The Geometry of Hidden Modes in Distance-Based Formation Control

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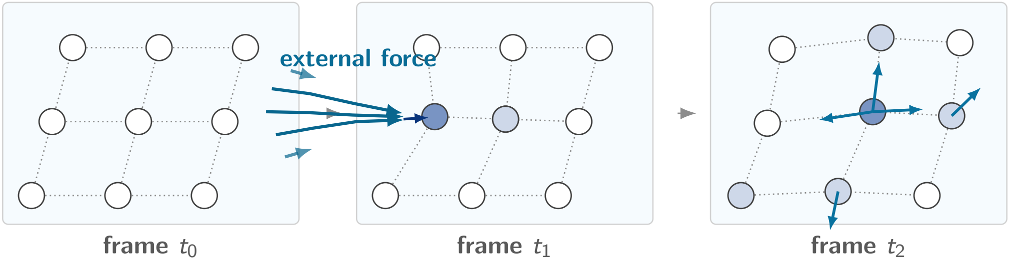
Motivation: What Can a Single Actuator Reach?



A localized input at one node propagating through a distance-constrained network.

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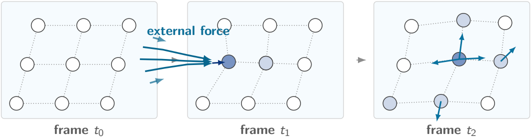
Framing Questions

- Can one actuator excite *every* mode, and if not, **which motions are hidden**?
- Of the motions it *can* excite, which restore the shape?



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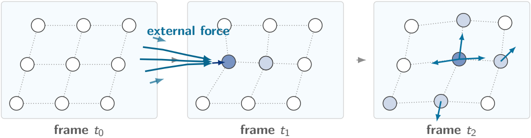

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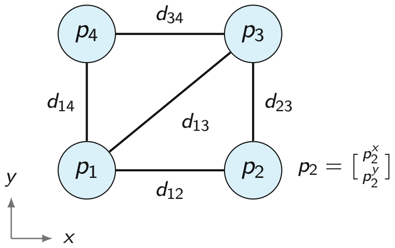
Some motions are always beyond a single actuator's reach — and the framework's geometry tells us exactly which.

The Formation Control Problem

Graphs and Frameworks

- **Graph** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$: $\mathcal{V} = \{1, \ldots, n\}$ agents (vertices), $\mathcal{E}$ sensing links ($m = |\mathcal{E}|$ edges)
- **Framework** $(\mathcal{G}, p)$: embedding of $\mathcal{G}$ in $\mathbb{R}^2$; $p = (p_1, \ldots, p_n) \in \mathbb{R}^{2n}$



A framework $(\mathcal{G}, p)$ in $\mathbb{R}^2$:

graph topology + positions

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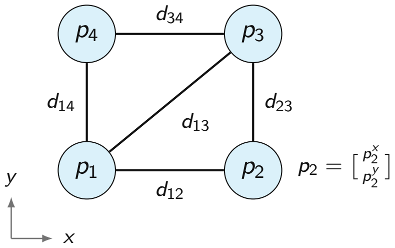
Control Objective

- **Dynamics:** $\dot{p}_i = u_i$ (single integrator)
- **Encode desired edge length:** $r_{ij}(p) = \frac{1}{2}\|p_i - p_j\|^2$
- **Goal:** converge to the *distance-constraint set*:

$$\mathcal{F}(p^*) = \left\{p : r_{ij}(p) = r_{ij}(p^*),\ \forall (i,j) \in \mathcal{E}\right\}$$



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Standard Approach: Gradient Control

Potential Function

$$V(p) = \frac{1}{2}\left\| r(p) - r(p^*) \right\|^2$$

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Gradient Descent Control Law

$$u_i = -\sum_{j \in \mathcal{N}_i} \underbrace{\left(r_{ij}(p) - r_{ij}(p^*)\right)}_{\text{edge error}} \underbrace{(p_i - p_j)}_{\text{rel. position}}$$

$$\dot{p} = -\left(\nabla_p r(p)\right)^\top \left(r(p) - r(p^*)\right)$$

- **Distributed:** agent i uses only local relative positions $(p_i - p_j)$ for $(i, j) \in \mathcal{E}$
- **Locally stable:** near p^*, the closed-loop is a gradient flow $\Rightarrow V(p(t))$ is non-increasing [Krick et al. 2009]

The Rigidity Matrix

$$R(p^*) := \nabla_p r(p)\Big|_{p^*} \in \mathbb{R}^{m \times 2n}$$

Graph Rigidity Theory

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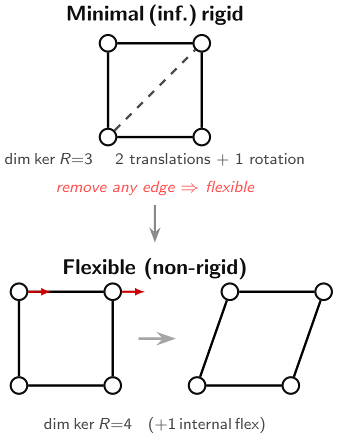
Fundamental Subspaces

- In edge space: $\operatorname{Im} R(p^*) \oplus \ker R(p^*)^\top = \mathbb{R}^m$
  - $\operatorname{Im} R(p^*)$: **infinitesimal strain**
  - $\ker R(p^*)^\top$: **self-stresses**
- In configuration space: $\ker R(p^*) \oplus \operatorname{Im} R(p^*)^\top = \mathbb{R}^{2n}$
  - $\operatorname{Im} R(p^*)^\top$: **deformations**
  - $\ker R(p^*)$: **rigid-body motions** + internal flexes



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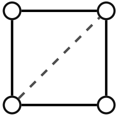
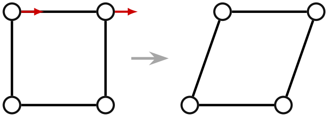
Minimal (inf.) rigid



dim ker R=3 2 translations + 1 rotation

remove any edge $\Rightarrow$ flexible

Flexible (non-rigid)



dim ker R=4 (+1 internal flex)

Next, we assume inf. rigid frameworks.

Localized Input Model

To study the influence of a localized input, we add an external input at agent i:

Nonlinear Input Model

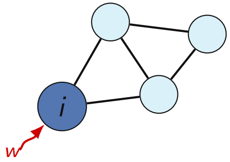
$$\dot{p} = -R(p)^\top \big(r(p) - r(p^*)\big) + B\,w$$

with $w \in \mathbb{R}^2$, $B = e_i \otimes I_2 \in \mathbb{R}^{2n\times 2}$.

Explicit form of B

$$B^\top = \big[\ \underbrace{0}_{\text{block }1}\ \cdots\ \underbrace{I_2}_{\text{block }i}\ \cdots\ \underbrace{0}_{\text{block }n}\ \big]$$

Only the i-th 2×2 block is non-zero: the input acts on agent i alone.



input w enters at agent i.

Linearization

We linearize around the target configuration p^*, with $x := p - p^*$ the **deviation** from the target:

$$\underbrace{\dot{p} = -R(p)^\top \big(r(p) - r(p^*)\big) + Bw}_{\text{Nonlinear model}} \xrightarrow{\text{linearize at } (p^*,\, w=0)} \underbrace{\dot{x} = A\,x + B\,w}_{\text{LTI model}}$$

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Eigenvalue structure:

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LTI model unlocks classical tools:

- Controllability
- Modal decomposition
- Transfer functions
- Steady-state analysis

Which Modes Stay Hidden?

Controllability Test (PBH)

Let v be a mode of the symmetric stiffness matrix: $A = -R(p^*)^\top R(p^*)$.

This mode is uncontrollable if and only if $B^\top v = 0$.

Since $B = e_i \otimes I_d$: $\quad B^\top v = 0 \iff v_i = 0$ — the mode is pinned at the actuator.

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Symmetry $\Rightarrow$ Modal Splitting

Because A is symmetric the hidden (uncontrollable) subspace splits as

$$\bar{\mathcal{C}} = \bigoplus_{\lambda} \left(E_\lambda \cap \ker B^\top\right).$$

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$$\bar{\mathcal{C}} = \underbrace{\left(\ker R(p^*) \cap \ker B^\top\right)}_{\textbf{Hidden RBMs } (\lambda=0)} \oplus \underbrace{\bigoplus_{\lambda<0} \left(E_\lambda \cap \ker B^\top\right)}_{\textbf{Hidden deformations}}$$

Unavoidable Hidden Modes

Question: Given an inf. rigid framework $(\mathcal{G}, p^*)$, how much of the rigid-body mode (RBM) eigenspace $E_0 = \ker R(p^*)$ can the actuator miss?

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RBM eigenspace dimension	3
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An RBM is d translations plus $\frac{d(d-1)}{2}$ rotations.

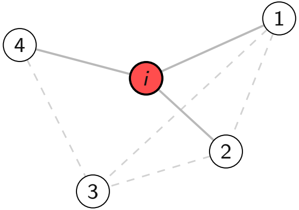
Are the hidden modes pure rotations? If yes, *which* ones?

Hidden RBMs Are Pure Rotations

Any RBM mode v decomposes into a translation ξ and a rotation about the centroid:

$$v_\ell = \xi + \Omega(p^*_\ell - p_{cm}), \quad \Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Pinning $v_i = 0$ forces: $\xi = -\Omega(p^*_i - p_{cm})$



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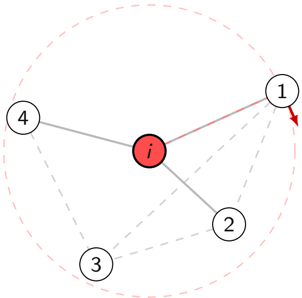

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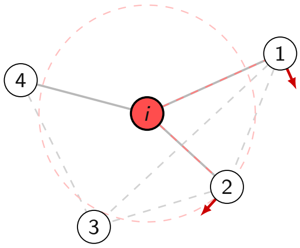

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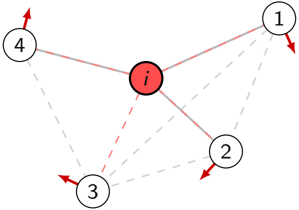

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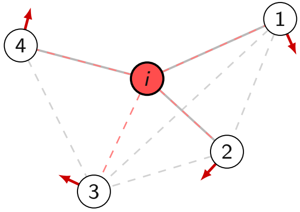

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Uncontrollable RBMs at Node i

$$\mathcal{R}_i(p^*) := \left\{ v : v_\ell = \Omega(p^*_\ell - p^*_i) \;\; \forall \ell \in \mathcal{V} \right\}$$

$\therefore \quad \mathcal{R}_i(p^*) = \ker R(p^*) \cap \ker B^\top$

The Local Rotational Subspace $\mathcal{T}_i$

Objective: the hidden deformations $\bar{\mathcal{C}} \cap \operatorname{Im} R(p^*)^\top$

Key Observation

The input acts at agent i and reaches the formation through i's incident edges. If those lengths are preserved, agent i produces no correction.

Definition: Local Rotational Subspace

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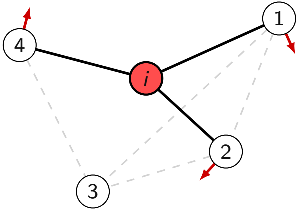

One representative motion; in general each neighbor may rotate independently about i.

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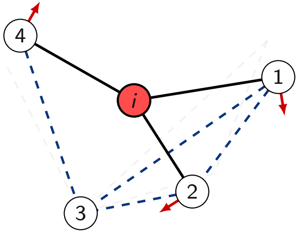

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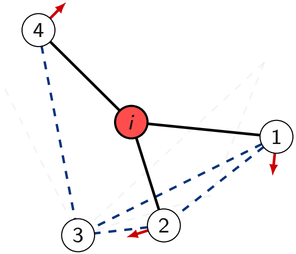

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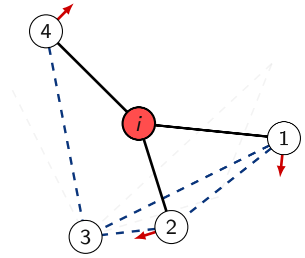
Geometric Inclusion

$\mathcal{R}_i(p^*) \subseteq \mathcal{T}_i(\mathcal{G}, p^*)$; $\mathcal{T}_i$ also contains **local deformations** not in $\mathcal{R}_i$.

Dimension: Counting Constraints

Pin i: $-d$; each neighbor j: -1; all other nodes free:

$$\dim \mathcal{T}_i = d(n-1) - \deg(i)$$



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$\mathcal{T}_i$ Is a Truncated Kalman Test

The hidden subspace is the intersection of *all* Kalman conditions:

$$\bar{\mathcal{C}} = \left(\mathrm{Im}\left[\, B,\ AB,\ \ldots,\ A^{dn-1}B \,\right]\right)^{\perp} = \bigcap_{k\geq 0} \ker\left(B^{\top}A^{k}\right)$$

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The first two conditions

- $k = 0$: $\quad B^\top v = 0 \iff v_i = 0$ *(pinning)*
- $k = 1$: $\quad B^\top A v = 0 \iff \sum_{j\in\mathcal{N}_i} (p_j^* - p_i^*)\, \underbrace{(p_j^* - p_i^*)^\top (v_j - \cancel{v_i})}_{\delta r_{ij}\text{ - first order edge length}} = 0$ *(no correction at i)*

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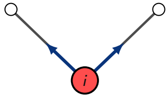
Minimal degree $(\deg(i)=d)$

For $\deg(i)=d$ the second forces each $\delta r_{ij}=0$, so

$$\mathcal{T}_i \;=\; \ker B^{\top}\,\cap\,\ker\big(B^{\top}A\big).$$

$\bar{\mathcal{C}}$ intersects over *all* k; $\mathcal{T}_i$ over $k\in\{0,1\}$ — fewer conditions, larger set.



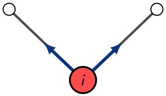
independent directions:
cancel only if **all** $\delta r_{ij}=0$

Geometric Confinement

Geometric Confinement on Hidden Modes ($\deg(i) = d$)

For an inf. rigid $(\mathcal{G}, p^*)$ actuated at a node of minimal degree:

$$\bar{\mathcal{C}} \subseteq \mathcal{T}_i(\mathcal{G}, p^*)$$



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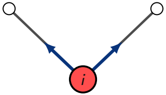

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Refined bound

$$\bar{\mathcal{C}} = \underbrace{\left(\bar{\mathcal{C}} \cap \ker R(p^*)\right)}_{\textbf{Hidden RBMs}} \oplus \underbrace{\left(\bar{\mathcal{C}} \cap \operatorname{Im} R(p^*)^\top\right)}_{\textbf{Hidden deformations}}$$

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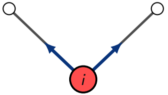

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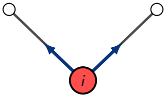


Refined bound

$$\bar{\mathcal{C}} \subseteq \underbrace{\mathcal{R}_i(p^*)}_{\text{Pure Rotation}} \oplus \underbrace{\left(\mathcal{T}_i \cap \operatorname{Im} R(p^*)^\top\right)}_{\text{Locally Hidden Deformations}} \subseteq \mathcal{T}_i$$

Geometric Confinement

Geometric Confinement on Hidden Modes ($\deg(i) = d$)

For an inf. rigid $(\mathcal{G}, p^*)$ actuated at a node of minimal degree:

$$\bar{\mathcal{C}} \subseteq \mathcal{T}_i(\mathcal{G}, p^*)$$

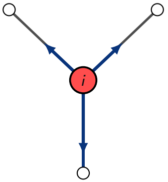
Refined bound

$$\bar{\mathcal{C}} \subseteq \underbrace{\mathcal{R}_i(p^*)}_{\text{Pure Rotation}} \oplus \underbrace{\left(\mathcal{T}_i \cap \operatorname{Im} R(p^*)^\top\right)}_{\text{Locally Hidden Deformations}} \subseteq \mathcal{T}_i$$

Beyond minimal degree ($\deg(i) > d$)

$$\deg(i) = d \quad \underset{\not\Leftarrow}{\Rightarrow} \quad \bar{\mathcal{C}} \subseteq \mathcal{T}_i$$

Sufficient, **not** necessary. Does confinement hold in general?



they cancel with
none vanishing

$$\sum_{j \in \mathcal{N}_i} (p_j^* - p_i^*)\delta r_{ij} = 0$$

How Big Is the Hidden Subspace?

$$\dim \bar{\mathcal{C}} \;=\; \frac{d(d-1)}{2} \;+\; \dim\big(\bar{\mathcal{C}} \cap \operatorname{Im} R(p^*)^\top\big)$$

$\mathcal{R}_i$ $\dim = \frac{d(d-1)}{2}$	$\bar{\mathcal{C}} \cap \operatorname{Im} R^\top$ **(unknown)**

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$$\dim \bar{\mathcal{C}} \;=\; \tfrac{d(d-1)}{2} \;+\; \dim\big(\bar{\mathcal{C}} \cap \operatorname{Im} R(p^*)^\top\big)$$

Minimal degree $(\deg(i) = d)$

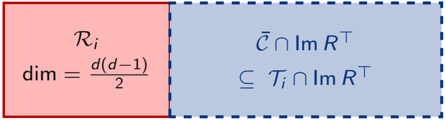
Bounded by locally hidden deformations:
$\bar{\mathcal{C}} \cap \operatorname{Im} R^\top \subseteq \mathcal{T}_i \cap \operatorname{Im} R^\top$.

$\mathcal{R}_i$
$\dim = \frac{d(d-1)}{2}$

$\bar{\mathcal{C}} \cap \operatorname{Im} R^\top$
$\subseteq \; \mathcal{T}_i \cap \operatorname{Im} R^\top$

How Big Is the Hidden Subspace?

$$\dim \bar{\mathcal{C}} = \frac{d(d-1)}{2} + \dim\left(\bar{\mathcal{C}} \cap \operatorname{Im} R(p^*)^\top\right)$$



Minimal degree $(\deg(i) = d)$

Bounded by locally hidden deformations:
$\bar{\mathcal{C}} \cap \operatorname{Im} R^\top \subseteq \mathcal{T}_i \cap \operatorname{Im} R^\top$.

$$\dim\left(\mathcal{T}_i \cap \operatorname{Im} R^\top\right) \geq d(n-2) - \frac{d(d+1)}{2}$$

a *lower* bound cannot cap $\bar{\mathcal{C}}$

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$\mathcal{R}_i$ $\dim = \tfrac{d(d-1)}{2}$	$\subseteq \mathcal{T}_i \cap \operatorname{Im} R^\top$ $\dim = d(n-2) - \tfrac{d(d+1)}{2}$

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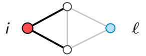
$$n \geq d+2 \implies \mathcal{T}_i + \operatorname{Im} R^\top = \mathbb{R}^{dn}$$

$$\dim\left(\mathcal{T}_i \cap \operatorname{Im} R^\top\right) = d(n-2) - \tfrac{d(d+1)}{2}$$



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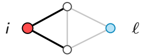

full table

$$\overbrace{\boxed{\begin{array}{c}\mathcal{R}_i \\ \dim = \frac{d(d-1)}{2}\end{array}}\;\boxed{\begin{array}{c}\subseteq \mathcal{T}_i \cap \operatorname{Im} R^\top \\ \dim = d(n-2) - \frac{d(d+1)}{2}\end{array}}}^{\frac{d(d-1)}{2} \,\leq\, \dim(\bar{\mathcal{C}}) \,\leq\, d(n-3)}$$

In the plane $(d = 2)$

$$1 \leq \dim \bar{\mathcal{C}} \leq 2n - 6$$

Which Inputs Recover the Shape?

Second Framing Question

Of the motions a single actuator *can* excite, which ones recover the target shape?

Final State Decomposition

An impulse $w(t) = w_0\,\delta(t)$ at agent i displaces the formation; the deformational modes ($\lambda < 0$) decay and the rigid-body modes ($\lambda = 0$) persist:

$$\lim_{t\to\infty} x(t) \;=\; P_0 B w_0 \;=\; c_x v_x + c_y v_y + c_r v_r$$

$$\text{Since } B = e_i \otimes I_d: \quad c_\bullet = \langle [v_\bullet]_i,\, w_0 \rangle.$$

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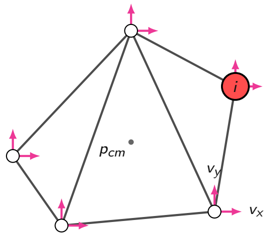

v_x, v_y: the *same* at every agent
$\Rightarrow$ no distance can change

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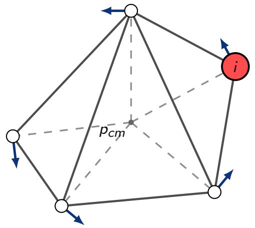

v_r: tangential, *different* at every agent
$\Rightarrow$ shape at risk

Which Inputs Recover the Shape?

Second Framing Question

Of the motions a single actuator *can* excite, which ones recover the target shape?

Final State Decomposition

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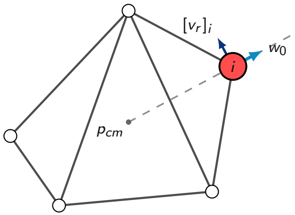
$$\lim_{t\to\infty} x(t) = P_0 B w_0 = c_x v_x + c_y v_y + c_r v_r$$

$$\text{Since } B = e_i \otimes I_d: \quad c_\bullet = \langle [v_\bullet]_i,\, w_0 \rangle.$$



recovery $\iff$ w_0 along $p_i^* - p_{cm}$

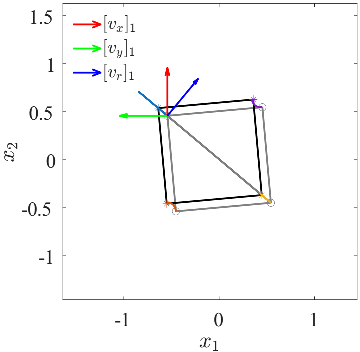
$$c_r = 0 \iff w_0 \parallel \left(p_i^* - p_{cm}\right)$$

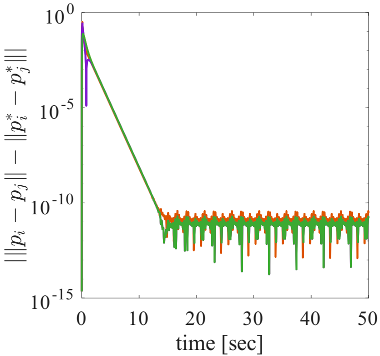
radial $\Rightarrow$ pure translation, shape restored

otherwise $\Rightarrow$ rotation persists

Case Study: Scenario A — Orthogonal Input

Setup: Input orthogonal to local rotational mode: $\langle [v_r]_1, w_0 \rangle = 0$.





Kinematic Result

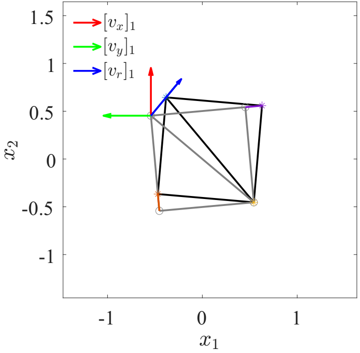
$c_r = 0 \Rightarrow$ settles into pure translation.

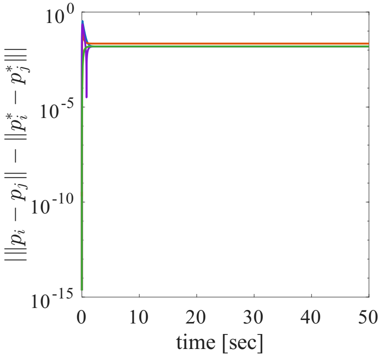
Dynamic Result

Edge errors $\to 0$ (shape recovered).

Case Study: Scenario B — Aligned Input

Setup: Identical formation. Input aligned with local rotational mode: $\langle [v_r]_1, w_0 \rangle \neq 0$.





Kinematic Result

$c_r \neq 0 \Rightarrow$ Translation + Rotation.

Dynamic Result

Errors settle at non-zero.

What Comes Next

Open Questions

1. **Beyond minimal degree ($\deg(i) > d$).** Does the inclusion $\bar{\mathcal{C}} \subseteq \mathcal{T}_i$ survive?

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Open Questions

1. **Beyond minimal degree ($\deg(i) > d$).** Does the inclusion $\bar{\mathcal{C}} \subseteq \mathcal{T}_i$ survive?
2. **Nonlinear controllability** *(ongoing).* linear to nonlinear — what survives for the exact dynamics?

Thank You

Questions?

CONNECT LAB
Cooperative Networks and Controls

Backup: Dimension Budget of the Hidden Subspaces

Subspace	general d	$d=2,\ \deg(i)=2$
$\ker R(p^*)$ (RBMs)	$\frac{d(d+1)}{2}$	3
$\operatorname{Im} R(p^*)^\top$ (deformations)	$dn - \frac{d(d+1)}{2}$	$2n-3$
$\mathcal{R}_i$ (hidden RBMs)	$\frac{d(d-1)}{2}$	1
$\mathcal{T}_i$ (locally hidden motions)	$d(n-1) - \deg(i)$	$2n-4$
$\mathcal{T}_i \cap \operatorname{Im} R(p^*)^\top$	$d(n-2) - \frac{d(d+1)}{2}$	$2n-7$
$\mathcal{R}_i \oplus (\mathcal{T}_i \cap \operatorname{Im} R(p^*)^\top)$	$d(n-3)$	$2n-6$

The last two rows and the upper bound $\dim \bar{\mathcal{C}} \leq d(n-3)$ require $\deg(i) = d,\ \ n \geq d+2$. The lower bound $\dim \bar{\mathcal{C}} \geq \frac{d(d-1)}{2}$ is unconditional.