On Internal Stability of Diffusive-Coupling and the Dangers of Cancel Culture

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Motivation: consensus protocol

An ensemble of ν independent integrator agents

$$P_i(s) = \frac{1}{s} \implies P(s) = \frac{1}{s} I_\nu$$

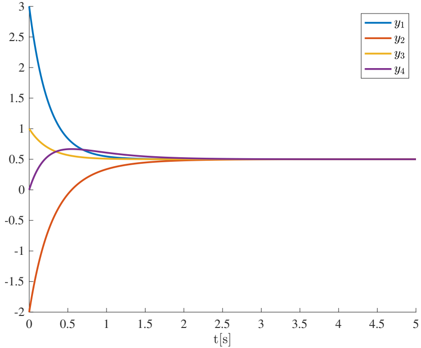
Goal: Asymptotic Agreement

$$\lim_{t\to\infty} (y_i(t) - y_j(t)) = 0, \quad \forall i, j.$$

The challenge: each agent can use only measurements relative to its neighbors.

The solution:

$$u_i(t) = -\sum_{j\in\mathcal{N}_i} k_{ij}(y_i - y_j)$$



Consensus Trajectories ($k_{ij} \equiv 1$)

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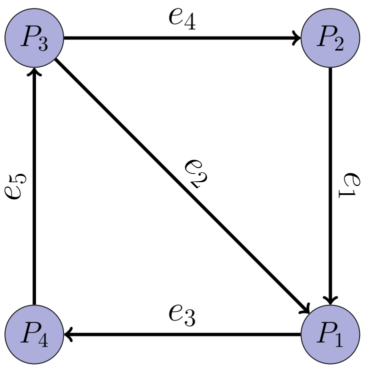
$$\mathcal{N}_1 = \{P_2, P_3, P_4\},\ \mathcal{N}_2 = \{P_1, P_3\}$$



An example coupling graph

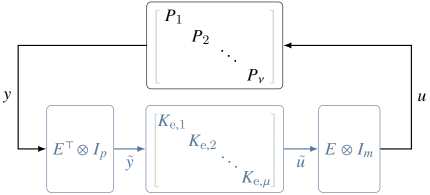
Diffusive coupling in cooperative control

An ensemble of general dynamic agents

$$P_i : u_i \mapsto y_i, \quad i \in [1, \ldots, \nu].$$

Controlled by a structured controller

$$K := (E \otimes I_m) K_e (E^\top \otimes I_p).$$



Canonical diffusivly-coupled control structure

Three-part controller:

- Difference operator.
- Edge controllers.
- Divergence operator.

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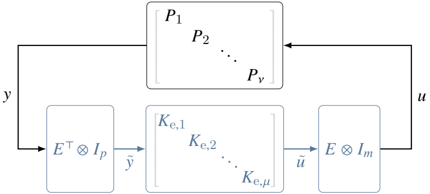
$$E(\mathcal{G}) = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \quad \mathbb{1}^\top E = 0$$



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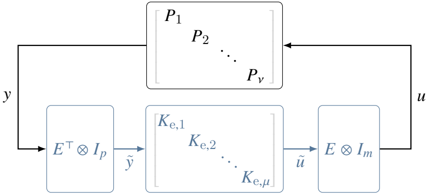

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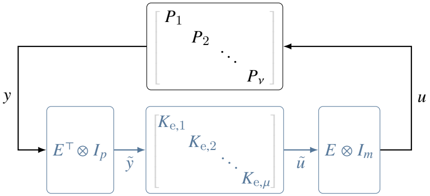

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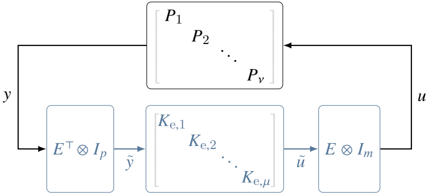


Canonical diffusivly-coupled control structure

Widespread in literature

- Vehicle formations (Fax and Murray, 2004)
- Consensus and synchronization (Li et al., 2010)
- Flow control (Bürger and De Persis, 2015)

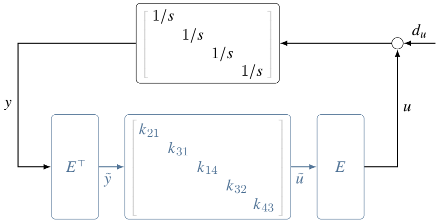
Three-part controller:

- Difference operator.
- Edge controllers.
- Divergence operator.

Disturbed consensus

What happens when disturbances are introduced?

$$\begin{cases} \dot{x}(t) = EK_eE^\top x(t) + d(t) \\ \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} I \\ EK_eE^\top \end{bmatrix} x(t) \end{cases}$$



Disturbed consensus block-diagram

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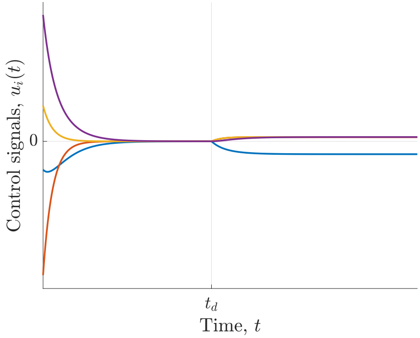

Control signals with one perturbed agent

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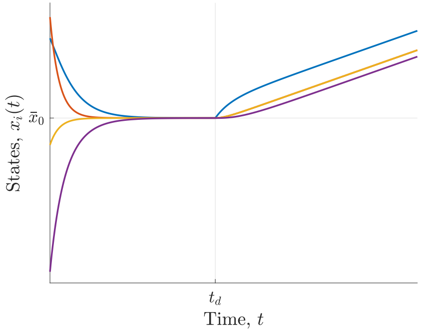
- Other signals diverge.



States with one perturbed agent

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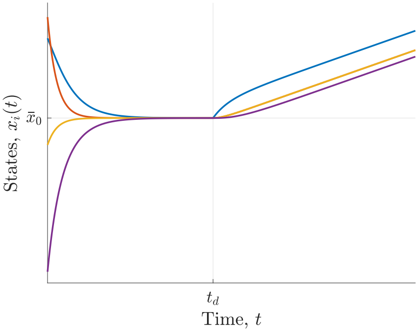

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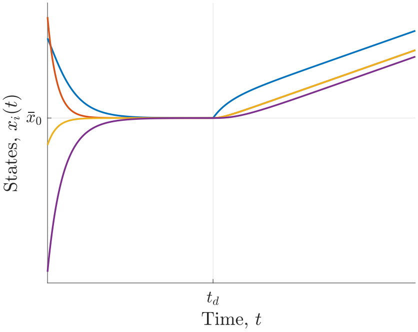


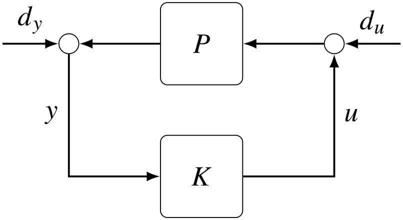
States with one perturbed agent

Motivates a deeper inspection of the internal stability of diffusively-coupled systems.

Internal stability

Definition

The interconnection depicted to the right is internally stable if all four subsystems mapping inputs $\begin{bmatrix} d_y \\ d_u \end{bmatrix}$ *to outputs* $\begin{bmatrix} y \\ u \end{bmatrix}$ *are causal and stable.*



Internal stability analysis framework.

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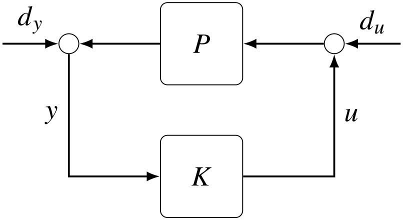
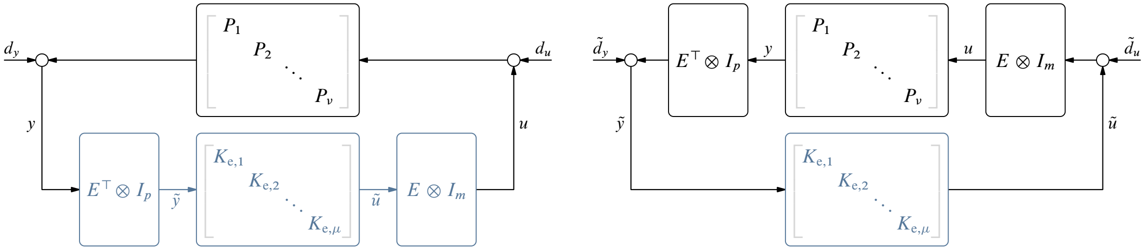


Internal stability analysis framework.

For finite-dimensional systems: iff $(I - PK)^{-1}$ is stable and PK and KP have no unstable cancellations.

Where should disturbances enter?



(a) Disturbances at the nodes. (b) Disturbances at the edges.

Two possible setups.

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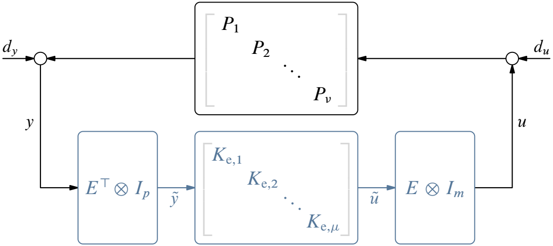

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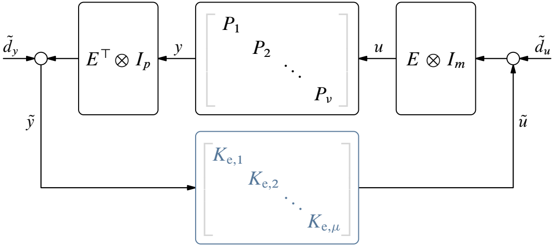


(b) Disturbances at the edges.

Two possible setups.

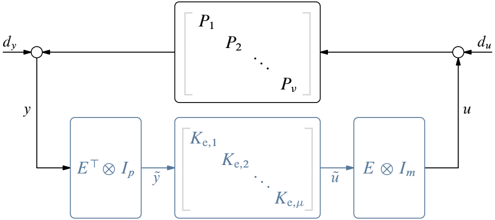
- In any realistic setup the input and outputs of the physical agents are the ones perturbed.

Problem setup

Agents: All P_i and $K_{e,i}$ are LTI and causal

Coprime factors: right coprime $M_i, N_i \in H_\infty$ and left coprime $\tilde{M}_i, \tilde{N}_i \in H_\infty$ such that

$$P_i = N_i M_i^{-1} = \tilde{M}_i^{-1} \tilde{N}_i, \quad \forall i.$$



Internal stability diagram for diffusive coupled systems.

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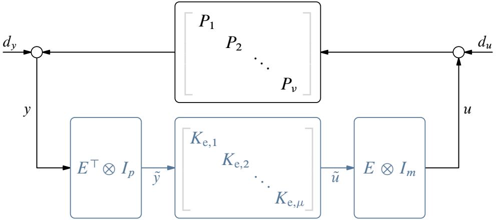


Internal stability diagram for diffusive coupled systems.

Under what conditions on the agents P_i are there edge controllers $K_{e,j}$ internally stabilizing the diffusively-coupled system?

The main result

Theorem 1

No LTI $K_{e,j}$ can internally stabilize the diffusively-coupled system if there is $\lambda \in \bar{\mathbb{C}}_0$, common to all agents, such that

$$\bigcap_{i=1}^{\nu} \ker\,[M_i(\lambda)]^\top \neq \{0\} \tag{1a}$$

or

$$\bigcap_{i=1}^{\nu} \ker \tilde{M}_i(\lambda) \neq \{0\}. \tag{1b}$$

where M_i and $\tilde{M}_i$ are denominators in coprime factorizations of P_i.

Two technical Lemmas

Relating stability with denominators.

Lemma 2

If $G(s)$ has coprime factorizations, then

$$G \in H_\infty \iff M_G^{-1} \in H_\infty \iff \tilde{M}_G^{-1} \in H_\infty.$$

The following Lemma is a consequence of the matrix corona theorem (Fuhrmann, 1968)

Lemma 3

If $G \in H_\infty^{n\times n}$, then

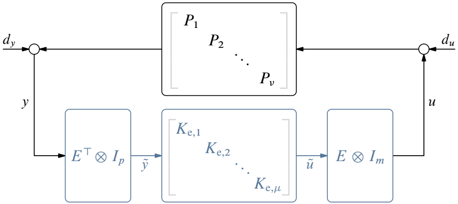
$$G^{-1} \in H_\infty^{n\times n} \iff \inf_{s\in\bar{\mathbb{C}}_0} \underline{\sigma}(G(s)) > 0.$$

Proof sketch

- The closed-loop system $T_4 := (d_y, d_u) \mapsto (y, u)$

$$T_4 = \begin{bmatrix} I \\ K \end{bmatrix} (I - PK)^{-1} \begin{bmatrix} I & P \end{bmatrix}$$
$$= \begin{bmatrix} M_K & 0 \\ N_K & 0 \end{bmatrix} \begin{bmatrix} M_K & -N_P \\ -N_K & M_P \end{bmatrix}^{-1} \quad (2)$$

where $K = N_K M_K^{-1}$ and $P = \operatorname{diag}\{N_i\} \operatorname{diag}\{M_i^{-1}\}$.



Internal stability diagram for diffusive coupled system.

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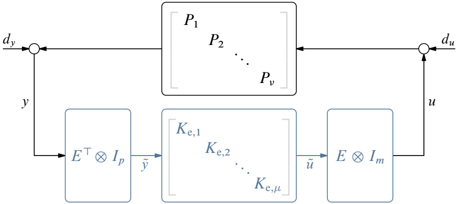
- From Lemma 2,

$$T_4 \in H_\infty \iff \begin{bmatrix} M_K & -N_P \\ -N_K & M_P \end{bmatrix}^{-1} \in H_\infty.$$



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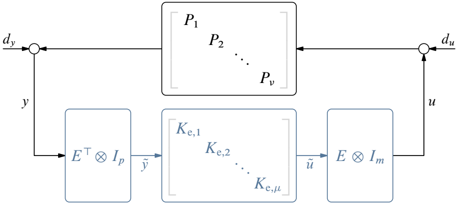
- From Lemma 2,

$$T_4 \in H_\infty \iff \begin{bmatrix} M_K & -N_P \\ -N_K & M_P \end{bmatrix}^{-1} \in H_\infty.$$

- From Lemma 3, T_4 is stable iff

$$\inf_{s \in \bar{\mathbb{C}}_0} \underline{\sigma}\left(\begin{bmatrix} M_K(s) & -N_P(s) \\ -N_K(s) & M_P(s) \end{bmatrix} \right) > 0 \qquad (3)$$



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Proof sketch

- Since $\mathbb{1}^\top E = 0$ and
$$N_K(s) = (E \otimes I_m) K_\mathrm{e}(s) (E^\top \otimes I_p) M_K(s),$$
$(\mathbb{1}^\top \otimes I_m) N_K(s) = 0$ for all s at which $K_\mathrm{e}(s)$ is finite.

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- From condition (1a), $\bigcap_{i=1}^{\nu} \ker\,[M_i(\lambda)]^\top \neq \{0\}$,
$$\exists v \neq 0 \text{ such that } v^\top M_i(\lambda) = 0\ \forall i \quad \Longrightarrow \quad (\mathbb{1} \otimes v)^\top M_P(\lambda) = 0.$$

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- Note that
$$(\mathbb{1} \otimes v)^\top N_K = v^\top (\mathbb{1} \otimes I_m)^\top N_K = 0,$$
thus
$$\begin{bmatrix} 0 & (\mathbb{1} \otimes v)^\top \end{bmatrix} \begin{bmatrix} M_K(\lambda) & -N_P(\lambda) \\ -N_K(\lambda) & M_P(\lambda) \end{bmatrix} = 0, \tag{4}$$
which violates (3).

Some corollaries

Intuitively: common unstable dynamics cannot be stabilized.

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Corollary 4 (Homogeneous agents)

If the agents are homogeneous, i.e. $P_i = P_0$ *for all* $i \in \mathbb{N}_\nu$, *and* $P_0(s)$ *has at least one pole in* $\bar{\mathbb{C}}_0$, *then no LTI* $K_{e,j}$ *can internally stabilize the diffusively-coupled system.*

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Corollary 5 (SISO agents)

If the agents are SISO and all have a pole at the same $\lambda \in \bar{\mathbb{C}}_0$, *regardless of multiplicities, then no LTI* $K_{e,j}$ *can internally stabilize the diffusively-coupled system.*

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Note: for MIMO agents, sharing an unstable pole is not equivalent to (1a) or (1b).

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Time-varying graphs: controllers of the form

$$K(t) = (E(t) \otimes I_m) K_e (E^\top(t) \otimes I_p),$$

for finite-dimensional agents using results by Verma, 1988.

Conditions for finite-dimensional agents

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Proposition 6

Let $P_i(s)$ have a minimal state-space realization (A_i, B_i, C_i, D_i) and let $\lambda \in \bar{\mathbb{C}}_0$ be a pole of $P(s)$.

i) (1a) *holds if and only if*

$$\bigcap_{i=1}^{\nu} B_i^\top \ker(\lambda I - A_i)^\top \neq \{0\}.$$

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Common dynamics imply a common pole λ and either of the above conditions.

Example

Consider a system with $\nu = 2$ first-order agents,

$$P_1(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2(s) = \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \frac{1}{s} \begin{bmatrix} 1 & \beta \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

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It can be verified that

$$B_1^\top \ker(\lambda I - A_1)^\top = \operatorname{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \operatorname{Im} \begin{bmatrix} 1 \\ \beta \end{bmatrix} = B_2^\top \ker(\lambda I - A_2)^\top, \quad \beta \neq 0$$

$$C_1 \ker(\lambda I - A_1) = \operatorname{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \operatorname{Im} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = C_2 \ker(\lambda I - A_2), \quad \alpha \neq 0$$

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It can be verified that

$$B_1^\top \ker(\lambda I - A_1)^\top = \operatorname{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \operatorname{Im} \begin{bmatrix} 1 \\ \beta \end{bmatrix} = B_2^\top \ker(\lambda I - A_2)^\top, \quad \beta \neq 0$$

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For edge controller

$$K_e(s) = \begin{bmatrix} (\alpha - \beta)\beta & -\alpha \\ \beta & 0 \end{bmatrix}$$

the closed-loop characteristic polynomial is then $(s + \alpha^2)(s + \beta^2)$, which is stable.

Internal stability of finite-dimensional systems

How to interpret these conditions for internal stability?

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Proposition 7 (Anderson and Gevers, 1981)

Let $P(s)$ and $K(s)$ be real-rational and proper transfer functions. If $(I - PK)^{-1}$ is stable, then $T_4(s)$ is unstable if and only if either $P(s)K(s)$ or $K(s)P(s)$ has an unstable pole-zero cancellation.

A well known criteria for the internal stability of an interconnection of real-rational transfer functions.

Pole cancellations

Definition

Given two systems G_1, G_2, their cascade $G_2 G_1$ has cancellations if

$$\deg(G_2 G_1) < \deg(G_1) + \deg(G_2).$$

We say that a pole of $G_1(s)$ and/or $G_2(s)$ is canceled if its multiplicity in $G_2(s)G_1(s)$ is smaller than the sum of its multiplicities in $G_1(s)$ and $G_2(s)$.

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SISO A pole is canceled iff there's a zero at the same location.

MIMO A pole can be canceled without the presence of zeroes.

MIMO pole cancellations

Consider

$$G_1(s) = \underbrace{\frac{1}{s}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\deg(G_1)=2} \quad \text{and} \quad G_2(s) = \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\deg(G_2)=0}.$$

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System G_2 is static and thus has no zeros, yet

$$\deg(G_2(s)G_1(s)) = \deg\left(\frac{1}{s}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = 1,$$

meaning that one of the poles of $G_1(s)$ is canceled.

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This cancellation was brought on by the normal rank deficiency of $G_2(s)$.

On the dangers of cancel culture

- Controllers of the form
$$K = (E \otimes I_m) K_e (E^\top \otimes I_p)$$
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Proposition 8

Let $P(s)$ and $K_e(s)$ be real-rational and proper and let $\lambda \in \bar{\mathbb{C}}_0$ be a pole of $P(s)$.

i) *If* (1a) *holds, then λ is canceled in $P(s)K(s)$.*

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- Diffusive coupling $\implies$ unavoidable cancellations of common dynamics.

Concluding remarks

- Intuitively - the controller is "blind" to common dynamics and disturbances.
- There is an inherent tradeoff between synchronization (common pole) and disturbance rejection.
- A clear explanation to phenomena observed in several scholarly works (e.g. Fax and Murray, 2004, Li et al., 2010, Ding, 2015).
- Future work: extending the results for non-linear P_i and $K_{e,j}$.

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