SYMMETRY-CONSTRAINED FORMATION MANEUVERING

64^{th} ISRAEL ANNUAL CONFERENCE ON AEROSPACE SCIENCES

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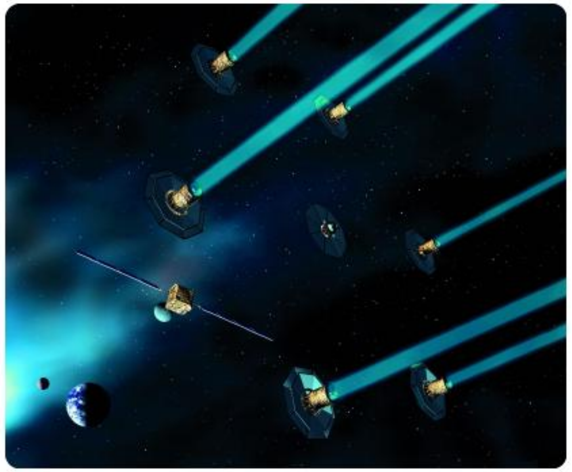
CONNECT LAB
Cooperative Networks and Controls

FORMATION CONTROL - INTRODUCTION

Many applications require multiple agents to organize into specific spatial formations.

- UAV Formations
  - Surveillance
  - Aerial Transportation
  - Communication Networks

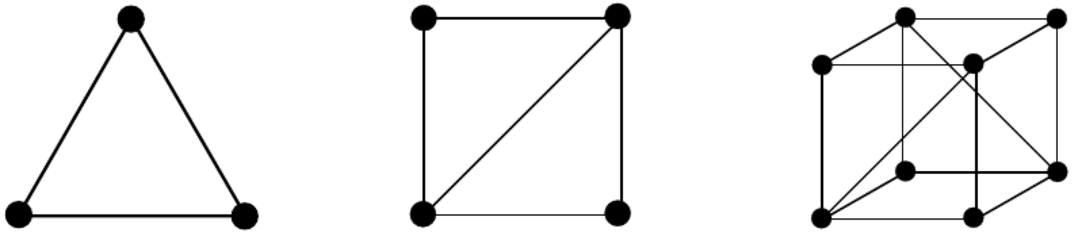
- Spacecraft formations
  - Interferometry
  - Constellations for sensing



FORMATION CONTROL - OBJECTIVE

Given a team of agents able to sense/communicate with neighboring agents:

- Design a control strategy for each agent by using *only* local information to achieve a desired spatial configuration - Formation Aquisition



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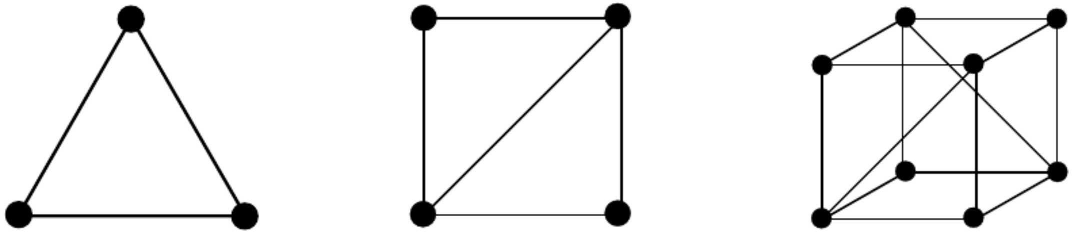
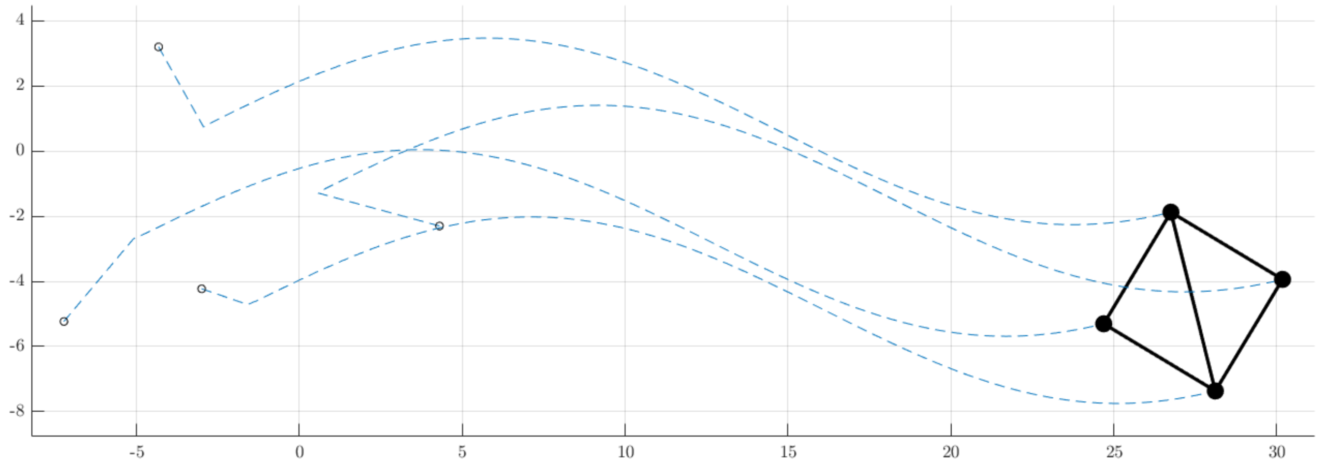


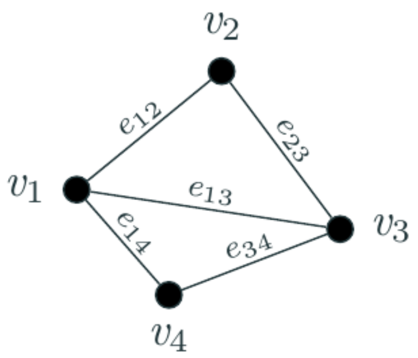
- Simultaneously move the formation through space as a rigid body - Formation Maneuvering



Consider a team of n agents, where the position of the ith agent is given by $p_i(t) \in \mathbb{R}^d$. Each follows the simple integrator dynamics:

$$\dot{p}_i(t) = u_i(t)$$

- The agents interact according to an information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$



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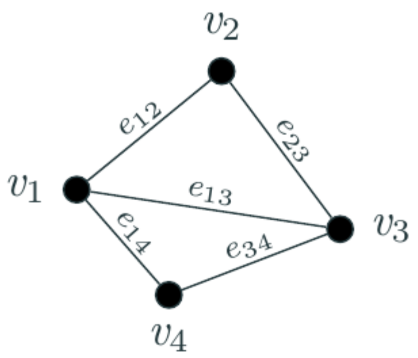
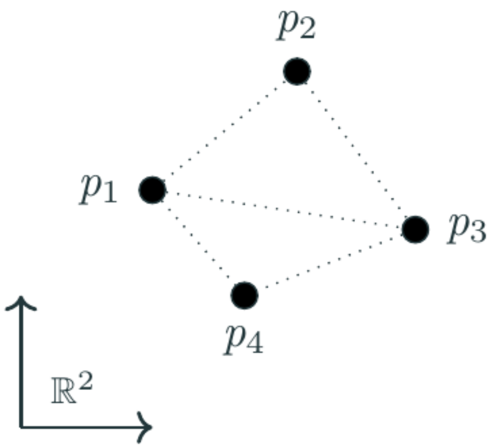


- The framework - $(\mathcal{G}, p)$ embeds the graph in Euclidean space



FORMATION CONTROL - AGENT CONFIGURATION

Consider a team of n agents, where the position of the ith agent is given by $p_i(t) \in \mathbb{R}^d$. Each follows the simple integrator dynamics:

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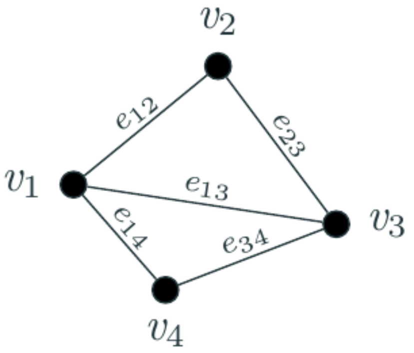

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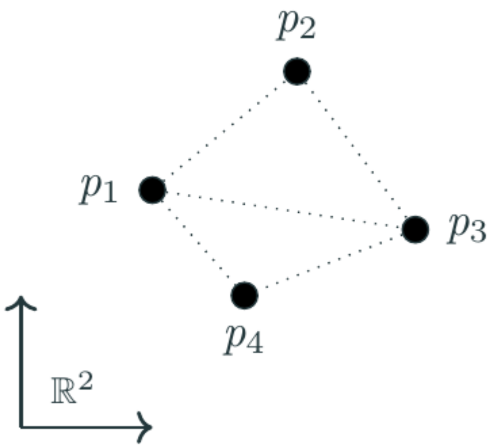
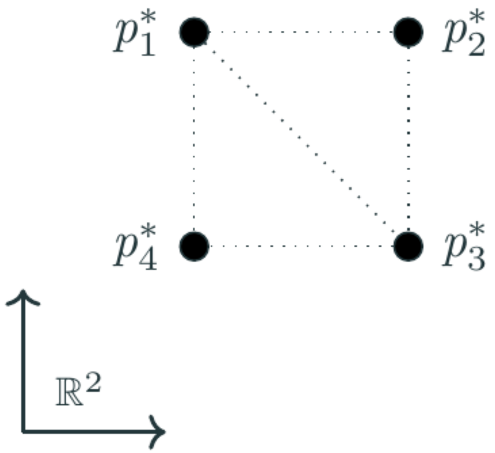


- The desired formation is represented by the framework $(\mathcal{G}, \mathbf{p}^\star)$



FORMATION CONTROL - CONSTRAINTS

- The desired formation is characterized by a set of M constraints, encoded in the function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, and a configuration $\mathbf{p}^\star$ satisfying the constraints.
- The set of all feasible formations is

$$\mathcal{F}(p) = \{p \in \mathbb{R}^{nd} \,|\, F(p) = F(\mathbf{p}^\star)\}$$

Formation Control Objective

For an ensemble of n agents with dynamics

$$\dot{p}_i = u_i,$$

with $p_i(t) \in \mathbb{R}^d$, an information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and formation constraint function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, design a distributed control law for each agent $i \in \{1, \ldots, n\}$ such that the set

$$\mathcal{F}(p) = \{p \in \mathbb{R}^{nd} \,|\, F(p) = F(\mathbf{p}^\star)\},$$

is asymptotically stable.

Theorem - Distance Constrained Formation Control [Krick 2009]

Consider the potential function

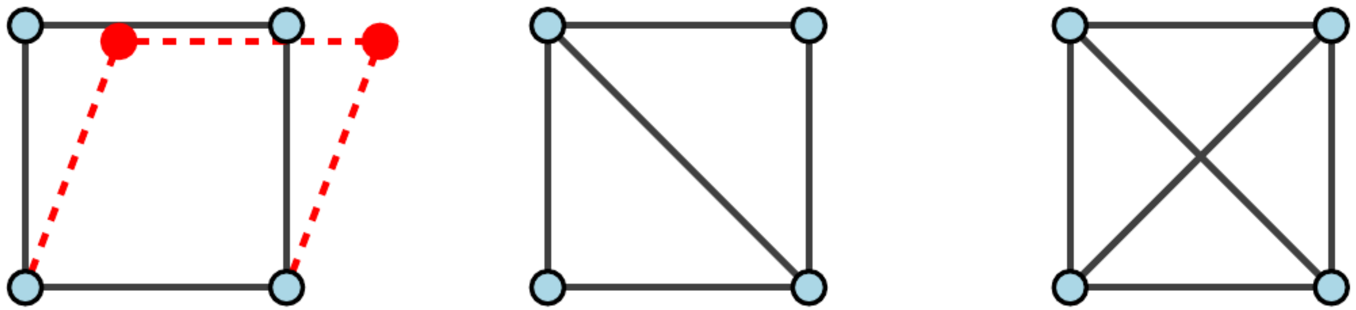
$$F_f(p) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2 \right)^2$$

and assume the desired distances $d_{ij}^\star$ correspond to a feasible formation. Then the gradient dynamical system

$$\dot{p}_i = u_i = -\nabla_{p_i} F_f(p) = \sum_{ij \in \mathcal{E}} \left(\|p_i - p_j\|^2 - (d_{ij}^\star)^2 \right) (p_j - p_i)$$

asymptotically converges to the critical points of the potential function, i.e., $\frac{\partial F_f(p)}{\partial p} = 0$.

How do we define shapes ?



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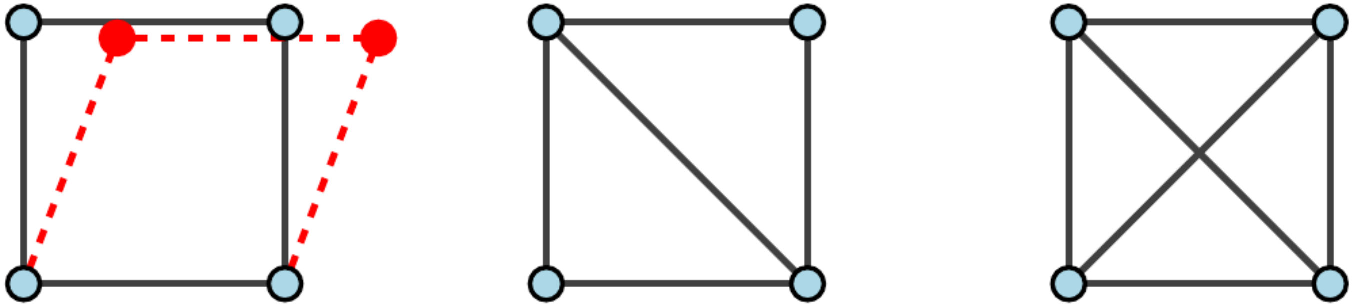

- Rigidity Theory allows us to determine:
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  - how the constraints should be distributed on the network.

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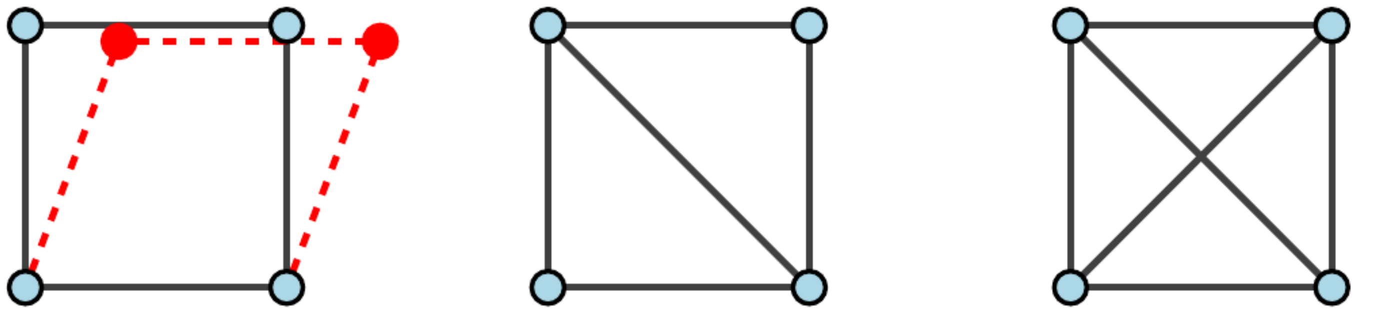

- Rigidity Theory allows us to determine:
  - the number of constraints required to ensure the desired shape.
  - how the constraints should be distributed on the network
- $R(p) = \frac{\partial F(p)}{\partial p} = \mathrm{diag}(p_i - p_j)(E^T \otimes I_d)$, the rigidity matrix of $(\mathcal{G}, p)$, where E is the incidence matrix of $\mathcal{G}$
  - A framework is infitesimally rigid if and only if $\mathrm{rk} R(p) = 2n - 3$ in $\mathbb{R}^2$
  - property that ensures formations defined properly

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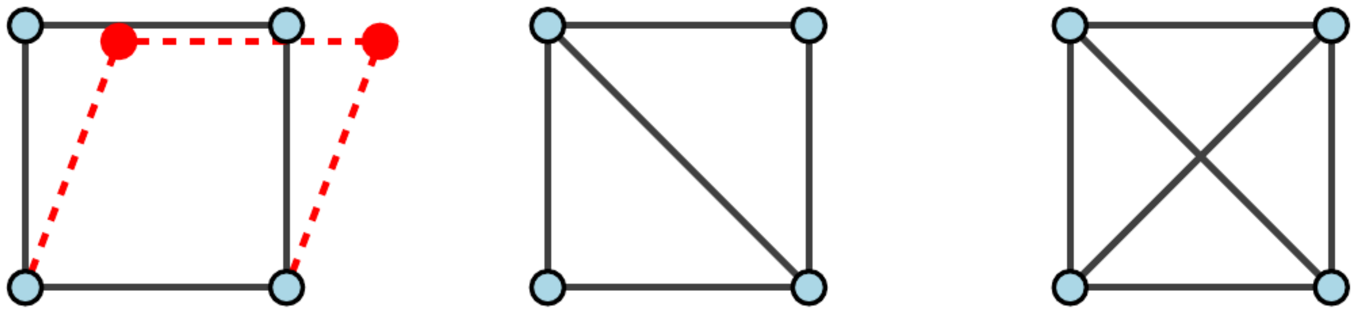

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$$\dot{p} = -\nabla_p F_f(p) = -R^T(p)\left(R(p)p - (d^\star)^2\right)$$

- Properties of the rigidity matrix lead to an architectural requirement for formation control problems, ensuring that the controller converges to the correct formation shape. Equivalent to:

$$\boxed{\mathrm{rk}\, R(p) = 2|\mathcal{V}| - 3 \text{ and } |\mathcal{E}| = 2|\mathcal{V}| - 3} \quad \text{(in } \mathbb{R}^2\text{)}$$

Q: Can the problem be solved with fewer constraints ?

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Q: Can the problem be solved with fewer constraints ?

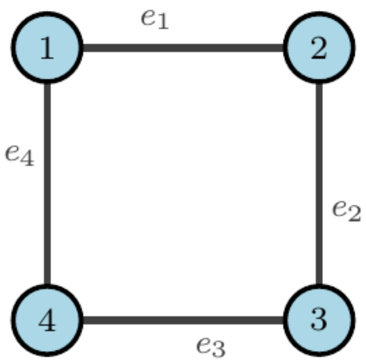
A: Yes, by additionally implementing symmetry constraints!

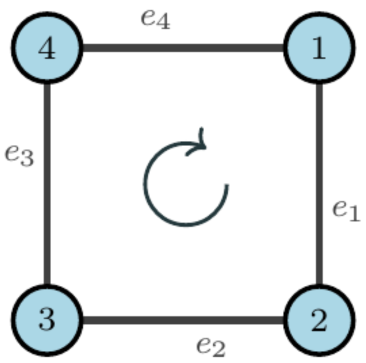
SYMMETRY AND GRAPH AUTOMORPHISMS

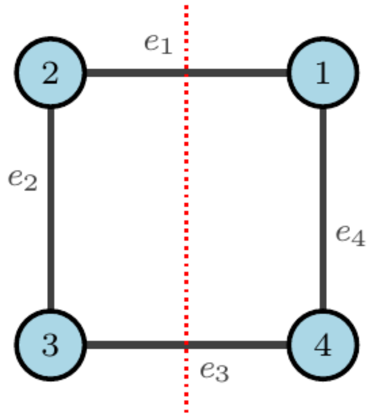
Graph Automorphism

An automorphism of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation $\psi : \mathcal{V} \rightarrow \mathcal{V}$ of of its vertex set such that

$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$







Identity:

$$\mathrm{Id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

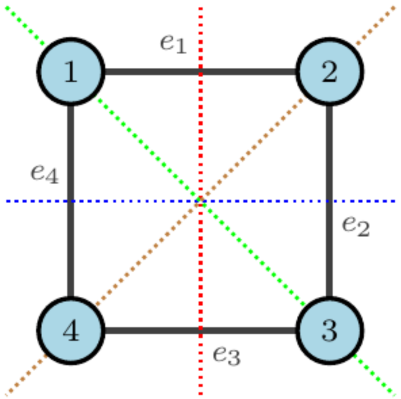
90° rotation:

$$\psi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

reflection:

$$\psi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

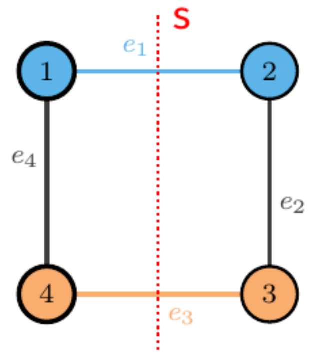
Automorphisms encode graph symmetries



- Additional permutations can be found for the given graph considering all possible reflections and rotations (by $180°$ and $270°$)
- The set of all automorphisms of $\mathcal{G}$ form a *group* - $\mathrm{Aut}(\mathcal{G})$
  - $\mathrm{Aut}(\mathcal{G}) = \{\mathrm{Id}, \psi_1, \psi_2, ...\}$
- A subgroup is a subset of a group, satisfying all properties of a group
  - $\{\mathrm{Id}, \psi_1\}$
  - $\{\mathrm{Id}, \psi_2\}$
- Subgroups of $\mathrm{Aut}(\mathcal{G})$ define specific symmetries in $\mathcal{G}$
- for any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$, we say that $\mathcal{G}$ is Γ-symmetric

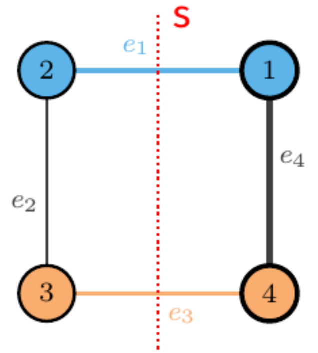
Definition

For a Γ-symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and vertex $i \in \mathcal{V}$, the set $\Gamma_i = \{\gamma(i) \mid \gamma \in \Gamma\}$ is called the vertex orbit of i. Similarly, for an edge $e = ij \in \mathcal{E}$, the set $\Gamma_e = \{\gamma(i)\gamma(j) \mid \gamma \in \Gamma\}$ is termed the edge orbit of e.



consider $\Gamma = \{\mathrm{Id}, \psi_2\}$ (reflection about mirror S)

- **Vertex Orbit:**
  $\Gamma_1 = \Gamma_2 = \{1, 2\}, \ \Gamma_3 = \Gamma_4 = \{3, 4\}$



- **Edge Orbit:**
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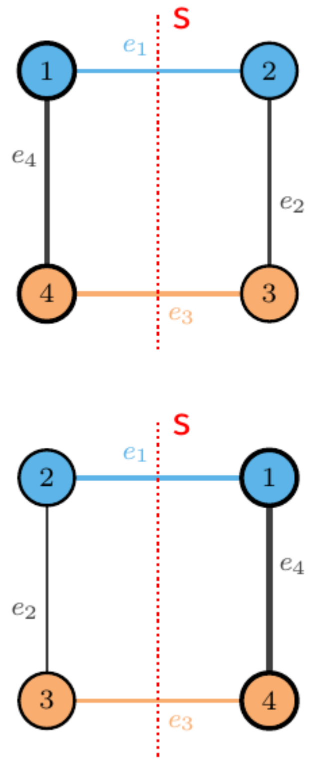

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  vertices inside a vertex orbit are equivalent

  representative vertex set: $\mathcal{V}_0 = \{1, 4\}$

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  representative edge set: $\mathcal{E}_0 = \{e_1, e_3, e_4\}$

Let Γ be represented as a point group.

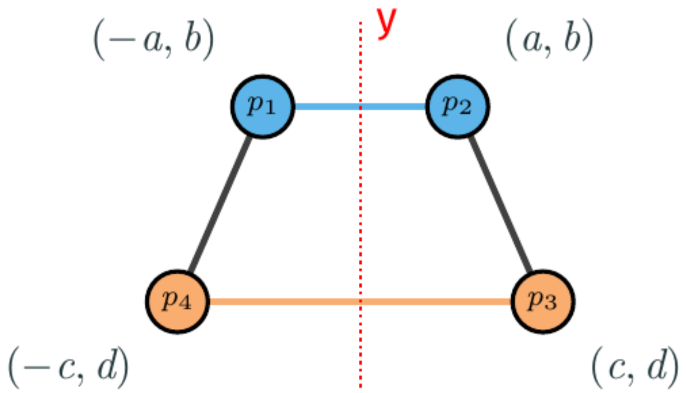
- homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$
- τ assigns an orthogonal matrix (describing an isometry of $\mathbb{R}^d$) to each element of Γ

Definition

A framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called $\tau(\Gamma)$-symmetric if

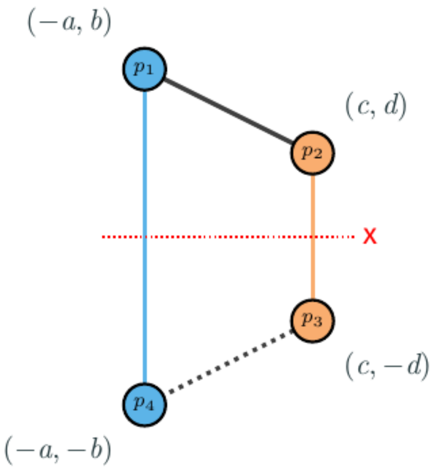
$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}.$$

For example



- consider $\Gamma = \{\mathrm{Id}, \psi_2\} \subseteq \mathrm{Aut}(\mathcal{G})$
- isometry $\tau(\psi_2) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$: $\tau(\psi_2)\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

isometries of the desired configuration coincide with symmetries of the automorphisms of $\mathcal{G}$



The rigidity matrix:

$$R(p) = \begin{bmatrix} (-a-c \quad b-d) & (c+a \quad d-b) & (0 \quad 0) & (0 \quad 0) \\ (0 \quad 2b) & (0 \quad 0) & (0 \quad 0) & (0 \quad -2b) \\ (0 \quad 0) & (0 \quad 2d) & (0 \quad -2d) & (0 \quad 0) \\ (0 \quad 0) & (0 \quad 0) & (c+a \quad -d+b) & (-a-c \quad -b+d) \end{bmatrix}$$

Symmetries make certain rows and columns of the rigidity matrix redundant

Orbit Rigidity Matrix $\mathcal{O}(\mathcal{G}_0, p)$ [Schulze 2011]

$$\mathcal{O}(\mathcal{G}_0, p) = \begin{bmatrix} (p_1 - p_2)^T & (p_2 - p_1)^T \\ (2p_1 - \tau_x p_1 - \tau_x^{-1} p_1)^T & (0 \quad 0) \\ (0 \quad 0) & (2p_2 - \tau_x p_2 - \tau_x^{-1} p_2)^T \end{bmatrix} = \begin{bmatrix} (-a-c \quad b-d) & (c+a \quad d-b) \\ (0 \quad 2b) & (0 \quad 0) \\ (0 \quad 0) & (0 \quad 2d) \end{bmatrix}$$

Describes the $\tau(\Gamma)$-symmetric infinitesimal rigidity properties of $\tau(\Gamma)$-symmetric frameworks.

The introduction of the orbit rigidity matrix suggests a further way to exploit symmetry in formation control

- representative edges used to maintain distances
- symmetry within vertex orbits have no need for distance constraints

A GRADIENT APPROACH

Similar to traditional rigidity approaches, define a *symmetric formation potential*

$$F_f(p(t)) = F_e(p(t)) + F_s(p(t))$$

where

- The representative edge formation potential:

$$F_e(p(t)) = \frac{1}{4}\sum_{ij\in\mathcal{E}_0}\left(\|p_i(t) - \tau(\gamma)p_j(t)\|^2 - (d^{\star}_{i\gamma(j)})^2\right)^2$$

- The symmetry potential:

$$F_s(p(t)) = \frac{1}{2}\sum_{i\in\mathcal{V}_0}\sum_{\substack{u,v\in\Gamma_i \\ uv\in\mathcal{E}}}\|p_u(t) - \tau(\gamma_{vu})p_v(t)\|^2$$

[Zelazo 25]

The states can be defined as $\tilde{p}(t) = Pp(t) = \begin{bmatrix} p_0^T(t) & p_f^T(t) \end{bmatrix}^T$, for some permutation matrix P.

- $p_0(t)$ - the restriction of the configuration vector $p(t)$ to agents in the representative vertex set $\mathcal{V}_0$.
- $p_f(t)$ - The remaining agents

Propose the gradient control

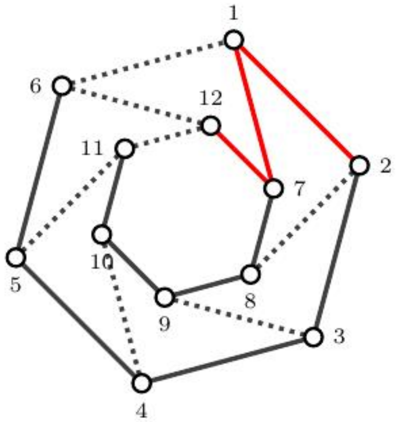
$$u(t) = -\nabla F_f(p(t))$$

The dynamics in state-space form become

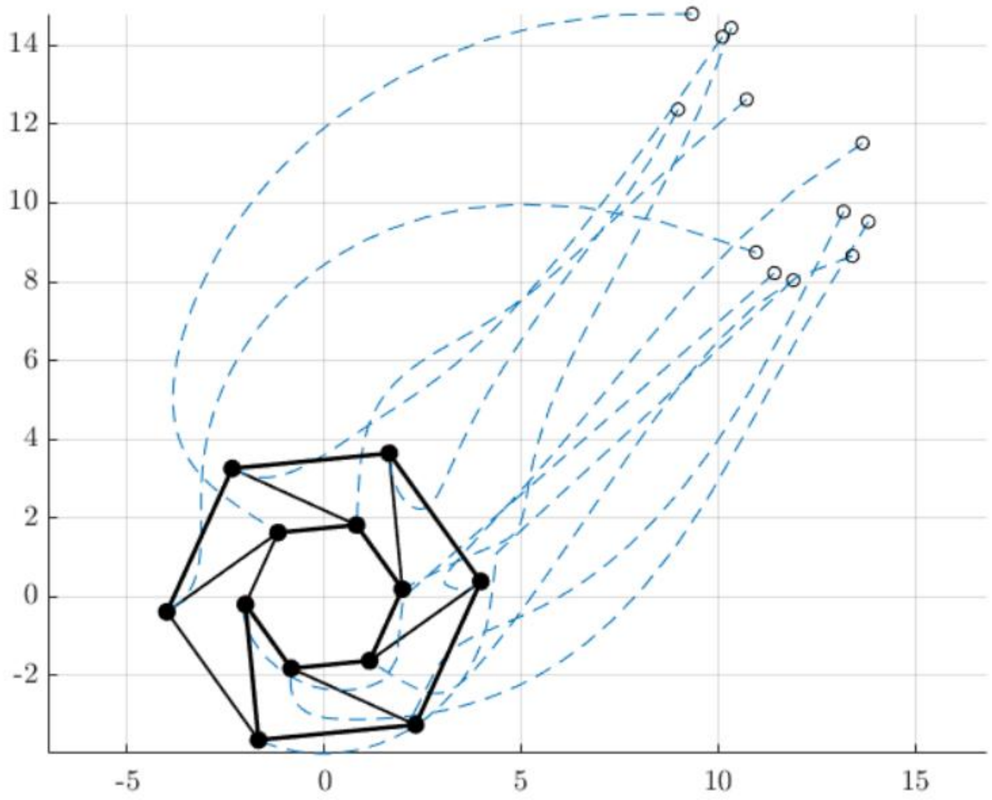
$$\begin{bmatrix} \dot{p}_0(t) \\ \dot{p}_f(t) \end{bmatrix} = \begin{bmatrix} -\mathcal{O}^T(\mathcal{G}_0, p_0(t))\left(\mathcal{O}(\mathcal{G}_0, p_0(t))p_0(t) - \mathbf{d}_0^2\right) \\ 0 \end{bmatrix} - PQP^T \begin{bmatrix} p_0(t) \\ p_f(t) \end{bmatrix}$$

[Zelazo 25]

SYMMETRIC FORMATION - EXAMPLE



- $2\pi/6$ rotational symmetry
- Requires at least 21 edges for ”classic” formation control
- Symmetry forced formation control requires only 11 edges



$\tau(\Gamma)$-symmetric frameworks by definition have point-group symmetries defined with respect to some fixed inertial point.

Q: Can the formation acquisition problem be achieved while simultaneously moving the formation through space as a rigid body ?

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Q: Can the formation acquisition problem be achieved while simultaneously moving the formation through space as a rigid body ?

A: Yes! By implementing a virtual state $r(t) \in \mathbb{R}^d$ as the reference signal for the agents to arrange themselves with respect to any point.

The trajectory consists only of a translation component, known by all agents.

Define the shifted state:

$$\bar{c}(t) = \begin{bmatrix} c_0^T(t) & c_f^T(t) \end{bmatrix}^T = P(p(t) - \mathbb{1} \otimes r(t))$$

choose $\begin{bmatrix} \dot{p}_0(t) & \dot{p}_f(t) \end{bmatrix}^T = u(t) = u_a(t) + u_m(t)$

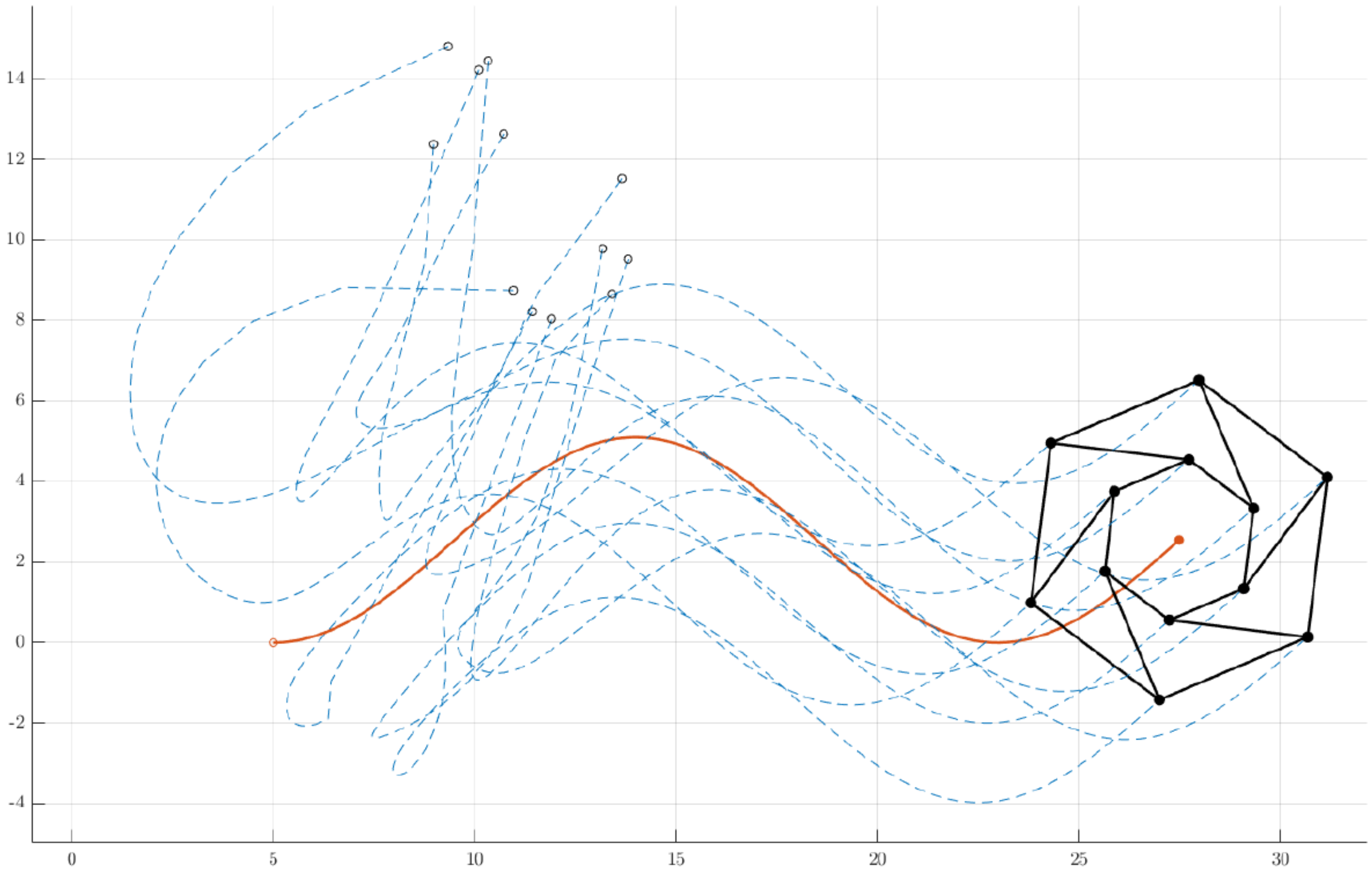
- Formation Acquisition

$$u_a(t) = \begin{bmatrix} -\mathcal{O}^T(\mathcal{G}_0, c_0(t))\left(\mathcal{O}(\mathcal{G}_0, c_0(t))c_0(t) - \mathbf{d}_0^2\right) \\ 0 \end{bmatrix} - PQP^T \begin{bmatrix} c_0(t) \\ c_f(t) \end{bmatrix}$$

- Formation Maneuvering

$$u_m(t) = \mathbb{1} \otimes \dot{r}(t)$$

FLOCKING – EXAMPLE



A single agent is subjected to the reference velocity input.

The modified control including a reference model takes the form:

$$\begin{bmatrix} \dot{p}_0(t) \\ \dot{p}_f(t) \end{bmatrix} = \begin{bmatrix} -\mathcal{O}^T(\mathcal{G}_0, c_0(t))\left(\mathcal{O}(\mathcal{G}_0, c_0(t))c_0(t) - \mathbf{d}_0^2\right) \\ 0 \end{bmatrix} - PQP^T \begin{bmatrix} c_0(t) \\ c_f(t) \end{bmatrix} + \dot{\bar{r}}(t)$$

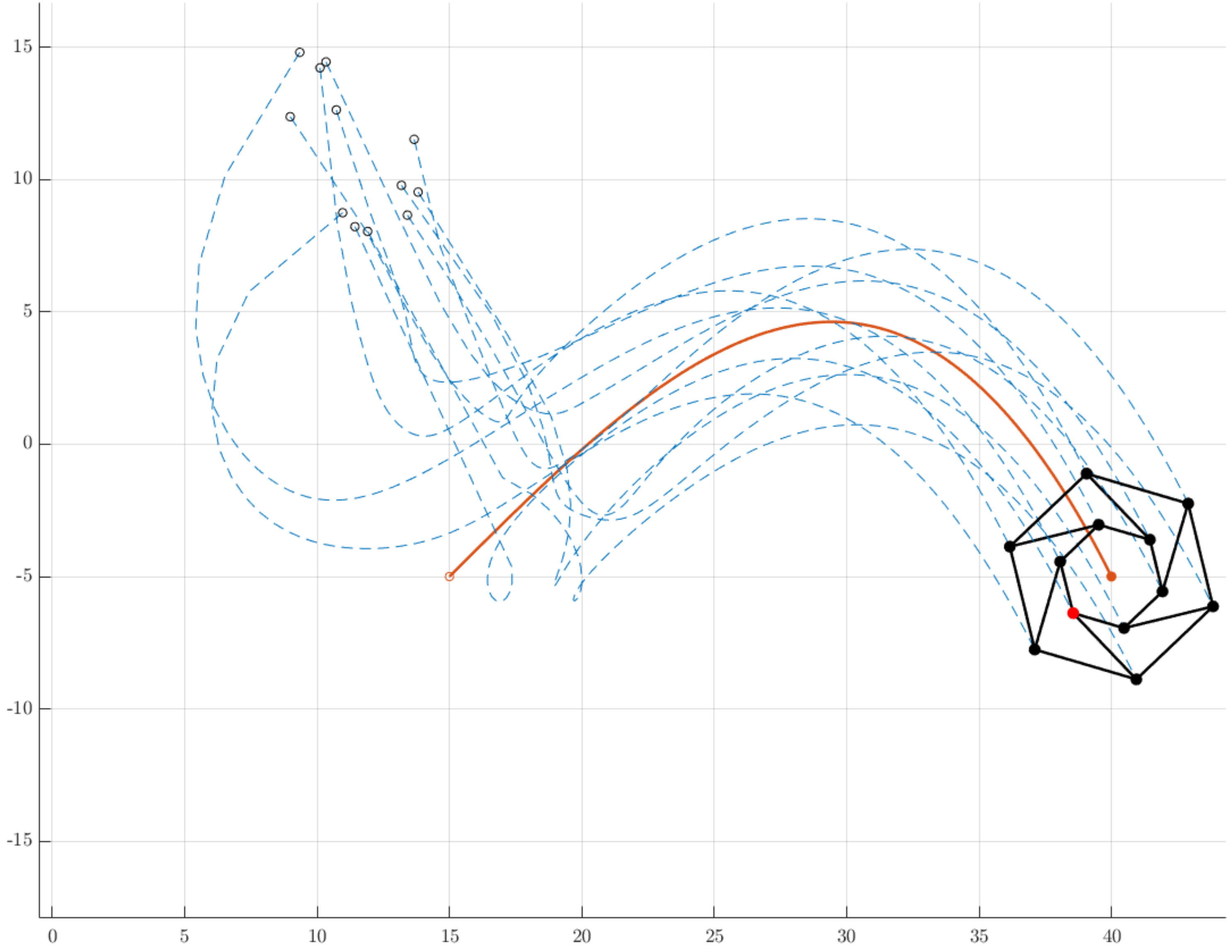
The trajectory is computed distributedly based to the consensus protocol:

$$\begin{cases} \dot{\bar{r}} & = -k_P \bar{L}(\mathcal{G})\bar{r} - k_I \bar{L}(\mathcal{G})\bar{\zeta} + nB \otimes v_0(t) \\ \dot{\bar{\zeta}} & = \bar{L}(\mathcal{G})\bar{r} \end{cases}$$

where:

- $v_0 \in \mathbb{R}^d$ is the reference velocity input
- $B \in \mathbb{R}^n$ is a standard base vector denoting which agent is subjected to $v_0(t)$

FLOCKING: DISTRIBUTED APPROACH - EXAMPLE



Symmetry-constrained formations undergoing rotations requires time-varying point group symmetries

A similarity transformation of a point group element $\tau(\gamma)$ by a rotation matrix $R(\theta(t))$ reorients the isometries about $\theta(t)$ in the original frame

$$\tau(\gamma, \theta(t)) = R(\theta(t))\tau(\gamma)R(\theta(t))^{-1}$$

Notations:

- $\theta(t)$ - The orientation of the rigid body
- $\omega_0(t)$ - The desired angular velocity vector

Assumption: The centroid of the formation is defined at the origin

Recall the defined shifted state:

$$\bar{c}(t) = \begin{bmatrix} c_0^T(t) & c_f^T(t) \end{bmatrix}^T = P(p(t) - \mathbb{1} \otimes r(t))$$

choose $\begin{bmatrix} \dot{p}_0(t) & \dot{p}_f(t) \end{bmatrix}^T = u(t) = u_a(t) + u_m(t)$

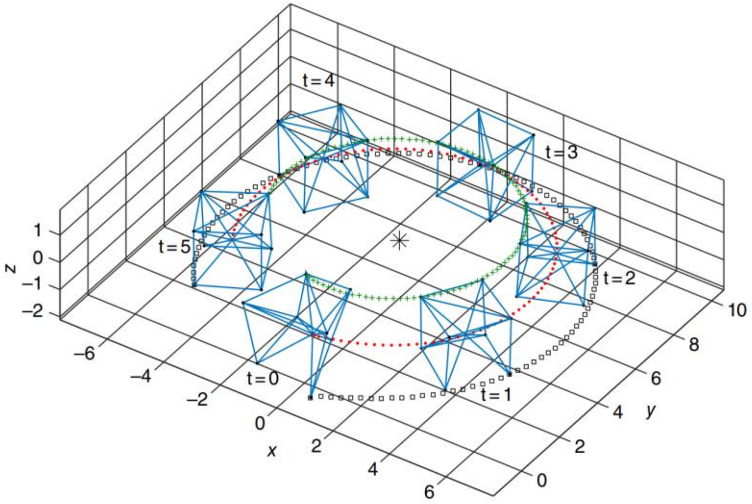
- Formation Aquisition

$$u_a(t) = \begin{bmatrix} -\mathcal{O}^T(\mathcal{G}_0, c_0(t), \tau(t)) \left(\mathcal{O}(\mathcal{G}_0, c_0(t), \tau(t)) c_0(t) - \mathbf{d}_0^2 \right) \\ 0 \end{bmatrix} - PQ(\tau(t)) P^T \begin{bmatrix} c_0(t) \\ c_f(t) \end{bmatrix}$$

- Formation Maneuvering

$$u_m(t) = \mathbb{1} \otimes \dot{r}(t) + \begin{bmatrix} \vdots \\ \omega_0(t) \times p_i(t) \\ \vdots \end{bmatrix}$$

SYMMETRY CONSTRAINED FORMATION MANEUVERING - EXAMPLE

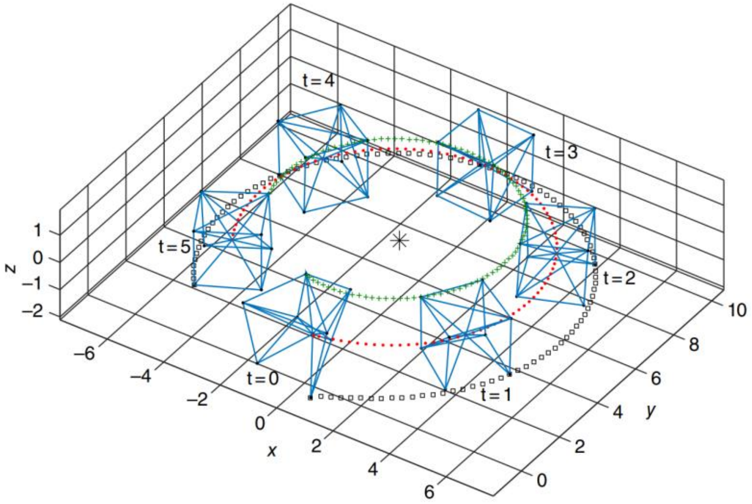


[Queiroz 18]

For classic formation control:

- A desired cube formation requires a known agent representing its geometric center
- At least 21 edges are required for infinitesimal rigidity

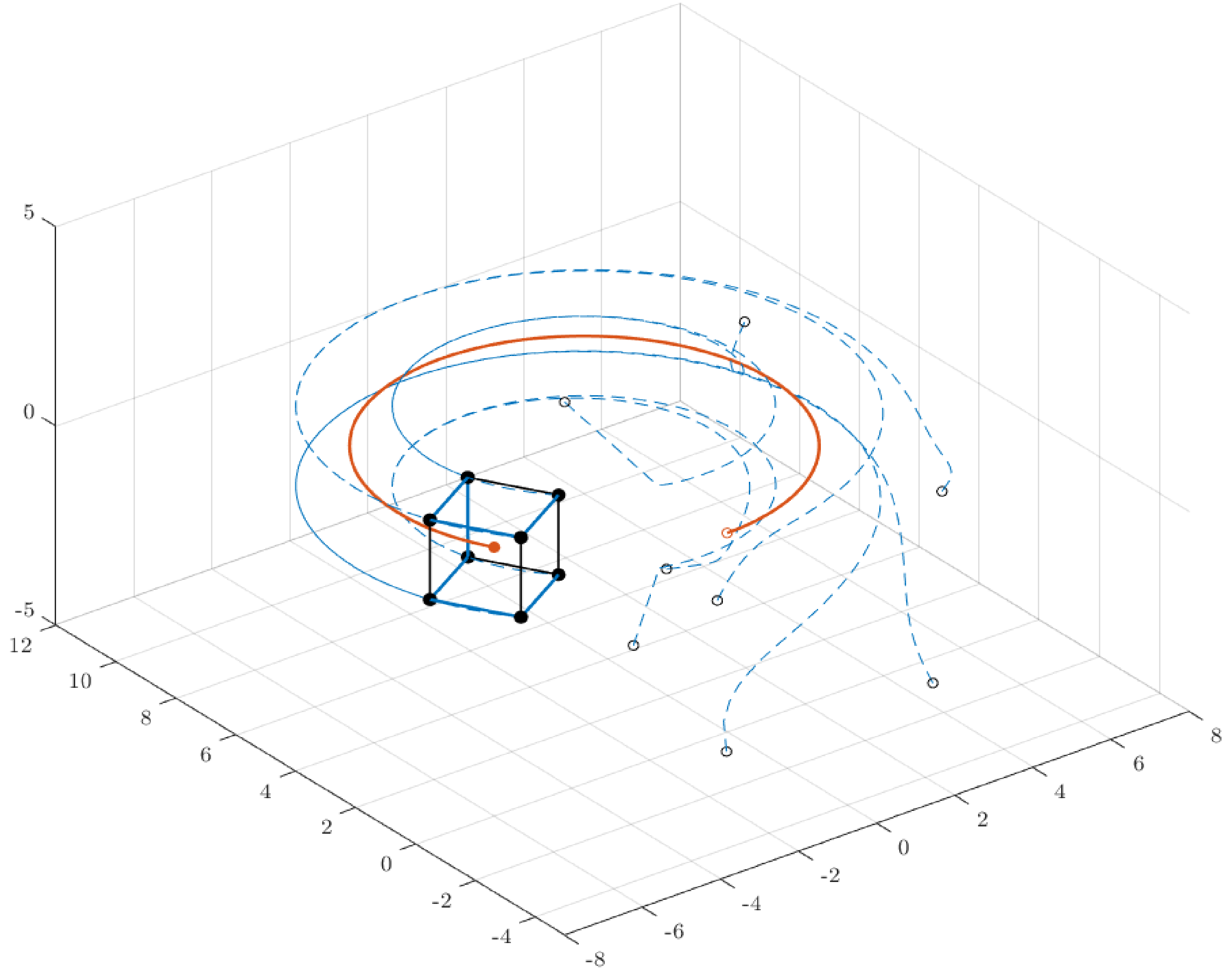
SYMMETRY CONSTRAINED FORMATION MANEUVERING - EXAMPLE



[Queiroz 18]

A symmetry constrained cube formation:

- has its geometric center at the origin
- requires 7 edges



Summary

- Symmetry-constrained formations require simpler graphs with significantly fewer information links compared to “classic” strategies
- The velocity reference command can be assigned to a single agent
- Point group symmetries can be conserved during rotations of the rigid body

Future Work

- Extending the approach to multi-agent systems with double integrator dynamics
- Exploring extensions for bearing rigidity

Questions?