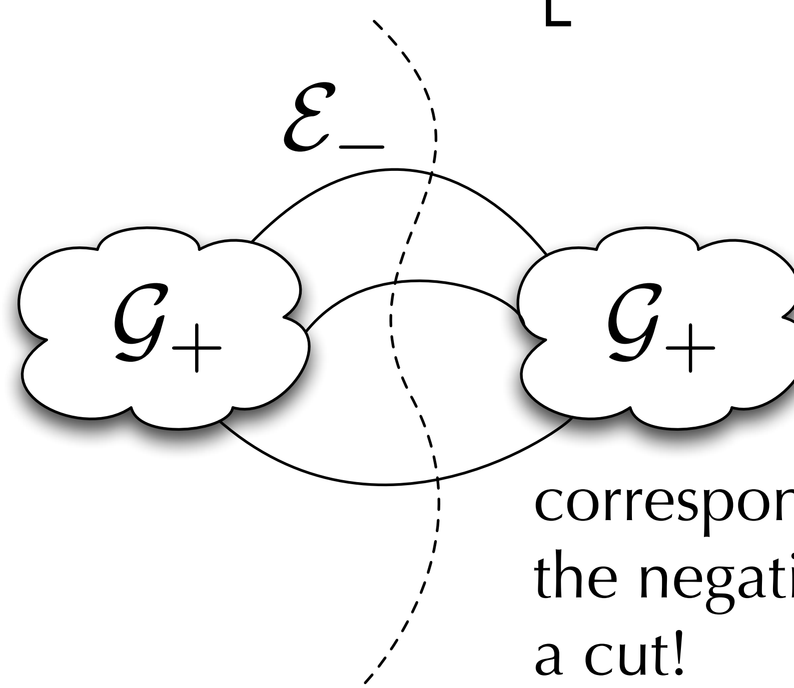


$$N_{\mathcal{F}_+} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$E_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$E_-^T N_{\mathcal{F}_+} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$



corresponds precisely to when the negative edge weights form a cut!

Spectral Properties of Signed Graphs

Corollary 1 *Assume that both $\mathcal{E}_+$ and $\mathcal{E}_-$ are not empty. If $\mathcal{G}_+$ is not connected, then $L(\mathcal{G})$ is indefinite for any choice of negative weights.*

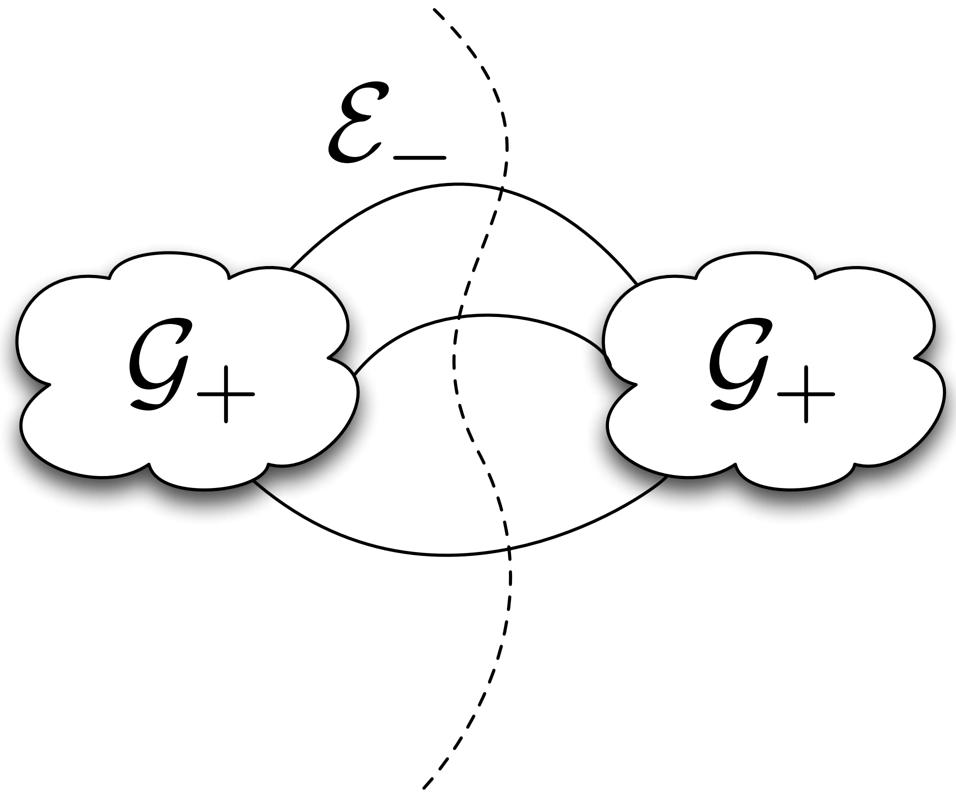
Proof:

$$x^T \begin{bmatrix} |W_-|^{-1} & E_-^T N_{\mathcal{F}_+} \\ N_{\mathcal{F}_+}^T E_- & \mathbf{0} \end{bmatrix} x = \sum_{i \in \mathcal{E}_-} |\mathcal{W}_-(i)|^{-1} x_i^2 + \sum_{k \in \mathrm{CUT}_1} \pm 2 x_k x_{m+1} + \cdots + \sum_{k \in \mathrm{CUT}_c} \pm 2 x_k x_{m+c}$$

$$< 0$$

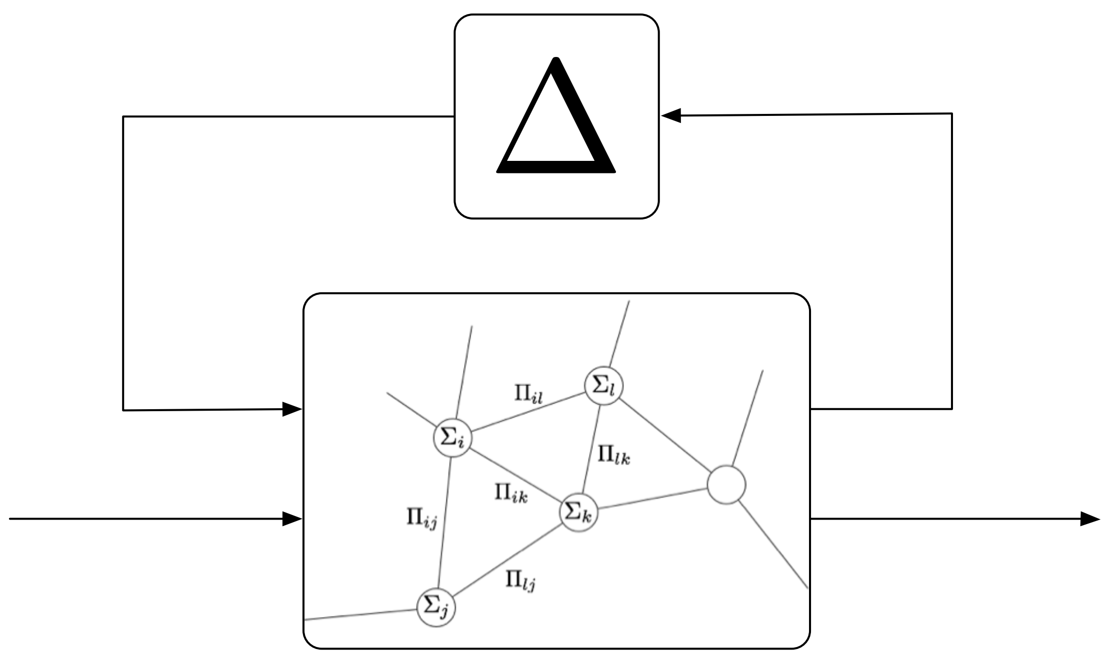
$x_i,\ i = m+1, \ldots, m+c$ can be arbitrarily chosen

Graph Cuts and Robustness



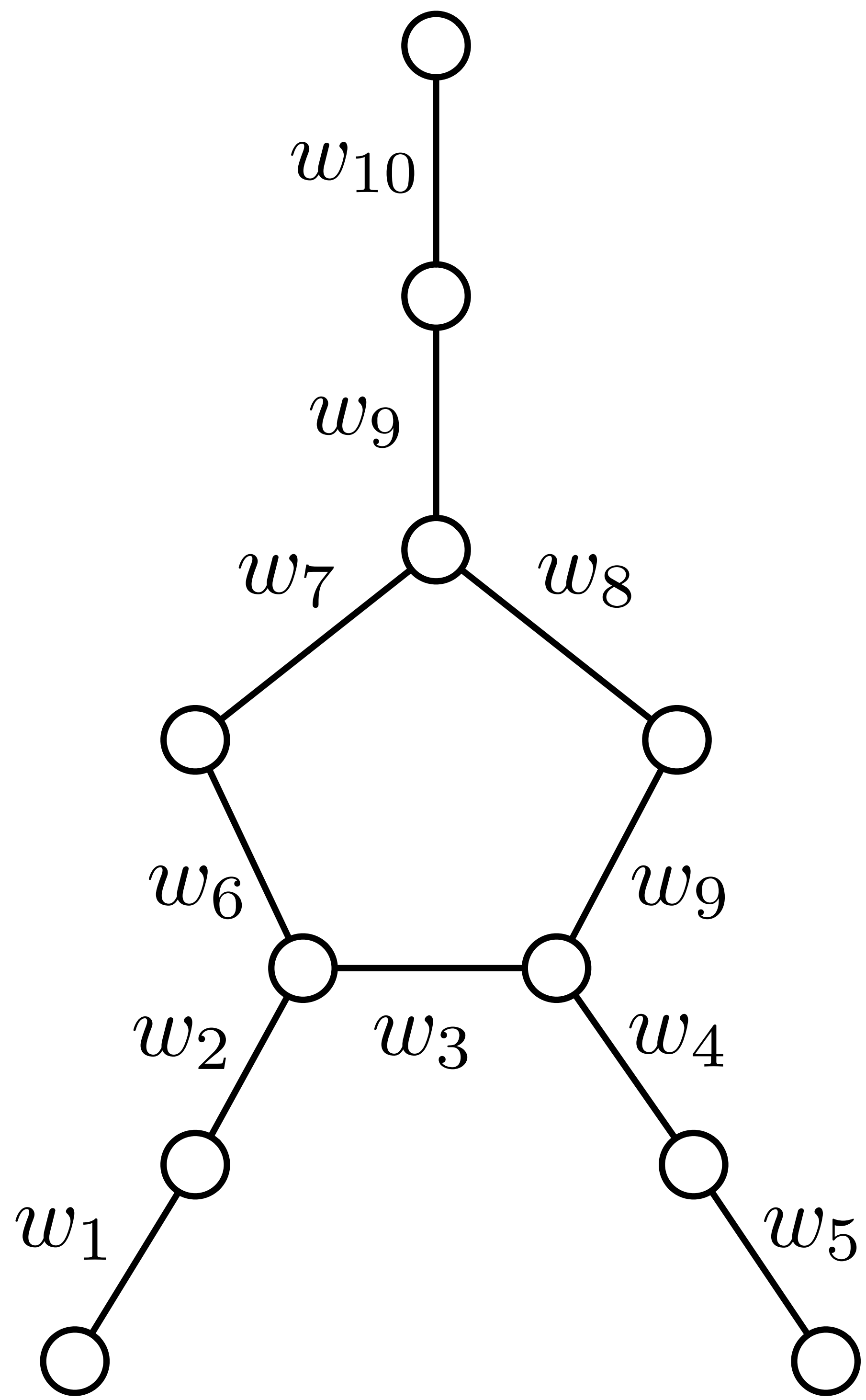
The smallest cardinality cut of a graph can be thought of as a **combinatorial robustness measure** for linear consensus protocols

As in the single negative weight edge example, graph cuts act to make an “open circuit”



- max-flow/min-cut algorithms
- minimum cardinality cut algorithms (Karger)

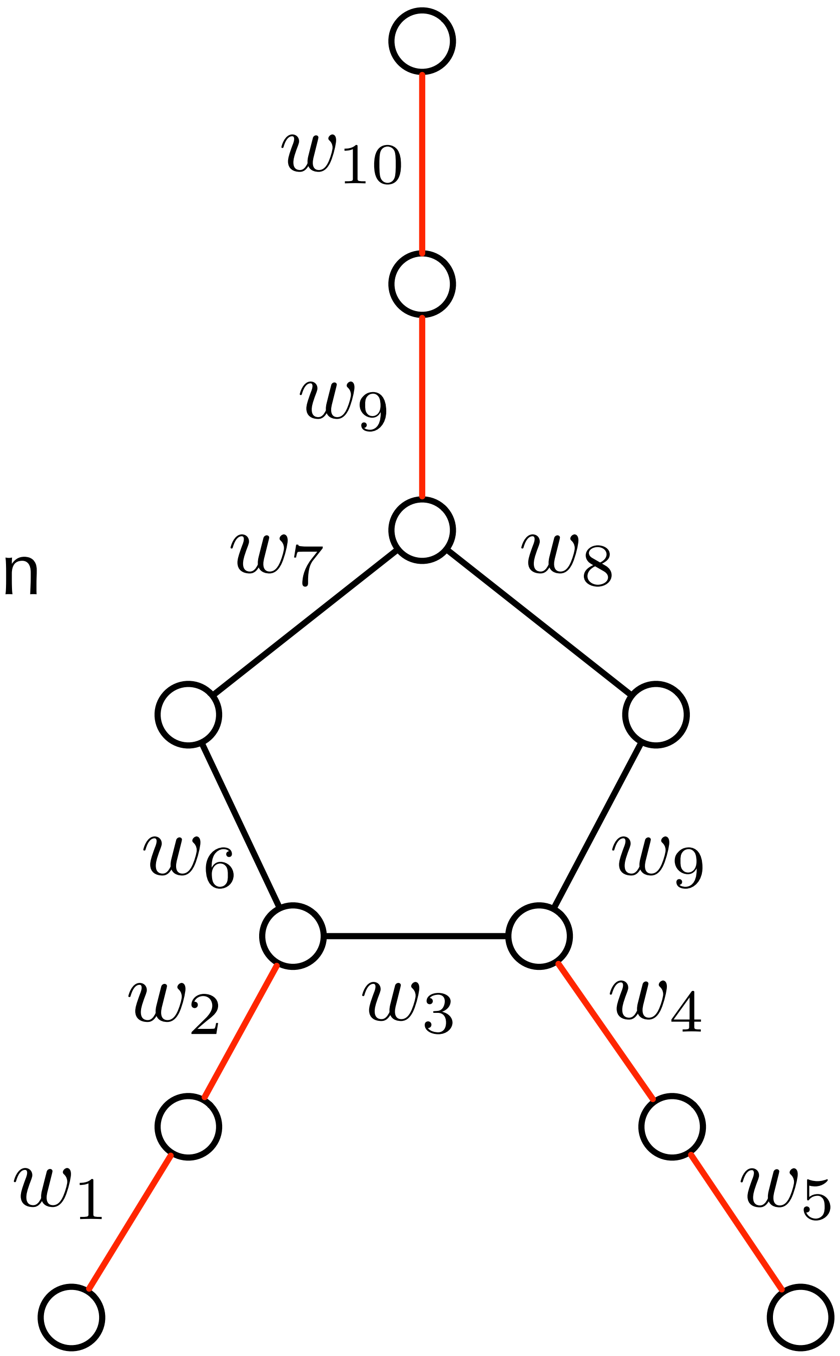
An Illustrative Example



An Illustrative Example

any single red edge is a cut in the graph

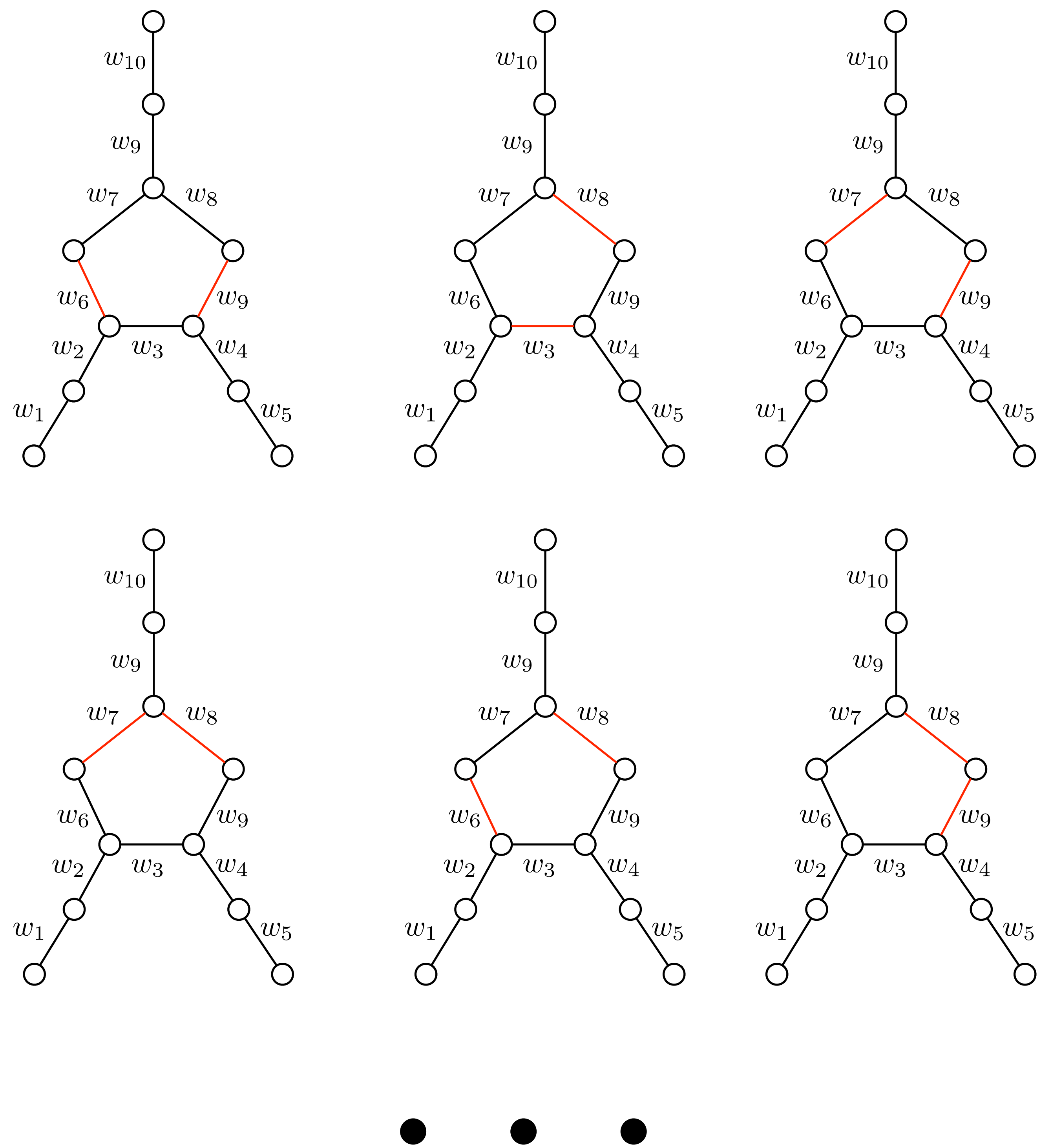
a negative weight on any red edge leads to an indefinite graph Laplacian



An Illustrative Example

any two edges on the cycle is a cut in the graph

a negative weight on any 2 red edge in a cycle leads to an indefinite graph Laplacian

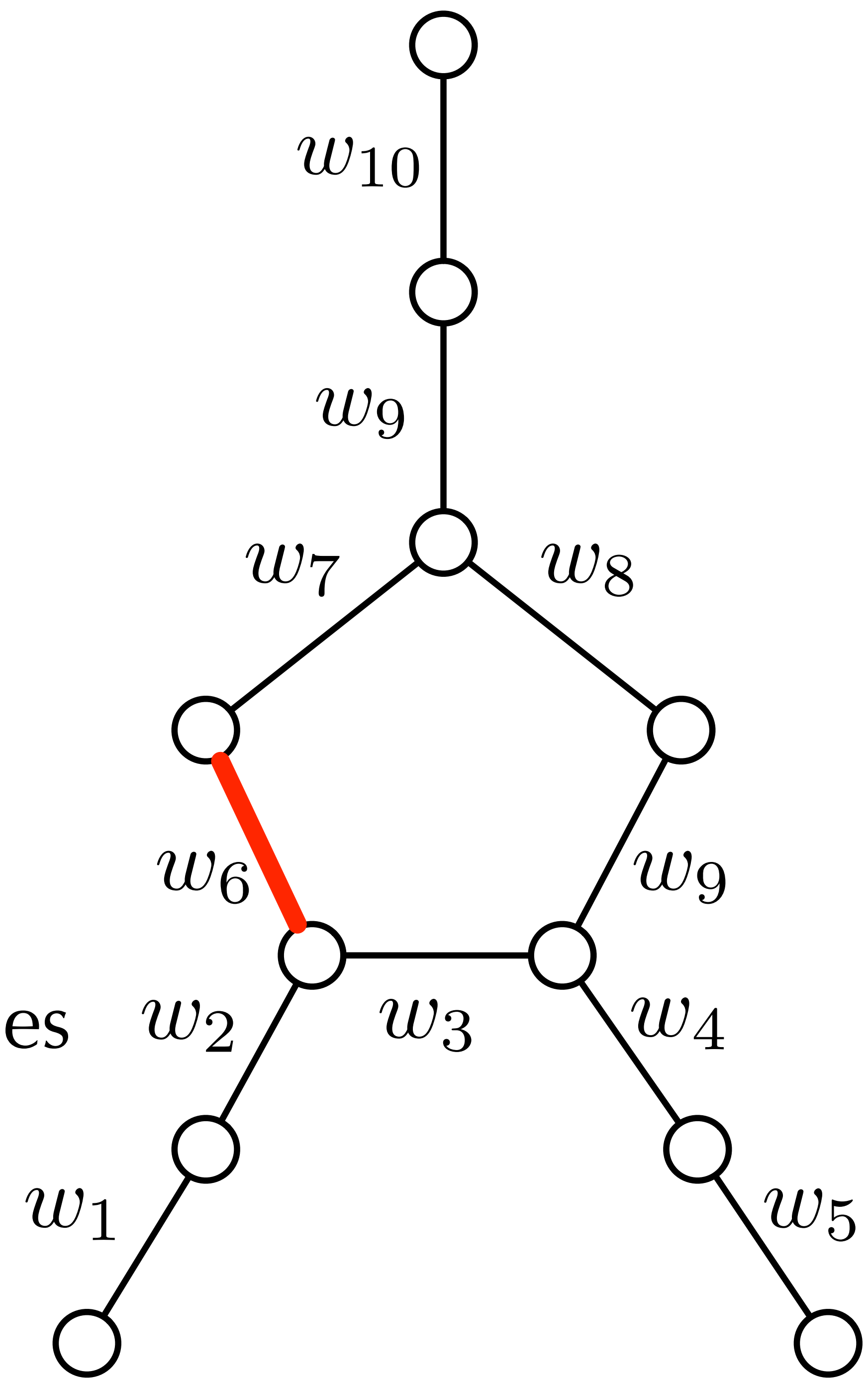


An Illustrative Example

any single edge in the cycle can make the Laplacian indefinite

$$w_6 = -\frac{1}{r_6} = -\frac{1}{4}$$

$L(\mathcal{G})$ has two eigenvalues at the origin



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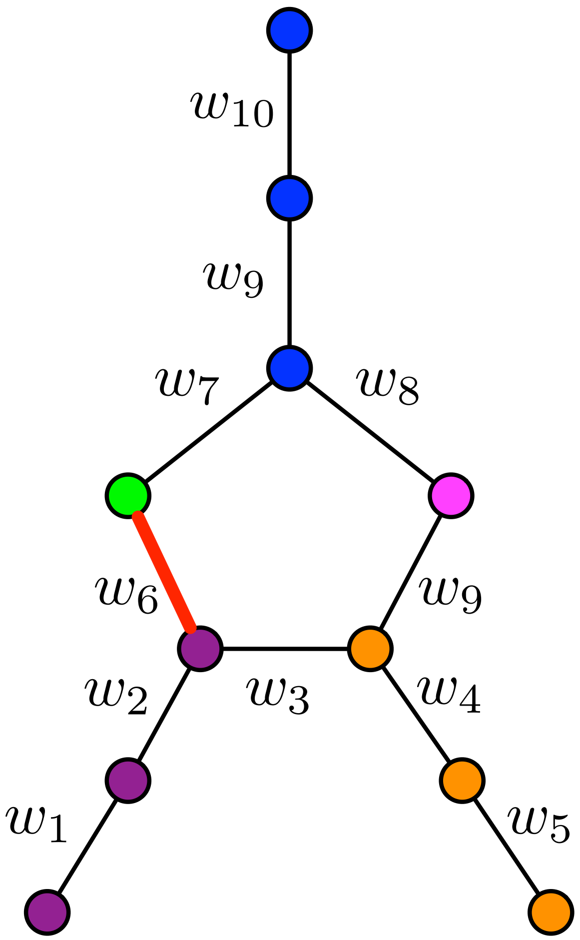
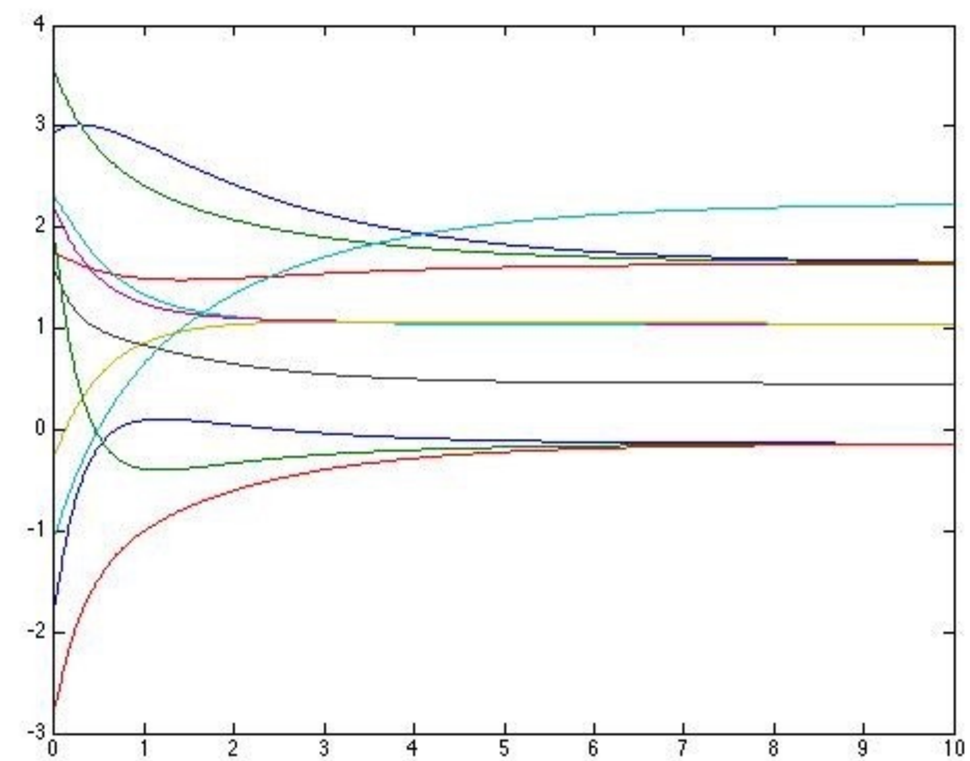




Computing Negative Weights

$$\min_{W_-} \quad \|W_-\|_p$$

$$\text{s.t.} \quad \begin{bmatrix} |W_-|^{-1} & E_-^T (E_{\mathcal{F}_+}^L)^T & E_-^T N_{\mathcal{F}_+} \\ E_{\mathcal{F}_+}^L E_- & R_{(\mathcal{F}_+,\mathcal{C}_+)} W_+ R_{(\mathcal{F}_+,\mathcal{C}_+)}^T & 0 \\ N_{\mathcal{F}_+}^T E_- & 0 & \mathbf{0} \end{bmatrix} \geq 0$$

- infeasible solution —> negative weight edges form a cut of the graph
- norm choice can influence structure of Laplacian spectrum

An Optimization Perspective

Consider the following optimization problem

$$\alpha = \min_{x} \frac{1}{2} x^T L(\mathcal{G}) x \quad = \min_{y,\zeta} \frac{1}{2} \zeta^T W \zeta$$

$$\text{s.t.} \zeta = E^T y$$

The consensus protocol corresponds to the gradient dynamics

$$\dot{x}(t) = -L(\mathcal{G}) x(t)$$

The optimization problem has a bounded solution if and only if the Laplacian is positive semi-definite (i.e. convexity!)

$$\alpha = 0$$

Negative edge weights influence the *convexity* of the quadratic program

Difference of Convex (DC) Program

$$\alpha = \min_{y,\zeta} \frac{1}{2}\zeta^T W \zeta$$
$$\text{s.t.}\ \zeta = E^T y$$

"edge" variable   "node" variable

$$\alpha = \min_{y,\zeta_+,\zeta_-} \frac{1}{2}\zeta_+^T W_+ \zeta_+ - \frac{1}{2}\zeta_-^T |W_-| \zeta_-$$
$$\text{s.t.}\ \zeta_+ = E_+^T y,\ \zeta_- = E_-^T y$$

$$g(y) = \min_{\zeta_+} \frac{1}{2}\zeta_+^T W_+ \zeta_+ \quad \text{s.t.}\ \zeta_+ = E_+^T y$$

$$h(y) = \min_{\zeta_-} \frac{1}{2}\zeta_-^T W_- \zeta_- \quad \text{s.t.}\ \zeta_- = E_-^T y$$

$$g^*(u) = \sup_y \{y^T u - g(y)\} = \min_{\lambda_+} \frac{1}{2}\lambda_+^T W_+^{-1} \lambda_+ \quad \text{s.t.}\ u = E_+ \lambda_+$$

$h^*(u) =$ same form

Difference of Convex (DC) Program

A Duality Result

Lemma 1

$$\alpha = \min_{u} \left\{ \left(\min_{\lambda_-} \frac{1}{2}\lambda_-^T W_-^{-1}\lambda_-\right) - \left(\min_{\lambda_+} \frac{1}{2}\lambda_+^T W_+^{-1}\lambda_+\right) \right\}$$

$$u = E_-\lambda_-, \quad u = E_+\lambda_+.$$

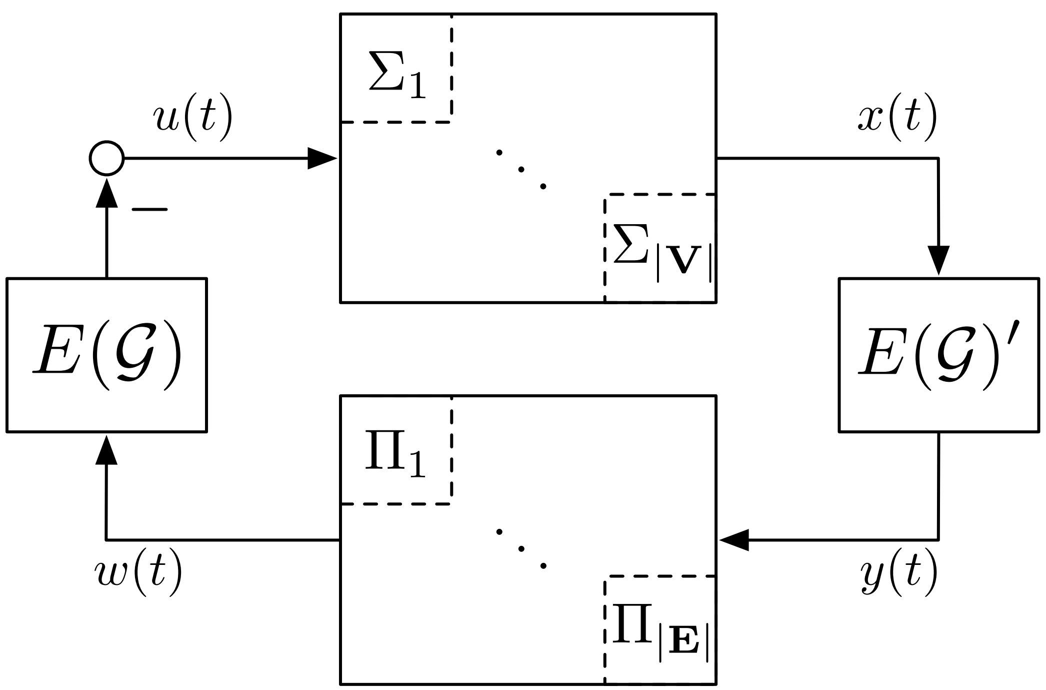
Theorem 1

$$L(\mathcal{G}) \geq 0 \Leftrightarrow \alpha = 0$$

What can the optimization perspective tell us?

Duality and Cooperative Control

a "canonical" networked dynamic system



- Passivity Theory
  - equilibrium independent passivity
  - maximal equilibrium independent passivity

(dynamic)

duality in convex optimization

$\mathcal{P}$

$$\min_x \sum_{i=1}^{n} J_i(x_i)$$
$$s.t.\ g(x) = 0$$

$$\mathcal{L}(x,\lambda) = \sum_{i=1}^{n} J_i(x_i) + \lambda^T g(x)$$

$\mathcal{D}$

$$\max_\lambda \inf_x \mathcal{L}(x,\lambda)$$

- Network Optimization Theory
  - optimal flow problems
  - optimal distribution problems

(static)

Network Optimization Problems

Optimal Flow Problem

$$\min_{\mathbf{u},\mu} \quad \sum_{i=1}^{|\mathbf{V}|} C_i^{div}(\mathrm{u}_i) + \sum_{k=1}^{|\mathbf{E}|} C_k^{flux}(\mu_k)$$

s.t. $\quad \mathbf{u} + E\mu = 0.$

- u_i: *divergence* (in/out-flow) at a node
- μ_k: *flow* on an edge

Optimal Potential Problem

$$\min_{\mathbf{y},\zeta} \quad \sum_{i=1}^{|\mathbf{V}|} C_i^{pot}(\mathrm{y}_i) + \sum_{k=1}^{|\mathbf{E}|} C_k^{ten}(\zeta_k)$$

s.t. $\quad \zeta = E^\top \mathbf{y}.$

- y_i: *potential* at a node
- ζ_k: *tension* (potential difference) on an edge

Dual Optimization Problems

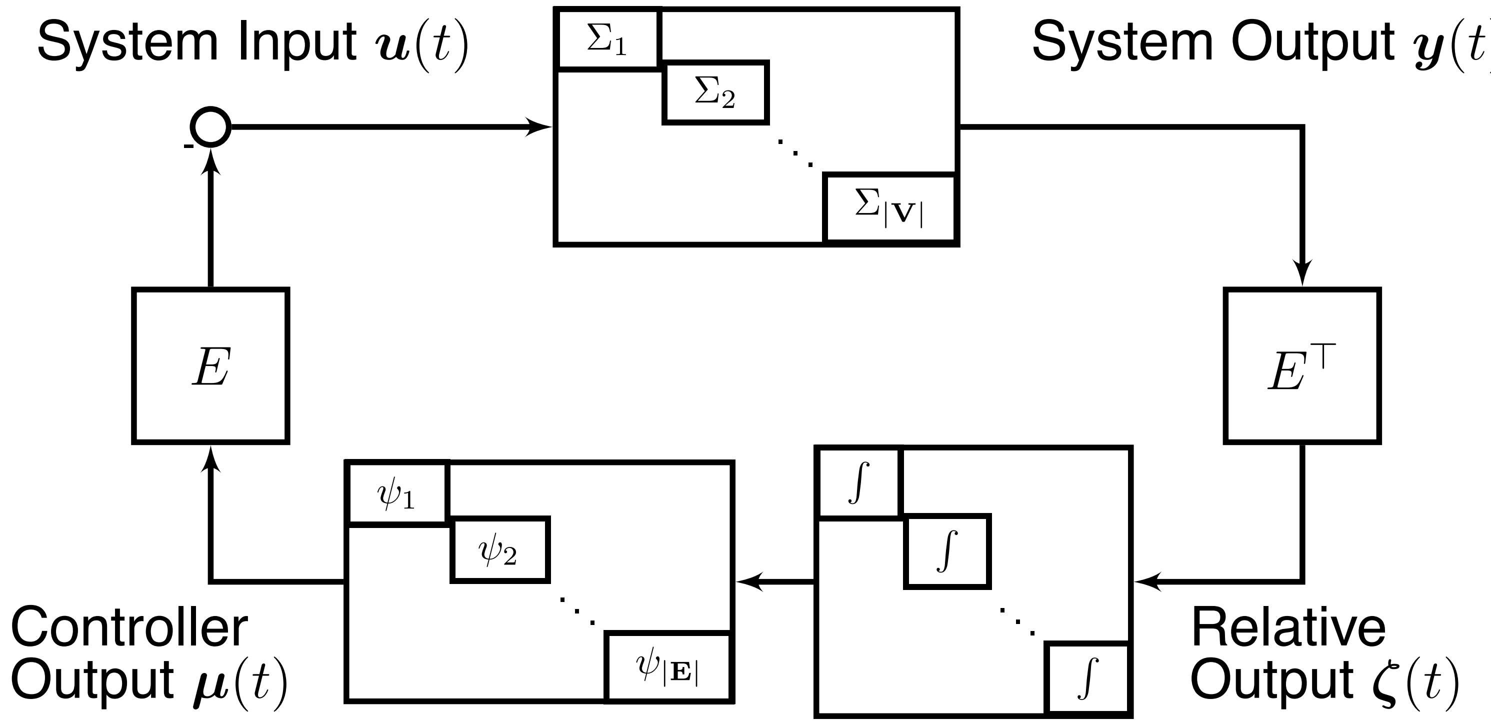
defined over the "same" network

$$C_i^{pot}(\mathrm{y}_i) := C_i^{div,*} = -\inf_{\tilde{\mathrm{u}}_i} \left\{ C_i^{div}(\tilde{\mathrm{u}}_i) - \mathrm{y}_i \tilde{\mathrm{u}}_i \right\}$$

Duality and Cooperative Control

$$\min_{\mathbf{u},\mu} \sum_{i=1}^{|\mathbf{V}|} K_i(\mathrm{u}_i)$$

s.t. $\mathbf{u} + E\mu = 0.$



$$\min_{\mathrm{y}_i} \sum_{i=1}^{|\mathbf{V}|} K_i^\star(\mathrm{y}_i),$$

s.t. $E^\top \mathbf{y} = 0.$

$\mathbf{y} = \partial \mathbf{K}(\mathbf{u})$

Divergence $\mathbf{u}$ —— Potential $\mathbf{y}$

$\mathbf{u} = -E\mu$

$\zeta = E^\top \mathbf{v}$

$$\min_{\mu} \sum_{k=1}^{|\mathbf{E}|} P_k^\star(\mu_k)$$

s.t. $\mathbf{u} + E\mu = 0,$

Flow μ —— Tension η

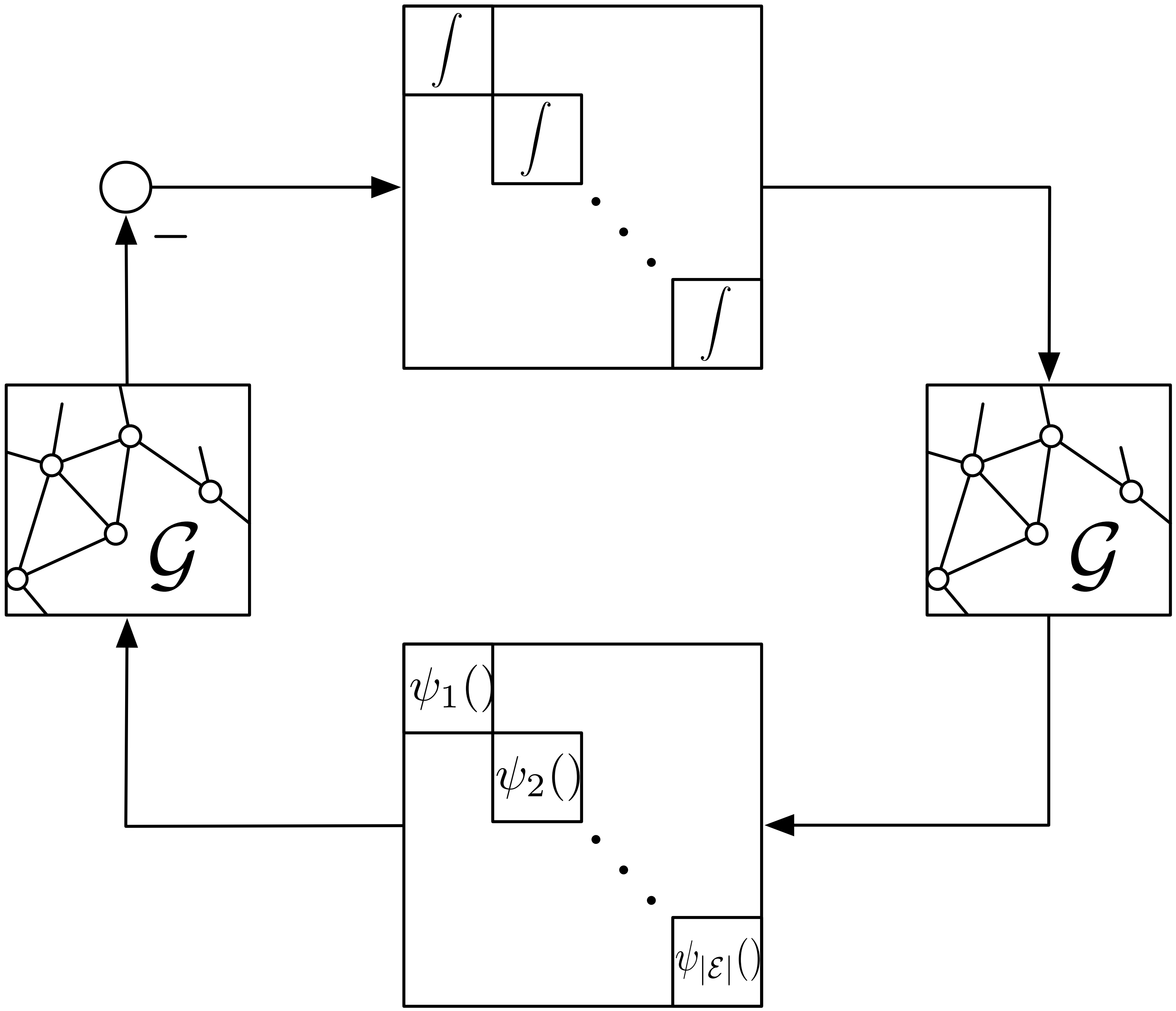
Tension ζ

$$\min_{\eta,\mathbf{v}} \sum_{k=1}^{|\mathbf{E}|} P_k(\eta_k) - \sum_{i=1}^{|\mathbf{V}|} \mathrm{u}_i \mathrm{v}_i,$$

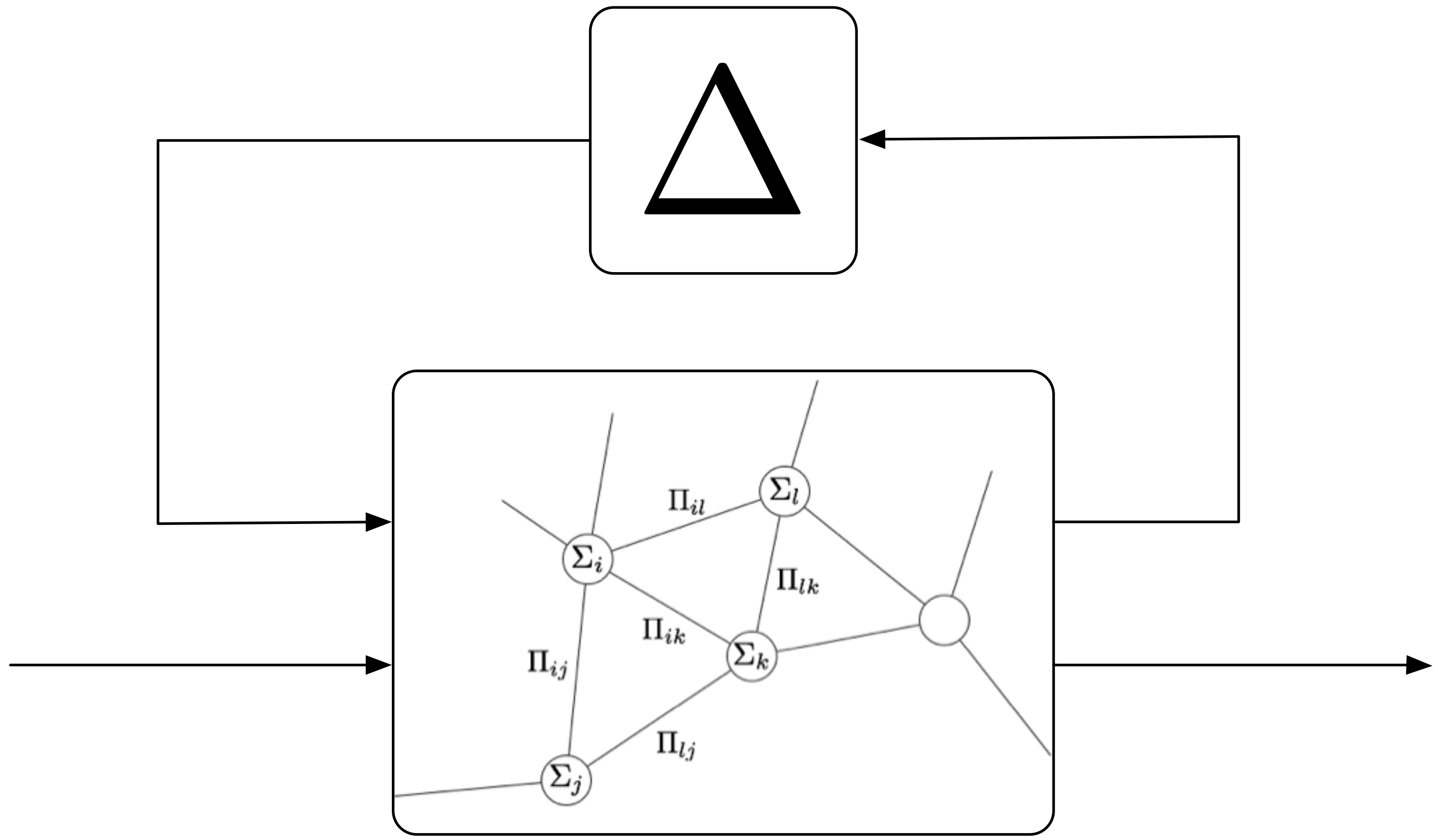
s.t. $\eta = E^\top \mathbf{v}.$

$\mu = \nabla \mathbf{P}(\eta)$

Nonlinear Consensus



Concluding Remarks



- networked dynamic systems require new tools for robustness analysis
- graph properties have real system theoretic implications

Acknowledgements

Thank-you!

Mathias Bürger

Questions?