Symmetry-Induced Clustering in Multi-Agent Systems using Network Optimization and Passivity

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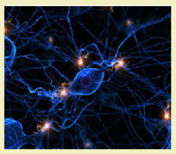
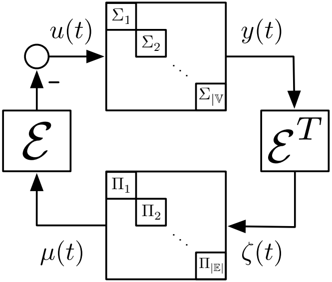
Multi-Agent Systems

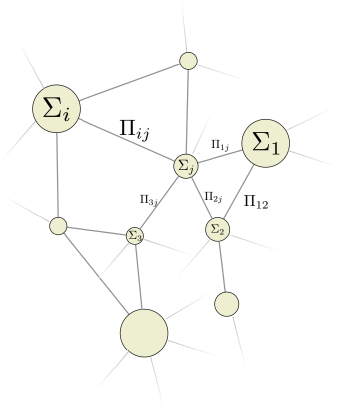


Overview

- Analysis of SISO Multi-Agent Systems using Network Optimization
- Weak Symmetries in Diffusively-Coupled Networks and Clustering
- A Brief on Cluster Synthesis

Closed-Loop System - Diffusive Coupling





- Σ_i are nonlinear dynamical systems representing the agents.
- Π_e are nonlinear dynamical system representing the edge controllers.
- $\mathcal{E}$ is the incidence matrix of the graph with arbitrary orientation.

Passivity

A Convergence Result and Passivity (Arcak,2007)

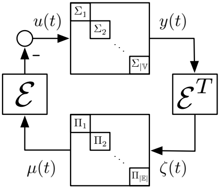
Assume

i) Agent dynamics Σ_i are output-strictly passive with respect to any steady-state,

ii) Controller dynamics Π_e are passive with respect to any steady-state

iii) There is a steady-state of the closed-loop system.

Then the closed-loop system converges to a constant steady-state.



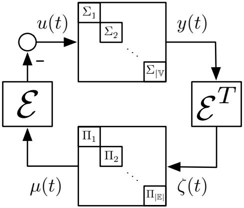
The Steady-state Input-Output Relation

- For the closed loop to reach a steady-state, each agent and controller must reach steady-state.

Definition

The collection of all steady-state input-output pairs of a system is called a *steady-state input-output relation*.

- Let k_i be the relations for the agents Σ_i, γ_e be the relations for the controllers Π_e and let k, γ be the stacked relation.



Assuring the Existence of a Consistent Steady-State

- How can we assure that there is a closed-loop steady-state?

Definition

A relation $r \subset \mathbb{R} \times \mathbb{R}$ is called *monotone* if for any $\mathrm{u}_1, \mathrm{u}_2$,

$$\mathrm{u}_1 < \mathrm{u}_2 \implies r(\mathrm{u}_1) \leq r(\mathrm{u}_2).$$

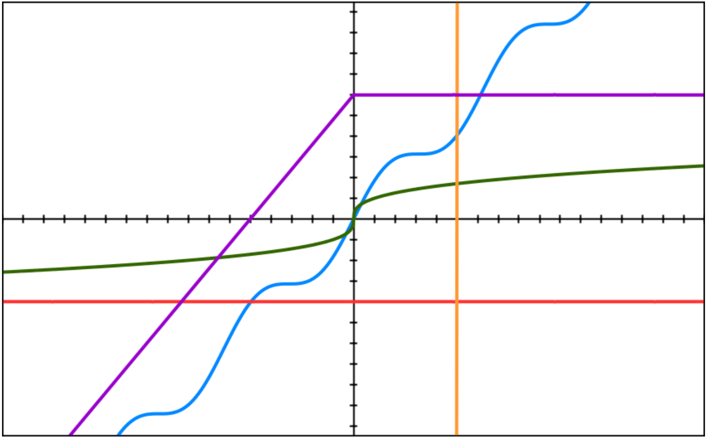
We say that r is *maximally monotone* if it is monotone and it is not contained in a larger monotone relation.

Theorem

Suppose all the relations k_i, γ_e are maximally monotone. Then there is a steady-state for the closed-loop system.

- Thus, we demand that k_i and γ_e are maximally monotone.

Maximally Monootone Relations



MEIP

We consider the following variant of passivity[1]

Definition (MEIP)

A SISO system is called *(output-strictly) maximal monotone equilibrium-independent passive* (MEIP) if:

1. The system is (output-strictly) passive with respect to any steady-state input-output pair.
2. The steady-state input-output relation is maximally-monotone.

[1] M. Burger, D.Zelazo and F. Allgower, "Duality and network theory in passivity-based cooperative control", Automatica, vol. 50, no. 8, pp, 2051–2061, 2014.

Example: Integrators

Consider the following SISO dynamical system:

$$\Upsilon : \begin{cases} \dot{x} = u \\ y = x \end{cases}$$

- The input-output steady-state relation k_Υ consists of all pairs $(0, y)$ where $y \in \mathbb{R}$. It's maximally monotone.
- The storage function for $(0, y_0)$ is $S_{y_0}(x) = 0.5(x - y_0)^2$.

Refinements of Passivity

	Scope	$\frac{1}{s}$	Ref
Passivity	Single s.s. input-output	Yes, separately	[1]
Equilibrium-Independent Passivity	s.s. input-output function $y_{ss} = f(u_{ss})$	No	[2]
MEIP	s.s. input-output relation	Yes	[3]

[1] H. Khalil, "Nonlinear Systems", Perason Education, Prentice Hall, 2002.

[2] G.H. Hines, M. Arcak and K. Packard, "Equilibrium-independent passivity: A new definition and numerical certifications", Automatica, vol.47. no.9. pp. 1949–1956, 2011.

[3] M. Burger, D.Zelazo and F. Allgower, "Duality and network theory in passivity-based cooperative control", Automatica, vol. 50, no. 8, pp, 2051–2061, 2014.

Integral Convex Function

Rockafellar's Theorem (Rockafellar,1969)

A relation is maximally monotone if and only if it is the subgradient of some convex function.

- Let $K_i, K_i^\star, \Gamma_e, \Gamma_e^\star$ be integral functions of $k_i, k_i^{-1}, \gamma_e, \gamma_e^{-1}$.
- Subgraident is a generalized form of the gradient. If k_i is smooth then $\nabla K_i = k_i$
- Let $K = \sum_i K_i$ and $\Gamma = \sum_e \Gamma_e$.

Analysis Result for SISO Systems

Theorem (Bürger, Zelazo and Allgöwer, 2014)

Consider the closed loop system, and suppose all nodal systems Σ_i are output-strictly MEIP and all edge controllers Π_e are MEIP.
Then the signals $u(t), y(t), \zeta(t)$ and $\mu(t)$ converge to constants $\hat{u}, \hat{y}, \hat{\zeta}$ and $\hat{\mu}$ which are optimal solutions to the problems (OFP) and (OPP):

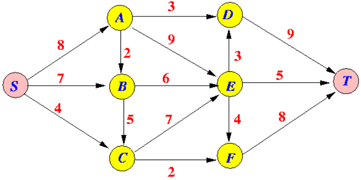
(OPP)	**(OFP)**
$\min_{y,\zeta} \quad \sum_i K_i^\star(y_i) + \sum_e \Gamma_e(\zeta_e)$	$\min_{u,\mu} \quad \sum_i K_i(u_i) + \sum_e \Gamma_e^\star(\mu_e)$
$s.t. \quad \mathcal{E}^T y = \zeta$	$s.t. \quad u + \mathcal{E}\mu = 0.$

The minimized functions are convex, so we can use gradient descent to solve these efficiently.

Network Optimization

- These problems are part of a field **Network Optimization** studying static optimization problems on graphs.
- Network Optimization has been extensively studied for decades, and found a range of uses in theoretical computer science, communication theory and operations research.

Optimal Potential Problem	**Optimal Flow Problem**
$\min_{\mathrm{y},\zeta} \quad \sum_i K_i^\star(\mathrm{y}_i) + \sum_e \Gamma_e(\zeta_e)$ $s.t. \quad \mathcal{E}^T \mathrm{y} = \zeta$	$\min_{\mathrm{u},\mu} \quad \sum_i K_i(\mathrm{u}_i) + \sum_e \Gamma_e^\star(\mu_e)$ $s.t. \quad \mathrm{u} + \mathcal{E}\mu = 0.$



Symmetries and Clustering

- Suppose we are given a multi-agent system $(\mathcal{G}, \Sigma, \Pi)$.
- Symmetries in the network structure should force some agents to act similarly.
- In the steady-state limit, this should lead to clustering.

Symmetries in Multi-Agent systems

- Symmetries are used throughout the control community for different applications, including designing observers, more efficient MPC and even bipedal locomotion.
- In multi-agent systems, they were used be Rahmani, Chapman and Mesbahi to study controlability and observability of networked linear systems.
- The idea - network symmetries force some agents to act identically, implying that the system is not controlable.

Weak Symmetries in Networks

- We want to understand how symmetries in a multi-agent system affect the steady-state of system.

Definition (Weakly Equivalent Systems)

Two systems are called *weakly equivalent* if they have the same steady-state relation

Definition (Weak Symmetries)

Let $(\mathcal{G}, \Sigma, \Pi)$ be a multi-agent system. A map $\psi : \mathbb{V} \to \mathbb{V}$ is called a weak automorphism if for any vertices i, j and any edge e,

1. If $i \to j$ is an edge, then so is $\psi(i) \to \psi(j)$.
2. Σ_i is weakly equivalent to $\Sigma_{\psi(i)}$.
3. Π_e is weakly equivalent to $\Pi_{\psi(e)}$.

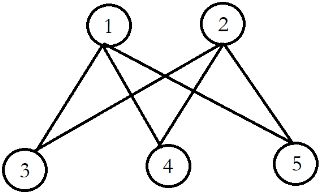
The group of weak symmetries is denoted $\mathrm{Aut}(\mathcal{G}, \Sigma, \Pi)$.

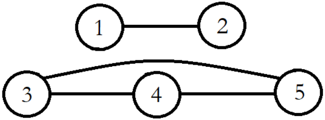
Exchangeability

- We say that the nodes i, j are exchangeable if there is a weak symmety $\psi \in \mathrm{Aut}(\mathcal{G}, \Sigma, \Pi)$ such that $\psi(i) = j$.
- The exchangeability relation can be viewed using the *exchangeability graph* $\mathcal{H} = \mathcal{H}(\mathcal{G}, \Sigma, \Pi)$.
  - $\mathcal{H}$ has the same vertices as $\mathcal{G}$.
  - Two vertices are connected by an edge in $\mathcal{H}$ if they are exchangeable.

Proposition

The exchangability graph is a union of disjoint cliques





Exchangeability

Theorem

Let $(\mathcal{G}, \Sigma, \Pi)$ be a diffusively-coupled system, and let $\mathcal{H}$ be the exchangeability graph. Assume the agents are output-strictly MEIP and that the controllers are MEIP, or vice versa. Then:

- *The closed-loop system converges to some ouput* y.
- *If i, j are connected in $\mathcal{H}$, then* $\mathrm{y}_i = \mathrm{y}_j$.
- *If i, j are not connected in $\mathcal{H}$, then* $\mathrm{y}_i \neq \mathrm{y}_j$.[a]

[a] More precisely, this happens if the controllers avoid some zero-measure set

In other words, generically, the closed-loop system converges to a clustered steady-state, with clusters corresponding to the cliques in the exchangeability graph $\mathcal{H}$.

Idea of Proof

- Let $\psi \in \mathrm{Aut}(\mathcal{G}, \Sigma, \Pi)$. Let P_ψ and Q_ψ be the corresponding permutation matrix on the nodes and edges, respectively.
- Show that $P_\psi \mathcal{E} = \mathcal{E} Q_\psi$.
- Conclude that the function $K(\mathrm{y}) + \Gamma(\mathcal{E}_\mathcal{G}^T \mathrm{y})$ is invariant under ψ.
- Conclude that the set of minimizers is invariant under ψ
- Use geometric understanding og the set of minimizers to conclude each minimizers y satisfies $P_\psi \mathrm{y} = \mathrm{y}$.

Homogenous Networks and Cluster Synthesis

Definition

We say that a network is weakly homogenous if all agents are weakly equivalent and all controllers are also weakly equivalent.

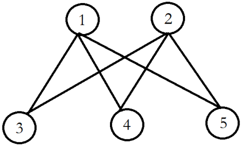
- In this case, $\mathrm{Aut}(G, \Sigma, \Pi)$ is just the automorphism of the oriented graph.

Problem (Cluster Synthesis)

Given fixed weakly homogenous agents, find a graph $\mathcal{G}$ and weakly homogenous controllers so that the closed-loop system converges to a clusterd steady-state, with prescribed cluster sizes at prescribed locations.

Cluster Synthesis - Example

- We are given 5 agents are LTI with TF $G(s) = \frac{1}{s+1}$.
- Goal - cluster of size 3 at $\mathrm{y} = 1$ and cluster of size 2 at $\mathrm{y} = 0$.



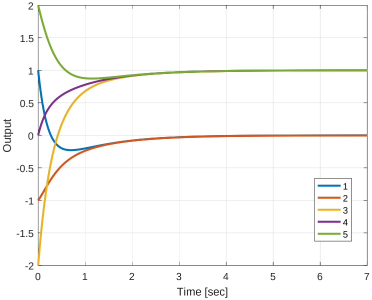
- Orient edges from $1, 2$ to $3, 4, 5$, and let γ_1 be the steady-state of the homogenous controller.
- The desired steady-state has $k^{-1}(\mathrm{y}) \notin \mathrm{Im}(\mathcal{E})$, so we need to use an external input, which will be identical to all agents.

Cluster Synthesis - Example

- The steady-state equation $\mathrm{w} = k^{-1}(\mathrm{y}) + \mathcal{E}\gamma(\mathcal{E}^T\mathrm{y})$ gives:

$$w = 0 - 3\gamma_1(1-0) = -3\gamma_1(1)$$
$$w = 1 + 2\gamma_1(1-0) = 1 + 2\gamma_1(1)$$

- A monotone relation solving the equations is $\gamma_1(x) = -1.2 + x$, together with $w = 0.6$
- We take all controllers equal to $\mu_e = -1.2 + \zeta_e$.



Conclusions

- The connection between diffusively-coupled systems and network optimization appears naturally when studying analysis of multi-agent systems.
- This connection and network symmetries prescribe a clustering structure using the exchangeability graph.
- One can use this idea to solve the cluster synthesis problem for homogenous agents using homogenous controllers.

Advancements

- A lower bound on the number of edges of graphs achieve a certain clustering structure was found.
- A methocial way of building graphs with a given clustering structure, and relatively few edges, was found.
  - For many interesting cases, we need no more than twice the edges as appearing in the lower bound.
- Given a fixed graph, finding a homogenous controller solving the cluster synthesis problem is equivalent to a LP problem.

Questions?

Weak Symmetries in Networks and Orientation

- We require that ψ preserves orientation as the controllers need not be "symmetric".
- If the controllers are chosen so that orientation does not matter, one can prove that the system converges to consensus.
- In order to achieve clustering, we have to take demand that the orientation is preserved.
- The orientation can be arbitrary, but different orientations will dictate different choices of controllers

Cluster Synthesis - Graph Synthesis

- How many edges to we need to achieve a certain clustering formation in a weakly homogenous network?

Theorem

Consider a collection of n homogenous agents, and let $r_1, \cdots r_k$ be the desired cluster sizes

- *Any connected graph $\mathcal{G}$ solving the probelm has at least m edges, where*

$$m = \min_{\mathcal{T} \text{ tree on } k \text{ vertices}} \sum_{e \in \mathcal{T}, e=\{i,j\}} \frac{r_i r_j}{\gcd(r_i, r_j)}$$

- *The exists a connected graph $\mathcal{G}$ solving the problem with M edges, where*

$$M = \min_{\mathcal{T} \text{ path on } k \text{ vertices}} \min_{e'=\{i',j'\} \in \mathcal{T}} \left[\left(\sum_{e' \neq e \in \mathcal{T}, e=\{i,j\}} \frac{r_i r_j}{\gcd(r_i, r_j)} \right) + 2 \frac{r_{i'} r_{j'}}{\gcd(r_{i'}, r_{j'})} \right]$$

Cluster Synthesis - Graph Synthesis

Corollary

For n agents, if the cluster sizes $r_1, \cdots, r_k$ are all equal, then n edges are enough to get the desired clusters

Corollary

For n agents, if the cluster sizes $r_1, \cdots, r_k$ are all bounded by q, then no more than $n + q^3$ edges are needed to get the desired clusters

Corollary

For n agents, if the cluster sizes $r_1, \cdots, r_k$ satisfy that for any i, j, either r_i divides r_j or vice versa, then no more than $2n$ edges are needed to get the desired clusters

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