Formation Control through Group Symmetries

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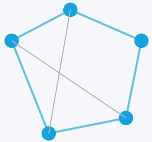
Introduction

Formation control arises naturally in a wide range of multi-agent applications

- UAV Swarms: mapping and surveillance
- Satellite constellations: uniform spacing for coverage
- Multi-robot systems: distributed sensing and transportation

Can We Exploit Structure in the Desired Formation?

A formation is a coordinated object

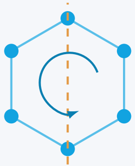


Agents must coordinate using only local information while maintaining a prescribed geometric arrangement.

The usual question is:

What information must the agents exchange to determine the formation?

Many formations are highly structured



Regular polygons, lattices, and many engineered formations contain repeated geometric patterns.

This suggests a different question:

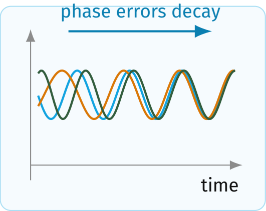
Can the symmetry of the target itself be used as coordination information?

Guiding idea

Rather than treating every geometric relation independently, **exploit repeated structure in the desired formation.**

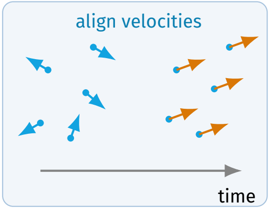
Consensus: A Canonical Model for Coordination

Consensus protocols aim to drive an ensemble of agents towards a common state or trajectory using only relative information from neighboring agents.



Synchronization

clocks, oscillators, generators, and rhythmic systems use relative phase errors to fall into step.



Flocking

Vehicles use only relative velocity measurements to achieve collective motion.

Consensus: The Model

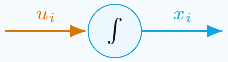
Agent dynamics

Consider n agents with scalar states

$$x_i \in \mathbb{R}, \qquad i \in \mathcal{V} := \{1, \dots, n\},$$

and single-integrator dynamics

$$\dot{x}_i = u_i.$$

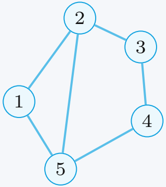


input determines agent velocity

Interconnection graph

Information exchange is described by an undirected graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}).$$



An edge $(i, j) \in \mathcal{E}$ means that agents i and j can exchange state information.

Consensus objective

Choose each u_i using only neighboring relative states so that

$$\boxed{\lim_{t \to \infty} \left(x_i(t) - x_j(t)\right) = 0 \qquad \text{for all } i, j \in \mathcal{V}.}$$

Consensus as a Gradient Flow

Network disagreement potential

Penalize disagreement across the edges:

$$V(x) = \frac{1}{2} \sum_{(i,j)\in\mathcal{E}} (x_i - x_j)^2.$$

Let L be the graph Laplacian. Then

$$\boxed{V(x) = \frac{1}{2} x^\top L x}$$

Gradient dynamics

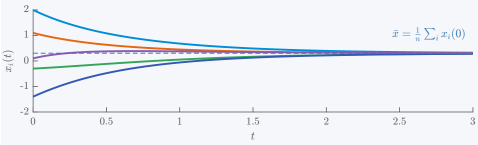
Follow the negative gradient:

$$\dot{x} = -\nabla V(x) = -Lx.$$

Equivalently, agent i implements

$$\boxed{\dot{x}_i = -\sum_{j\in\mathcal{N}_i} (x_i - x_j)}$$

Consensus trajectories



Since $L\mathbf{1} = 0$, the consensus direction lies in the kernel:

$$\boxed{\mathbf{1} \in \ker L.}$$

Hence, for a connected undirected graph,

$$\boxed{x(t) \longrightarrow \bar{x}\,\mathbf{1}, \qquad \bar{x} = \frac{1}{n}\mathbf{1}^\top x(0).}$$

Consensus: A marriage of graph theory and systems theory

Connectivity $\iff$ agreement

For undirected consensus,

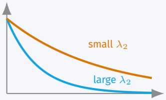
$$x(t) \to \bar{x}\,\mathbf{1}, \qquad \bar{x} = \frac{1}{n}\mathbf{1}^\top x(0),$$

for every $x(0)$ if and only if

$$\lambda_2(L) > 0,$$

equivalently, $\mathcal{G}$ is connected.

Algebraic connectivity $\iff$ speed



The slowest disagreement mode decays at rate

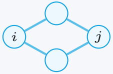
$$\lambda_2(L).$$

Hence

$$\|x(t) - \bar{x}\mathbf{1}\| \leq e^{-\lambda_2}\|x(0) - \bar{x}\mathbf{1}\|$$

Larger λ_2 means faster convergence.

Effective resistance $\iff$ robustness



$$R_{ij} = (e_i - e_j)^\top L^\dagger (e_i - e_j).$$

Small R_{ij} means stronger coupling and lower sensitivity of relative states to disturbances or edge loss.

Cycles $\iff$ disturbance performance

For noisy first-order consensus,

$$\|\Sigma\|_{\mathcal{H}_2}^2 \propto \operatorname{tr}(L^\dagger) = \sum_{k=2}^{n} \frac{1}{\lambda_k(L)}.$$

Cycles add alternative paths and typically improve disturbance attenuation.

Central lesson

A powerful model highlighting where networks and dynamic systems intersect

From Consensus to Formation Control

Distance-error potential

For $p_i \in \mathbb{R}^2$, prescribe $d_{ij} > 0$ on each edge:

$$e_{ij}(p) = \|p_i - p_j\|^2 - d_{ij}^2$$

The network potential is

$$\boxed{V(p) = \frac{1}{4} \sum_{(i,j)\in\mathcal{E}} e_{ij}(p)^2}$$

Gradient formation dynamics

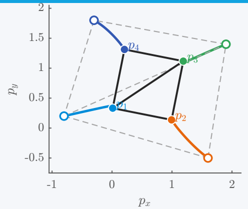
Follow the negative gradient:

$$\dot{p} = -\nabla V(p) = -R(p)^\top R(p)p + R(p)^\top d^2$$

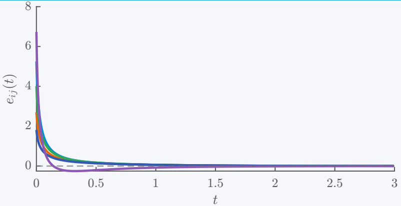
Agent i implements

$$\boxed{\dot{p}_i = -\sum_{j\in\mathcal{N}_i} e_{ij}(p)(p_i - p_j)}$$

Formation trajectories and distance errors



Agent trajectories and achieved formation



Squared-distance errors $e_{ij}(t) \to 0$

Distance-Based Formation Control: What Changes?

Nonlinear dynamics

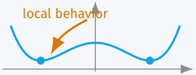
The distance errors are quadratic:

$$e_{ij}(p) = \|p_i - p_j\|^2 - d_{ij}^2,$$

so the closed loop

$$\dot{p} = -R(p)^\top e(p)$$

is nonlinear.



multiple critical configurations

Unlike consensus, convergence is generally local and depends on the initial configuration.

Rigidity governs dynamics

The rigidity matrix replaces the incidence matrix:

$$\dot{p} = -R(p)^\top e(p)$$

For a planar framework,

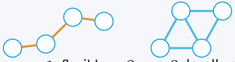
$$\operatorname{rank} R(p) = 2n - 3$$

means infinitesimal rigidity.

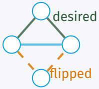
How many distances are needed?

For a generic planar formation, minimal rigidity requires

$$\boxed{|\mathcal{E}| = 2n - 3}$$



$n - 1$: flexible $2n - 3$: locally rigid

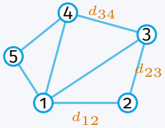


Central lesson

Distance constraints turn a linear consensus problem into a nonlinear rigidity problem:
more edges are needed, guarantees become local, and uniqueness requires stronger rigidity.

A Different Way to Describe a Formation

Describe the shape by distances



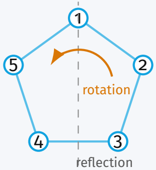
Each edge prescribes a metric constraint

$$\|p_i - p_j\| = d_{ij}.$$

For a generic planar formation, local rigidity requires

$$|\mathcal{E}| \geq 2n - 3.$$

Describe the shape by symmetry



Instead, prescribe relations generated by **rotations and reflections**.

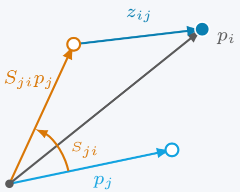
The geometry is encoded through repeated symmetry operations rather than through every required distance.

Motivating question

Can symmetry describe and control the desired formation using fewer pairwise constraints than rigidity-based methods?

Symmetry Constraints Look Like Consensus Constraints

A symmetry disagreement



- On edge (i,j), prescribe a rotation or reflection S_{ji}:
$$p_i = S_{ji} p_j .$$
- Define the symmetry error
$$\boxed{z_{ij} = p_i - S_{ji} p_j .}$$
- $z_{ij} = 0$ exactly when the relation holds.

A consensus-like gradient flow

Penalize symmetry disagreement:

$$F(p) = \frac{1}{2} \sum_{(i,j)\in\mathcal{E}_I} \|p_i - S_{ji} p_j\|^2$$

Follow the negative gradient:

$$\dot{p} = -\nabla F(p)$$

At the agent level:

$$\boxed{\dot{p}_i = \sum_{j\in\mathcal{N}_i} (S_{ji} p_j - p_i)}$$

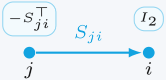
Consensus compares neighbor states; symmetry control compares transformed neighbor states.

From Symmetry Errors to a Matrix-Weighted Laplacian

Stacking the edge errors

For an oriented edge $j \to i$,

$$z_{ij} = p_i - S_{ji} p_j = E_{ij}^\top p.$$



$$E_{ij} = \begin{bmatrix} \vdots \\ -S_{ji}^\top \\ \vdots \\ I_2 \\ \vdots \end{bmatrix}.$$

Only the j- and i-blocks are nonzero.

Stacking the edge columns,

$$E_S = \begin{bmatrix} E_{e_1} & \cdots & E_{e_m} \end{bmatrix}, \qquad z = E_S^\top p.$$

Quadratic potential and linear dynamics

The total symmetry potential is

$$F(p) = \frac{1}{2}\|z\|^2 = \frac{1}{2}\|E_S^\top p\|^2.$$

Therefore

$$F(p) = \frac{1}{2} p^\top E_S E_S^\top p.$$

Define

$$\boxed{Q := E_S E_S^\top.}$$

Then

$$\nabla F(p) = Qp$$

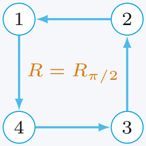
and the gradient flow is

$$\boxed{\dot{p} = -Qp.}$$

Q is the symmetry analogue of the graph Laplacian.

What Does Q Look Like? A C_4 Example

Cycle graph C_4



Each edge imposes a quarter-turn relation:

$$z_{12} = p_1 - Rp_2$$
$$z_{23} = p_2 - Rp_3$$
$$z_{34} = p_3 - Rp_4$$
$$z_{41} = p_4 - Rp_1$$

The resulting block matrix

Since each node has two incident constraints, the diagonal blocks are $2I_2$. Neighboring agents contribute rotation blocks:

$$Q = \begin{bmatrix} 2I_2 & -R & 0 & -R^\top \\ -R^\top & 2I_2 & -R & 0 \\ 0 & -R^\top & 2I_2 & -R \\ -R & 0 & -R^\top & 2I_2 \end{bmatrix}$$

The dynamics are

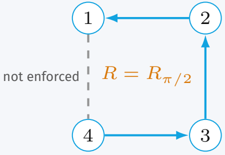
$$\boxed{\dot{p} = -Qp}$$

$$L(C_4) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

The sparsity pattern is exactly that of the ordinary graph Laplacian. The scalar entries -1 are replaced by 2×2 rotation blocks.

How Many Symmetry Constraints Are Needed?

From the cycle to a spanning tree



Only three symmetry equations are imposed:

$$p_1 = Rp_2$$
$$p_2 = Rp_3$$
$$p_3 = Rp_4$$

Symmetry propagates along the tree

Combining the three relations gives

$$p_1 = R^3 p_4$$

Since $R = R_{\pi/2}$,

$$R^4 = I_2, \qquad R^3 = R^{-1} \Rightarrow p_4 = Rp_1$$

the missing cycle relation is recovered automatically.

$$\boxed{|\mathcal{E}_I| = n - 1}$$

More generally, on a spanning tree every agent is connected to a root by a unique path, and products of the prescribed rotations determine the remaining symmetry relations.

A Connectivity Requirement

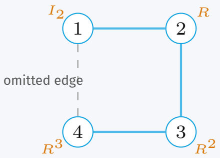
A spanning tree is sufficient to enforce a cyclically symmetric formation: only $n-1$ pairwise constraints are required.

Rotational Symmetry Is Consensus in New Coordinates

Rotate each state into a common frame

For $R = R_{\pi/2}$, define
$$\mathcal{T} = \operatorname{diag}(I_2, R, R^2, R^3), \qquad \tilde{p} = \mathcal{T}p.$$

Equivalently,
$$\tilde{p}_1 = p_1, \quad \tilde{p}_2 = Rp_2, \quad \tilde{p}_3 = R^2 p_3, \quad \tilde{p}_4 = R^3 p_4.$$



The coordinate change removes the quarter-turn offsets:
$$\dot{\tilde{p}} = -\mathcal{T}Q\mathcal{T}^\top \tilde{p}.$$

A common transformed state becomes a C_4-symmetric formation.

Consensus in the transformed coordinates

For the three-edge spanning tree,
$$\mathcal{T}Q\mathcal{T}^\top = L(P_4) \otimes I_2.$$

Hence
$$\boxed{\dot{\tilde{p}} = -\big(L(P_4) \otimes I_2\big)\tilde{p}.}$$

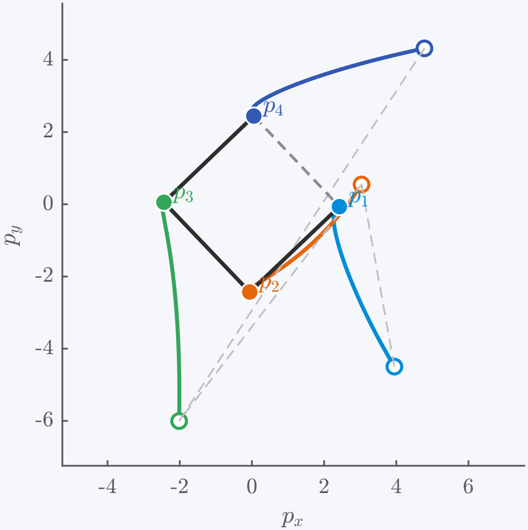
This is ordinary consensus in $\mathbb{R}^2$, so
$$\tilde{p}_i(t) \longrightarrow \bar{p},$$
where the consensus point is
$$\boxed{\begin{aligned} \bar{p} &= \frac{1}{4}\sum_{k=1}^{4} \tilde{p}_k(0) \\ &= \frac{1}{4}\Big(p_1(0) + Rp_2(0) + R^2 p_3(0) + R^3 p_4(0)\Big). \end{aligned}}$$

Transforming back,
$$p_i(t) = R^{-(i-1)}\tilde{p}_i(t) \longrightarrow R^{-(i-1)}\bar{p}.$$

Therefore,
$$\boxed{p_\infty = \operatorname{col}\left(\bar{p},\ R^{-1}\bar{p},\ R^{-2}\bar{p},\ R^{-3}\bar{p}\right).}$$

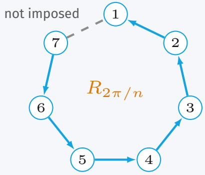
Simulation: Cyclic Formation from Three Symmetry Constraints

Simulation of the C_4 rotational controller



Cyclic Formations from a Spanning Tree

A C_n-symmetric target



Assign to each tree edge a rotation consistent with the cyclic target:

$$p_i = R_{ij} p_j \,, \qquad (i,j) \in \mathcal{E}_I$$

The interaction graph may be any spanning tree:

$$\boxed{|\mathcal{E}_I| = n - 1}$$

The missing symmetry relations follow by composing rotations along the unique tree paths.

Main result

Let

$$F(p) = \frac{1}{2} \sum_{(i,j)\in\mathcal{E}_I} \|p_i - R_{ij} p_j\|^2, \qquad \dot{p} = -Qp, \qquad Q \succeq 0.$$

Assume that $\mathcal{G}_I$ is a spanning tree and that the edge rotations are consistent with a C_n-symmetric target.

Then

$$p(t) \longrightarrow \ker Q$$

exponentially, and

$$\boxed{\ker Q = \{p : \; p_i = R_{ij} p_j \text{ for all } (i,j) \in \mathcal{E}_I\}\,.}$$

For consistent cyclic labels, this kernel is exactly the family of C_n-symmetric configurations.

Distance constraints	**Symmetry constraints**
$2n-3$ edges	$n-1$ edges
nonlinear flow	linear flow

The desired cyclic formation is the null space of Q.

Martinez and DZ, “Formation control with rotational symmetry,” American Control Conference,

From Cyclic to Dihedral Symmetry

The symmetry group of a regular n-gon

The cyclic group C_n contains only rotations.

The full symmetry group is the dihedral group

$$\boxed{D_n = C_n \rtimes C_2.}$$

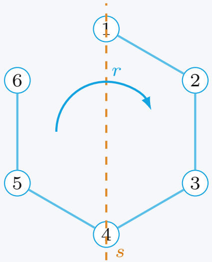
It is generated by a rotation r and a reflection s, with

$$\boxed{r^n = \mathrm{id}, \qquad s^2 = \mathrm{id}, \qquad srs = r^{-1}.}$$

The last relation says that reflecting reverses the direction of rotation.

C_n	D_n
rotations only	rotations and reflections

Geometric interpretation



The cyclic theory uses the rotational subgroup C_n.

The next step is to incorporate the reflection generator s and recover the full dihedral symmetry.

Can the consensus construction be extended from C_n to D_n?

Encoding Reflection Symmetry

Reflection across a line

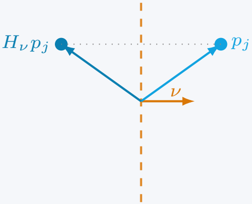
Let $\nu \in \mathbb{R}^2$ be a unit normal to the mirror line.

The corresponding reflection is the Householder matrix

$$\boxed{H_\nu = I_2 - 2\nu\nu^\top.}$$

It satisfies

$$H_\nu^\top = H_\nu, \qquad H_\nu^2 = I_2.$$



Reflection disagreement

On an interaction edge (i, j), prescribe the relation

$$p_i = H_{ij} p_j.$$

Define the edge error

$$\boxed{z_{ij} = p_i - H_{ij} p_j.}$$

The reflection potential is

$$F_{\mathrm{ref}}(p) = \frac{1}{2} \sum_{(i,j)\in\mathcal{E}_I} \|p_i - H_{ij} p_j\|^2.$$

Its gradient flow again has the form

$$\boxed{\dot{p} = -Q_{\mathrm{ref}} p, \qquad Q_{\mathrm{ref}} \succeq 0.}$$

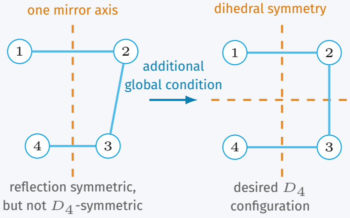
Rotational edge	**Reflection edge**
$p_i = R_{ij} p_j$	$p_i = H_{ij} p_j$

The same matrix-weighted consensus structure survives.

Why Pairwise Reflection Constraints Are Not Enough

Pairwise reflections do not enforce D_4

Pairwise reflection constraints can enforce symmetry about one axis without selecting the full dihedral symmetry of the target.



The reflection-edge potential alone may have a larger zero-error set than the desired dihedral formation family.

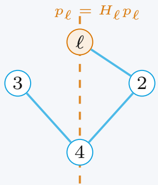
Fix one agent to a mirror line

Choose one agent ℓ and one prescribed reflection H_ℓ. Add the stabilizer constraint

$$\boxed{p_\ell = H_\ell p_\ell.}$$

Since the fixed-point set of a Householder reflection is its mirror line,

$$p_\ell = H_\ell p_\ell \quad \iff \quad p_\ell \in \mathrm{Fix}(H_\ell).$$

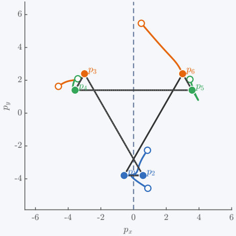


The tree propagates the relative reflection relations, while this one local condition selects the intended global dihedral configuration.

Spanning tree + one stabilizer constraint.

Why One More Reflection Constraint Is Needed

Pairwise reflections only



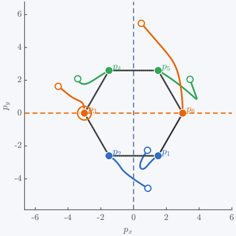
Impose only the $n-1$ inter-agent constraints

$$p_i = H_{ij} p_j, \qquad (i,j) \in \mathcal{E}_I.$$

These relations are all satisfied at equilibrium, but they still leave one additional geometric degree of freedom.

The limit can be reflection-consistent without having full C_{6v} symmetry.

Fix one agent to its mirror line



Choose one agent ℓ and impose one additional condition

$$\boxed{p_\ell = H_\ell p_\ell.}$$

Since the fixed-point set of H_ℓ is its mirror line, this forces agent ℓ to lie on the prescribed axis.

That single local condition removes the ambiguity and recovers the desired C_{6v}-symmetric family.

Martinez and DZ, "Symmetry-Based Formation Control on Cycle Graphs Using Dihedral Point Groups," IFAC, 2026.

Reflection Constraints: The General Result

Pairwise reflections

For a spanning tree

$$\mathcal{G}_I = (\mathcal{V}, \mathcal{E}_I), \qquad |\mathcal{E}_I| = n-1,$$

consider the potential

$$F_H(p) = \frac{1}{2} \sum_{(i,j)\in\mathcal{E}_I} \|p_i - H_{ij} p_j\|^2.$$

Its gradient flow is linear:

$$\dot{p} = -Q_H p, \qquad Q_H \succeq 0.$$

The equilibrium set is

$$\ker Q_H = \{p : \; p_i = H_{ij} p_j, \; (i,j) \in \mathcal{E}_I\},$$

and

$$\boxed{\dim \ker Q_H = 2.}$$

Pairwise reflections leave one extra geometric degree of freedom.

One additional mirror-line condition

Choose one agent ℓ and impose

$$p_\ell = H_\ell p_\ell.$$

Equivalently, augment the closed-loop matrix:

$$\boxed{\widetilde{Q}_H = Q_H + \operatorname{diag}(e_\ell) \otimes (I_2 - H_\ell).}$$

Then

$$\widetilde{Q}_H \succeq 0, \qquad \boxed{\dim \ker \widetilde{Q}_H = 1.}$$

Moreover,

$$p(t) \longrightarrow \ker \widetilde{Q}_H$$

exponentially, and this one-dimensional kernel is exactly the family of desired C_{nv}-symmetric configurations.

One local mirror constraint selects the full dihedral symmetry.

$n-1$ **inter-agent constraints** $+$ **one local self-reflection constraint**

Martinez and DZ, "Symmetry-Based Formation Control on Cycle Graphs Using Dihedral Point Groups," IFAC, 2026.

Reflection Constraints Become Vector Consensus

Change coordinates

Root the spanning tree at the anchored agent ℓ. Define $S_\ell = I_2$ and choose $S_i \in O(2)$ so that on each edge

$$S_i = H_{ij} S_j .$$

Set

$$\boxed{\widetilde{p}_i = S_i^\top p_i .}$$

Then

$$p_i - H_{ij} p_j = S_i(\widetilde{p}_i - \widetilde{p}_j),$$

so

$$\|p_i - H_{ij} p_j\|^2 = \|\widetilde{p}_i - \widetilde{p}_j\|^2 .$$

Consensus in transformed coordinates

The pairwise-reflection potential becomes

$$\frac{1}{2} \sum_{(i,j)\in\mathcal{E}_I} \|\widetilde{p}_i - \widetilde{p}_j\|^2 .$$

Hence

$$\boxed{\dot{\widetilde{p}} = -(L \otimes I_2)\widetilde{p}.}$$

This is ordinary vector consensus:

$$\widetilde{p}_i(t) \longrightarrow q, \qquad q \in \mathbb{R}^2 .$$

The two-dimensional ambiguity is just the consensus value q.

The Mirror Constraint Pins One Coordinate

Coordinates aligned with fixed mirror

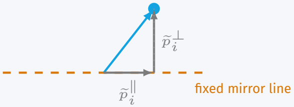
Let $\hat{\ell}$ be a unit vector along the fixed mirror line and $\hat{\nu}$ its unit normal.

Write

$$\widetilde{p}_i = \hat{\ell}\,\widetilde{p}_i^{\parallel} + \hat{\nu}\,\widetilde{p}_i^{\perp}.$$

Equivalently,

$$\widetilde{p}_i^{\parallel} = \hat{\ell}^{\top}\widetilde{p}_i, \qquad \widetilde{p}_i^{\perp} = \hat{\nu}^{\top}\widetilde{p}_i.$$



In this basis,

$$H_\ell = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Hence

$$p_\ell = H_\ell p_\ell \implies \widetilde{p}_\ell^{\perp} = 0.$$

What happens to the two coordinates?

Collect the scalar coordinates across all agents:

$$\widetilde{p}^{\parallel} = \begin{bmatrix} \widetilde{p}_1^{\parallel} \\ \vdots \\ \widetilde{p}_n^{\parallel} \end{bmatrix}, \qquad \widetilde{p}^{\perp} = \begin{bmatrix} \widetilde{p}_1^{\perp} \\ \vdots \\ \widetilde{p}_n^{\perp} \end{bmatrix}.$$

The transformed dynamics split into

$$\boxed{\dot{\widetilde{p}}^{\parallel} = -L\widetilde{p}^{\parallel}}$$

and

$$\boxed{\dot{\widetilde{p}}^{\perp} = -(L + 2e_\ell e_\ell^{\top})\widetilde{p}^{\perp}.}$$

Therefore,

$$\widetilde{p}_i^{\parallel}(t) \longrightarrow \bar{p}^{\parallel} := \frac{1}{n}\sum_{k=1}^{n}\widetilde{p}_k^{\parallel}(0),$$

while

$$\widetilde{p}_i^{\perp}(t) \longrightarrow 0.$$

Thus every transformed agent converges to

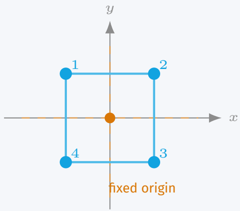
$$\boxed{\widetilde{p}_i(t) \longrightarrow \bar{p}^{\parallel}\,\hat{\ell}.}$$

Consensus is preserved along the mirror line; the normal component is pinned to zero.

From Formation Acquisition to Maneuvering

So far: symmetry about a fixed origin

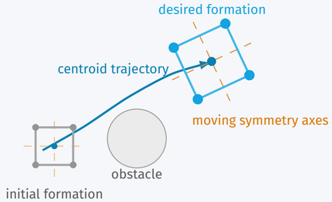
The symmetry constraints are defined with respect to a fixed inertial frame.



The controller drives the agents to the correct symmetric shape, but its position and orientation are tied to the global frame.

What we actually want

The same formation should be able to move through space while preserving its symmetry.



We want to prescribe

$\boxed{\text{translation} \quad + \quad \text{rotation} \quad + \quad \text{scaling}}$

without destroying the formation symmetry.

Can the symmetry constraints themselves move with the formation?

Symmetry in a Moving Reference Frame

Translate, rotate, and scale the frame

Let the desired maneuver be described by

$$r(t) \in \mathbb{R}^2, \qquad \theta(t) \in \mathbb{R}, \qquad s(t) > 0,$$

with

$$\dot{r} = v, \qquad \dot{\theta} = \omega, \qquad \dot{s} = \alpha s.$$

Define positions relative to the moving center:

$$c_i = p_i - r.$$

As the formation rotates, each reflection constraint rotates with it:

$$\boxed{H_{ij}(t) = R(\theta) H_{ij} R(\theta)^\top.}$$

Thus the desired symmetry is always expressed in the *current formation frame*, rather than the inertial frame.

$\boxed{\text{static symmetry control} \;\rightarrow\; \text{translation + rotation + scaling}}$

Return to the static problem

Express the configuration in the translating, rotating, and scaling frame:

$$\boxed{\zeta = \frac{1}{s}\left(I_n \otimes R(\theta)^\top\right) c.}$$

The maneuvering controller compensates for the prescribed translation, rotation, and scaling.

After the change of coordinates, all reference-dependent terms cancel:

$$\boxed{\dot{\zeta} = -Q\zeta.}$$

This is exactly the same static symmetry-control system analyzed earlier.

Therefore, if the static controller converges to the desired dihedral formation, then

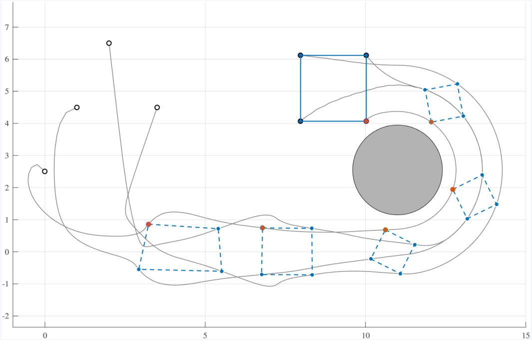
$$\zeta(t) \longrightarrow \ker Q$$

exponentially, while $p(t)$ follows the prescribed maneuver.

The formation moves, but the convergence problem does not change.

Simulation: Maneuvering While Preserving Symmetry

A moving symmetric formation



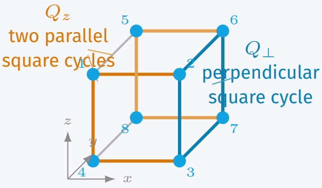
The agents follow a prescribed reference motion while preserving the reflection constraints in the moving formation frame.

translation rotation scaling

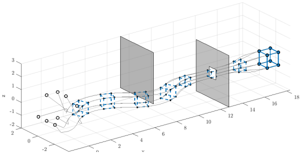
Outlook: Rotation Symmetry in $\mathbb{R}^3$

Cube as coupled planar symmetries

The 3D example builds a cube by combining several planar quarter-turn symmetry constraints.



Think of Q_z as enforcing two horizontal square cycles, and $Q_\perp$ as enforcing an additional square cycle through a perpendicular face.



Composing the 3D constraints

The cube example combines two rotation-symmetry Laplacians:

$$Q_z \quad \text{and} \quad Q_\perp.$$

The full matrix is assembled as

$$\boxed{Q = I_2 \otimes Q_z + P \begin{bmatrix} Q_\perp^\top & 0 \end{bmatrix}^\top P^\top.}$$

$I_2 \otimes Q_z$	:	copies the z-axis square constraint onto the two parallel layers of the cube,
$Q_\perp$	:	adds a perpendicular square constraint,
P	:	reorders the agent coordinates so the blocks act on the correct cube vertices.

By construction,

$$Q \succeq 0, \qquad \ker Q = \{\text{cube-symmetric configurations}\}.$$

Numerical evidence: convergence to a moving cube formation.

Outlook: Distributed Maneuvering Commands

Reference known only to some agents

Suppose only a subset of agents receives the maneuvering command

$$r_{\mathrm{ref}}(t), \qquad \dot{r}_{\mathrm{ref}}(t).$$

A naive translation controller,

$$\dot{p} = -Q(p - B \otimes r_{\mathrm{ref}}) + B \otimes \dot{r}_{\mathrm{ref}},$$

is insufficient: uninformed agents still enforce symmetry about the wrong reference.

The reference must first be distributed through the network.

PI reference filter

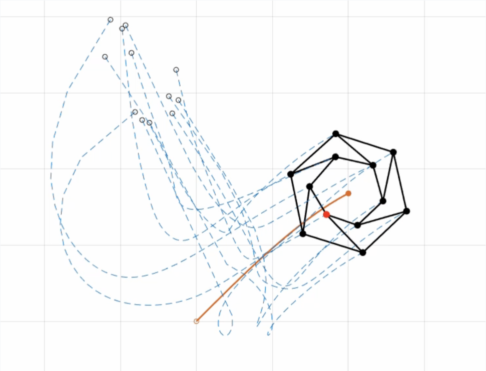
Each agent maintains a local estimate $r_i(t)$:

$$\begin{cases} \dot{r} = -k_p \bar{L} r - k_I \bar{L} \zeta + nB \otimes \dot{r}_{\mathrm{ref}}, \\ \dot{\zeta} = \bar{L} r. \end{cases}$$

Then use

$$\boxed{\dot{p} = -Q(p - r) + \dot{r}.}$$

Distributed tracking example



The informed agents inject the reference command; the PI filter spreads it through the network while the formation controller maintains the symmetric shape.

Cascade: reference filter $\longrightarrow$ formation dynamics.

Summary: From Consensus to Symmetry-Based Formations

A common structure

formation control as a sequence of increasingly rich consensus-like gradient flows:

edge potential $\Longrightarrow$ matrix-weighted Laplacian $\Longrightarrow$ symmetric formation

Model	**Constraint**	**Equilibrium set**
Consensus	$x_i = x_j$	agreement
Rigidity-based formation	$\|p_i - p_j\| = d_{ij}$	distance-realizing shapes
Rotation symmetry	$p_i = R_{ij} p_j$	cyclic formations
Reflection symmetry	$p_i = H_{ij} p_j$	dihedral formations

The key idea: replace metric constraints by symmetry constraints.

Takeaway

For highly symmetric targets, formation control can be solved with connectivity-level information:

$n - 1$ inter-agent constraints

Future Directions

Theory

▷ **Complete 3D analysis**

Extend the planar rotation and reflection theory to spatial point groups and characterize the resulting kernels.

▷ **General graph structures**

Move beyond spanning trees: directed graphs, switching graphs, redundant constraints, and robustness.

▷ **Rigidity versus symmetry**

Clarify when symmetry constraints can replace rigidity constraints, and when both are needed.

Control and applications

▷ **Distributed maneuvering**

Analyze cascades where only selected agents receive translation, rotation, or scaling commands.

▷ **Quotient-level control**

Exploit orbit representatives to reduce communication and computation:

$$n \longrightarrow |\mathcal{V}_0|.$$

▷ **Experiments**

Test symmetry-based architectures on robotic swarms, sensing networks, and aerial formations.

The long-term goal is a control theory for formations whose structure is specified by symmetry rather than by a large set of metric constraints.

Concluding Remarks

Thank you

- Z. Martinez and D. Zelazo, “Formation Control via Rotation Symmetry Constraints,” in American Control Conference, New Orleans, LA, USA, May 2026.
- Z. Martinez and D. Zelazo, “Symmetry-Based Formation Control on Cycle Graphs Using Dihedral Point Groups,” in IFAC World Congress, Busan, South Korea, Aug. 2026.