Symmetry-Based Formation Control on Cycle Graphs Using Dihedral Point Groups

IFAC World Congress
Busan, Republic of Korea

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August 24, 2026

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Introduction

Formation control arises naturally in a wide range of multi-agent applications

UAV teams

Coordinated swarms for mapping and surveillance.

Satellite constellations

Coverage, distributed sensing and communication.

Multi-robot systems

Distributed sensing and transportation.

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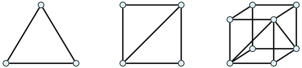
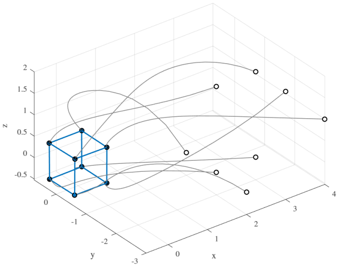
Goal: Steer the entire team toward a desired formation using only local interactions.

How can we define formations?

Formation Control Objective

A team of n agents able to sense or communicate with neighboring agents must coordinate to achieve a desired spatial configuration using only local information.





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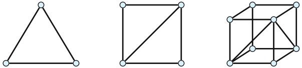
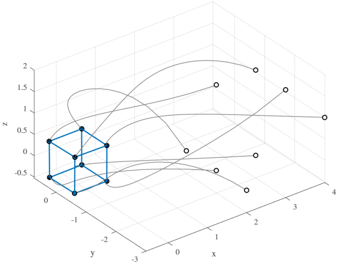
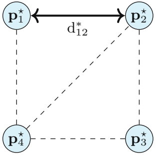


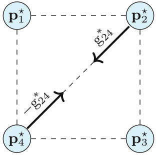


- Standard approaches impose explicit geometric constraints between agents
- The number of inter-agent links depend on the imposed constraints

In $\mathbb{R}^2$, $2n-3$ constraints are required to characterize a rigid formation



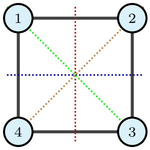
Distances



Bearings

Alternative: Symmetry Constraints

- Many practical formations exhibit inter-agent symmetries
- A formation on a cycle graph exhibits **dihedral symmetry** - n rotations *and* n reflections



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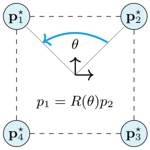
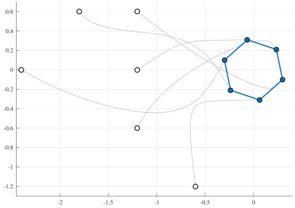
- Enforcing only **rotations** reduces the required constraints to $n-1$ ([1])



Rotations



[1] Z. Martinez and D. Zelazo, "Formation control via rotation symmetry constraints," in American Control Conference, New Orleans, LA, USA, May 2026, pp. 3469–3474.

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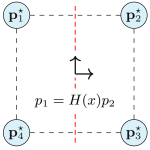
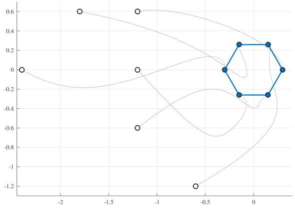
- A formation on a cycle graph exhibits **dihedral symmetry** - n rotations *and* n reflections
- Enforcing only **rotations** reduces the required constraints to $n-1$ ([1])
- Can **reflections** alone enforce the full dihedral symmetry?



Reflections

$$H(v) = I - 2vv^\top$$

The householder matrix



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The Formation Control Problem

Formation Control Problem

For n agents with single-integrator dynamics

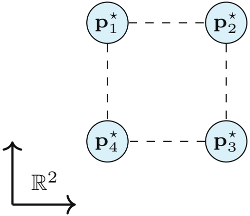
$$\dot{p}_i = u_i, \qquad p_i(t) \in \mathbb{R}^d,$$

an interaction graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and a constraint map $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, design distributed control laws u_i such that

$$\mathcal{F} = \{p \in \mathbb{R}^{nd} \mid F(p) = F(\mathrm{p}^\star)\}$$

is asymptotically stable.

- A feasible formation is defined as a framework $(\mathcal{G}, \mathrm{p}^\star)$



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- $\mathcal{F}$ is the set of feasible configurations, defined only up to global transformations

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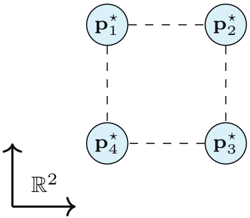

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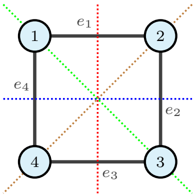
We require a formal language to talk about symmetries

Symmetry in Graphs

Formation objectives are defined over the underlying graph $\mathcal{G}$

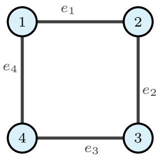


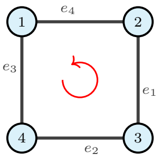
Graph Automorphism

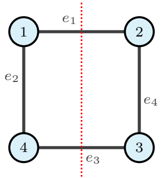
An **automorphism** of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation $\psi : \mathcal{V} \to \mathcal{V}$ of its vertex set such that

$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$

Example: Cycle graph C_4







Identity:

$$\mathrm{Id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

90° rotation:

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

reflection:

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

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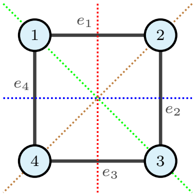

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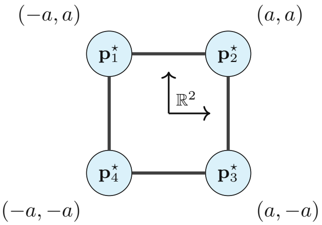
- Additional permutations can be found for the given graph considering all possible reflections and rotations
- The set of all automorphisms of $\mathcal{G}$ form a *group* - $\mathrm{Aut}(\mathcal{G})$
- For any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$, we say that $\mathcal{G}$ is **Γ-symmetric**, which define specific symmetries in $\mathcal{G}$

Definition

A framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called **$\tau(\Gamma)$-symmetric** if

$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all} \quad \gamma \in \Gamma \quad \text{and all} \quad i \in \mathcal{V}$$

Graph symmetries can be realized in $\mathbb{R}^d$ by assigning to each element of Γ an orthogonal matrix τ representing a point group isometry.



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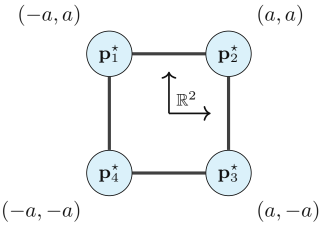
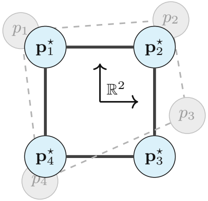


Example: $\Gamma = \{\mathrm{Id}, \psi_x, \psi_y\}$ (Reflections about x and y of C_4)

- $\tau(\psi_y)p_2 = H(x)p_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} -a \\ a \end{bmatrix} = p_1$
- $\tau(\psi_x)p_3 = H(y)p_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -a \\ -a \end{bmatrix} = \begin{bmatrix} -a \\ a \end{bmatrix} = p_1$

Symmetry in Frameworks

Consider formations on a cycle graph C_n with full dihedral symmetry D_n



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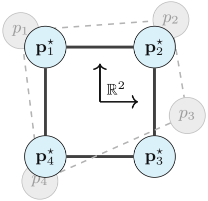

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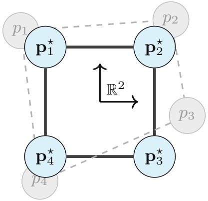

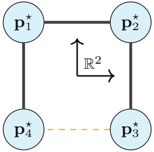

$$\mathbf{p}_4^\star = \tau(\psi_x)\tau(\psi_y)\tau(\psi_x)\,\mathbf{p}_3^\star$$
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- A $\tau(\Gamma)$-symmetric framework is called **D_n-symmetric** when all symmetries match those of C_n
- $\Gamma = \mathrm{Aut}(C_n)$

Any *interaction graph* $\mathcal{G}_I = (\mathcal{V}, \mathcal{E}_I)$, chosen as a **spanning tree** of C_n is sufficient to describe the shape.

Reflection Based Formation Control

The control objective is to render the set of D_n-symmetric configurations

$$\mathcal{F} = \{p \in \mathbb{R}^{2n} \mid \tau(\gamma)p_i = p_{\gamma(i)},\ \forall\gamma \in \mathrm{Aut}(C_n),\ i \in \mathcal{V}\}$$

asymptotically stable by using *reflections* alone.

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Formation Control Problem

Let Γ_r be the rotational subgroup of $\mathrm{Aut}(C_n)$, and define

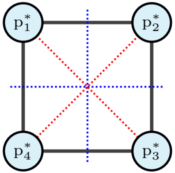
$$\mathscr{F}_s := \{\gamma \in \Gamma \setminus \Gamma_r \mid \gamma(i) \neq i,\ \forall i \in \mathcal{V}\}\,,$$

$$\mathscr{S}_s := \{\gamma \in \Gamma \setminus \Gamma_r \mid \gamma(i) = i,\ \text{for some } i \in \mathcal{V}\}\,.$$

Over the interaction graph $\mathcal{G}_I = (\mathcal{V}, \mathcal{E}_I)$, chosen as a spanning tree of C_n, design u_i such that

(i) $\lim_{t\to\infty} \|p_i(t) - \tau(\gamma_{ji})p_j(t)\| = 0, \qquad \gamma_{ji} \in \Gamma_r \cup \mathscr{F}_s$, for every $ij \in \mathcal{E}_I$;

(ii) $\lim_{t\to\infty} \|p_i(t) - \tau(\gamma_i)p_i(t)\| = 0, \qquad \gamma_i \in \mathscr{S}_s$.



A Gradient Approach

For $\gamma_{ij} \in \mathscr{F}_s$ define a symmetry-forcing potential and take its gradient flow $\dot{p}_i = -\partial F / \partial p_i$:

$$F(p) = \frac{1}{2} \sum_{ij \in \mathcal{E}_I} \|p_i - \tau(\gamma_{ji}) p_j\|^2, \qquad \dot{p}_i = \sum_{ij \in \mathcal{E}_I} \big(\tau(\gamma_{ji}) p_j - p_i\big).$$

Stacking all agents p_i, we get the compact form:

$$\dot{p}(t) = -Q(\mathscr{S}_s)\, p(t),$$

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$Q(\mathscr{S}_s)$ is a **symmetry-constrained matrix-weighted Laplacian** — the matrix analogue of the scalar graph Laplacian L:

$$\underbrace{[Q]_{ij} = \begin{cases} d(i)I_2, & i = j \\ -\tau(\gamma_{ji}), & ij \in \mathcal{E}_I \\ 0, & \text{otherwise} \end{cases}}_{\text{symmetry-constrained}} \quad \longleftrightarrow \quad \underbrace{[L]_{ij} = \begin{cases} d(i), & i = j \\ -1, & ij \in \mathcal{E}_I \\ 0, & \text{otherwise} \end{cases}}_{\text{scalar Laplacian}}$$

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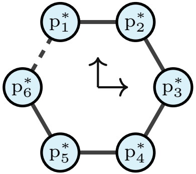
Q is PSD and has the same sparsity as L $\Rightarrow$ $Q(\mathscr{S}_s) = \bar{E}(\Gamma_r) \bar{E}(\Gamma_r)^\top$

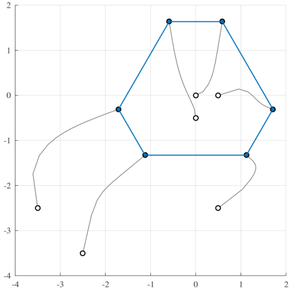
Inter-agent Reflection Constraints Are Not Enough

Pairwise reflection constraints can enforce the required symmetry about specific axes without achieving the full dihedral symmetry of the target.

Example



Target: D_6-symmetric formation (hexagon with predefined orientation)



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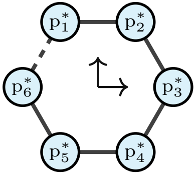

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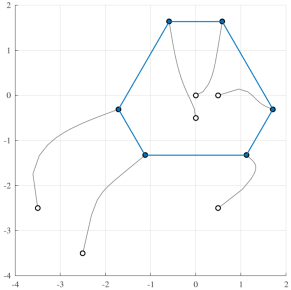


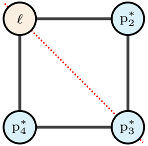
Results in a flexible D_3-symmetric configuration

Full Reflection Based Formation Control

Free reflections alone are insufficient: $p_i = \tau(\gamma_{ji})p_j$ is satisfied while $p_j \notin L_j$ (L_j is the mirror line of p_j).

Idea: Anchor a single agent ℓ to L_ℓ:

$$Q(\mathscr{R}_s) = Q(\mathscr{F}_s) + \mathrm{diag}(e_\ell) \otimes (I_2 - \tau(\gamma_\ell)).$$



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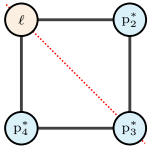
$\tau(\gamma_\ell)$ is a Householder reflection. Using $I_2 - \tau(\gamma_\ell) = 2\hat{n}_\ell \hat{n}_\ell^\top$,

$$p^\top Q(\mathscr{R}_s)p = \underbrace{\|E(\mathscr{F}_s)^\top p\|^2}_{\text{pairwise mirrors}} + \underbrace{2|\hat{n}_\ell^\top n_\ell|^2}_{\text{anchor}} \geq 0.$$

So $p^\top Q(\mathscr{R}_s)p = 0$ iff *both* terms vanish:

$$E(\mathscr{F}_s)^\top p = 0 \;\text{ and }\; (I_2 - \tau(\gamma_\ell))p_\ell = 0.$$



Lemma (mirror propagation)

On a spanning tree, fixing one mirror line L_ℓ and the free reflections γ_{ji} *uniquely* determines every L_i via $L_j = \tau(\gamma_{ji})L_i$.

Full Reflection Based Formation Control

Theorem

for any initial condition $p(0) \in \mathbb{R}^{2n}$, the control $\dot{p} = -Q(\mathscr{R}_s)p$ renders the set

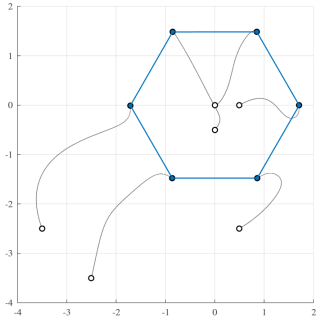
$$\mathcal{F} = \{p \in \mathbb{R}^{2n} \mid \tau(\gamma)p_i = p_{\gamma(i)},\ \forall\gamma \in \mathrm{Aut}(C_n),\ i \in \mathcal{V}\}$$

exponentially stable with $p(\infty)$ as the orthogonal projection of $p(0)$ onto $\mathcal{F}$,

$$\lim_{t\to\infty} p(t) = \frac{1}{n} V_{s_0} V_{s_0}^T p(0),$$

where $V_{s_0} = \begin{bmatrix} (S_1\hat{\mathcal{L}}_\ell)^T & \cdots & (S_n\hat{\mathcal{L}}_\ell)^T \end{bmatrix}^T \in \mathbb{R}^{2n}$, $\hat{\mathcal{L}}_\ell$ is a unit direction vector along the mirror line $\mathcal{L}_\ell$, and $S_j = \tau(\gamma_{ji}) \in O(2)$ for all $j = 1, \ldots, n$, with $S_i = I_2$.

Example continuation:



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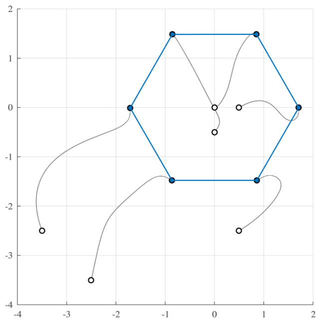
exponentially stable with $p(\infty)$ as the orthogonal projection of $p(0)$ onto $\mathcal{F}$,

$$\lim_{t\to\infty} p(t) = \frac{1}{n} V_{s_0} V_{s_0}^T p(0),$$

where $V_{s_0} = \begin{bmatrix} (S_1\hat{\mathcal{L}}_\ell)^T & \cdots & (S_n\hat{\mathcal{L}}_\ell)^T \end{bmatrix}^T \in \mathbb{R}^{2n}$, $\hat{\mathcal{L}}_\ell$ is a unit direction vector along the mirror line $\mathcal{L}_\ell$, and $S_j = \tau(\gamma_{ji}) \in O(2)$ for all $j = 1, \ldots, n$, with $S_i = I_2$.

Example continuation:



The correct D_6-symmetric configuration is achieved!

Maneuvering of Symmetric Formations

We augment the control law $\dot{p} = -Q(\Gamma)p$ so a reflection-constrained formation can translate, rotate, and scale along a time-varying reference.

Virtual reference trajectory

Define the state $\chi = [\,r^\top \; \theta \; s\,]^\top$ — translation $r \in \mathbb{R}^2$, rotation $\theta \in \mathbb{R}$, scale $s \in \mathbb{R}^+$, evolving as

$$\dot{r} = v, \qquad \dot{\theta} = \omega, \qquad \dot{s} = \alpha s,$$

with rotation $\mathcal{R}(\theta) \in SO(2)$ satisfying

$$\dot{\mathcal{R}} = \Omega\,\mathcal{R}, \qquad \Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \omega.$$

v, ω, α are the translational velocity, angular velocity, and scaling rate of the desired formation.

Augmented control law

With shifted states $c = p - \mathbb{1}_n \otimes r$,

$$u_{aug} = -Q(\mathscr{R}_s\,\theta)\,c + \mathbb{1}_n \otimes v + (I_n \otimes \Omega + \alpha)\,c,$$

where the rotated blocks satisfy

$$[Q(\Gamma, \theta)]_{ij} = \begin{cases} d(i) I_2, & i = j, \\ -\mathcal{R}(\theta)\,\tau(\gamma_{ji})\,\mathcal{R}(\theta)^\top, & ij \in \mathcal{E}_I, \\ 0, & \text{o.w.} \end{cases}$$

The actions $\tau(\gamma_{ji})$ undergo a **similarity transformation** by $\mathcal{R}(\theta)$, so the symmetry relations rotate *with* the formation.

Maneuvering of Symmetric Formations

Main Result

For n integrators on a spanning tree $\mathcal{G}_I \subset C_n$, define the symmetric configuration set

$$\mathcal{F}_c = \{p \mid \tau(\gamma)c_i = c_{\gamma(i)},\ \forall \gamma \in \mathrm{Aut}(C_n)\}.$$

Then for any $p(0)$, the control law u_{aug} renders $\mathcal{F}_c$ exponentially stable.

Maneuvering of Symmetric Formations

Main Result

For n integrators on a spanning tree $\mathcal{G}_I \subset C_n$, define the symmetric configuration set

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Then for any $p(0)$, the control law u_{aug} renders $\mathcal{F}_c$ exponentially stable.

Proof sketch:

Express the configuration in the trajectory frame:

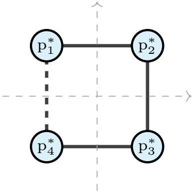
$$\zeta = \frac{1}{s}\left(I_n \otimes R(\theta)^\top\right) c$$

After the change of coordinates, all reference-dependent terms cancel returning the problem to the static case

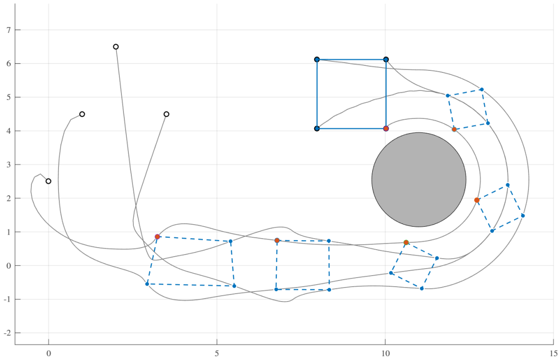
$$\boxed{\dot{\zeta} = -Q\zeta, \quad \zeta(t) \longrightarrow \ker Q}$$

Simulation: Formation Maneuvering

Example: Agents converge to a D_4 formation along a trajectory.

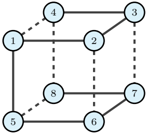


Target: D_4-symmetric formation

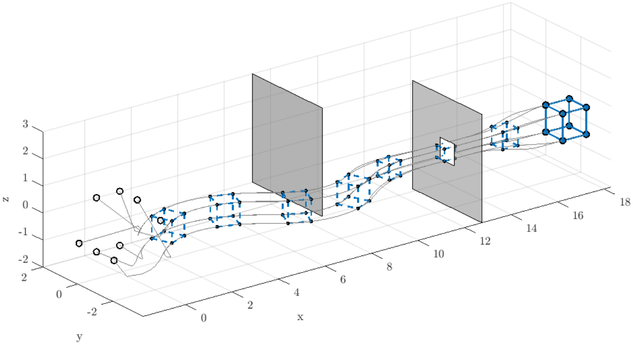


Simulation: Formation Maneuvering

Can be naturally extended to $\mathbb{R}^3$ (e.g. a moving cube)



Target: $\mathcal{O}_h$-symmetric formation (cube with predefined orientation)



Concluding Remarks

Summary

- Cyclic formations with prescribed orientation can be achieved using reflection symmetry constraints alone
- The proposed method requires only $n-1$ edges, corresponding to minimal connectivity
- The framework provides a direct foundation for coordinated formation maneuvering while preserving the desired symmetry

Future Work

- Extension to general point-groups in $\mathbb{R}^d$
- Explore leader-follower architectures enabling fully distributed agreement on time-varying virtual trajectories
- Incorporate additional geometric constraints alongside symmetry to enable a broader class of target formations

Thank you