STABILIZATION OF SYMMETRIC FORMATIONS

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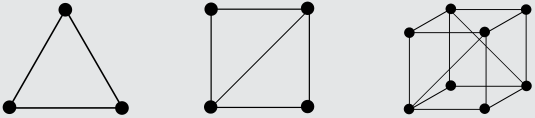
14.07.2023
IFAC World Congress
Yokohama, Japan

Formation Control Objective

Given a team of robots endowed with the ability to sense/ communicate with neighboring robots, design a control for each robot using only local information that moves the team into a desired spatial configuration.



FORMATION CONSTRAINTS

- The desired formation is characterized by a set of M constraints, encoded in the function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, and a configuration $\mathbf{p}^\star$ satisfying the constraints.
- The set of all feasible formations is

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \mid F(p) = F(\mathbf{p}^\star)\}$$

Formation Control Objective

For an ensemble of n agents with dynamics

$$\dot{p}_i = u_i,$$

with $p_i(t) \in \mathbb{R}^d$, an information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and formation constraint function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, design a distributed control law for each agent $i \in \{1, \ldots, n\}$ such that the set

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \mid F(p) = F(\mathbf{p}^\star)\},$$

is asymptotically stable.

Theorem - Distance Constrained Formation Control [Krick 2009]

Consider the potential function

$$F_f(p) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2 \right)^2$$

and assume the desired distances $d_{ij}^\star$ correspond to a feasible formation. Then the gradient dynamical system

$$u_i = -\nabla_{p_i} F_f(p) = \sum_{ij \in \mathcal{E}} \left(\|p_i - p_j\|^2 - (d_{ij}^\star)^2 \right) (p_j - p_i)$$

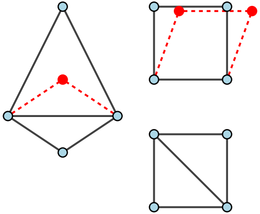
$$\dot{p} = -\nabla_p F_f(p) = -R^T(p)R(p)p + R^T(p)(d^\star)^2$$

asymptotically converges to the critical points of the potential function, i.e., $\frac{\partial F_f(p)}{\partial p} = 0$.

- $R(p)$ is the *rigidity matrix* for the framework $(\mathcal{G}, p)$
- rigidity theory used to understand more about the equilibrium sets

Rigidity theory helps us understand

- how many constraints are required to ensure uniqueness of formation shape (modulo translations, rotations, and flip ambiguities)
- how the constraints should be distributed in the network



A widely accepted architectural requirement for distance constrained formation control is that minimally infinitesimally rigid frameworks are required. Equivalent to:

$$\operatorname{rk} R(p) = 2|\mathcal{V}| - 3 \text{ and } |\mathcal{E}| = 2|\mathcal{V}| - 3 \quad (\text{in } \mathbb{R}^2)$$

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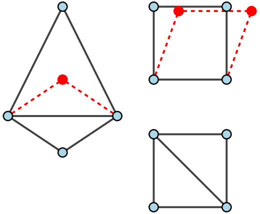

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Q: is this a necessary condition? (can we solve the problem with fewer edges?)

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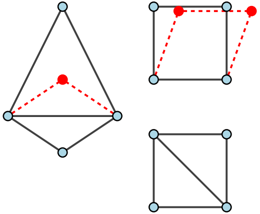

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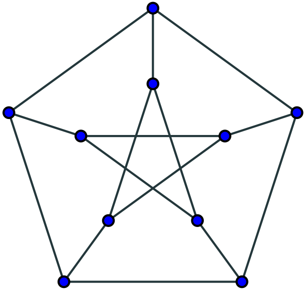
Q: is this a necessary condition? (can we solve the problem with fewer edges?)

A: Impose additional symmetry constraints without requiring more information exchange (in fact, less!)

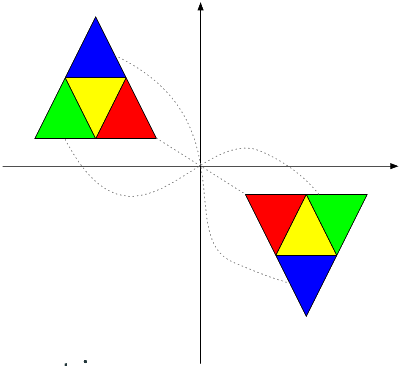
GRAPH SYMMETRIES AND POINT GROUPS

Graph Symmetries



- graph automorphisms

Point Groups

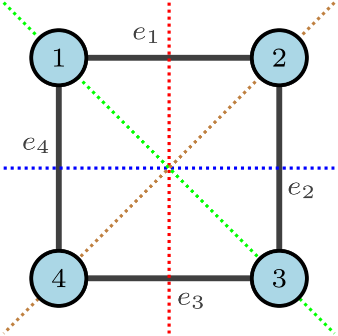


- isometries

Graph Automorphism

An automorphism of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of of its vertex set such that

$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$



Automorphisms encode graph symmetries

- identity: $\mathrm{Id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$
- $90°$ rotation: $\psi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
- $180°$ rotation: $\psi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
- $270°$ rotation: $\psi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

Graph Automorphism

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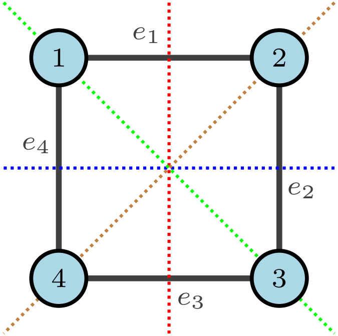

Automorphisms encode graph symmetries

- reflection: $\psi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
- reflection: $\psi_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
- reflection: $\psi_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$
- reflection: $\psi_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

Definition

Let X be a set, and let Γ be a collection of invertible functions $X \to X$. Then Γ is called a group if the identity map, Id, belongs to Γ, and for any $\Gamma \ni f, g : X \to X$, both the composite function $f \circ g$ and the inverse function f^{-1} belong to Γ.

Automorphisms of a graph form a *group* - $\mathrm{Aut}(\mathcal{G})$

- $\mathrm{Aut}(\mathcal{G}) = \{\mathrm{Id}, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\}$

A subgroup is a subset of a group, and also satisfies all properties of a group

- $\{\mathrm{Id}, \psi_1, \psi_2, \psi_3\}$
- $\{\mathrm{Id}, \psi_2, \psi_4, \psi_5\}$
- $\{\mathrm{Id}, \psi_2\}$
- $\{\mathrm{Id}, \psi_6\}$
- $\{\mathrm{Id}, \psi_7\}$

Γ-SYMMETRIC GRAPHS

- Subgroups of $\mathrm{Aut}(\mathcal{G})$ define specific symmetries in $\mathcal{G}$
- for any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$, we say that $\mathcal{G}$ is Γ-symmetric

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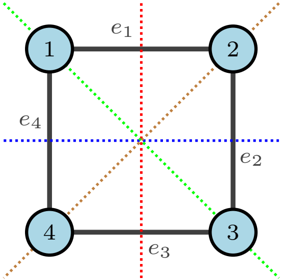
Definition

For a Γ-symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and vertex $i \in \mathcal{V}$, the set $\Gamma_i = \{\gamma(i) \,|\, \gamma \in \Gamma\}$ is called the vertex orbit of i. Similarly, for an edge $e = ij \in \mathcal{E}$, the set $\Gamma_e = \{\gamma(i)\gamma(j) \,|\, \gamma \in \Gamma\}$ is termed the edge orbit of e.



Consider $\Gamma = \{\mathrm{Id}, \psi_2\}$ (ψ_2 is the 180° rotation)

- **Vertex Orbit:**
  $\Gamma_1 = \Gamma_3 = \{1, 3\}, \; \Gamma_2 = \Gamma_4 = \{2, 4\}$
  vertices inside a vertex orbit are equivalent
  representative vertex set: $\mathcal{V}_0 = \{1, 2\}$
- **Edge Orbit:**
  $\Gamma_{e_1} = \Gamma_{e_3} = \{e_1, e_3\}$,
  $\Gamma_{e_2} = \Gamma_{e_4} = \{e_2, e_4\}$

combine notions of graph symmetries with point groups

- let $\mathcal{G}$ be a Γ-symmetric graph
- Γ also represented as a *point group*
  - homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$
  - τ assigns an orthogonal matrix (describing an isometry of $\mathbb{R}^d$ such as a rotation or reflection) to each element of Γ

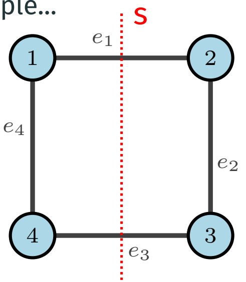
Definition

A framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called $\tau(\Gamma)$-symmetric if

$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}.$$

$\tau(\Gamma)$-SYMMETRIC FRAMEWORK

example...



- consider $\Gamma = \{\mathrm{Id}, \psi_4\} \subseteq \mathrm{Aut}(\mathcal{G})$
- $\gamma = \psi_4 \in \Gamma$ (reflection about mirror S)
- isometry $\tau(\gamma) : (a, b) \mapsto (-a, b)$

satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$.

- note: for a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ and for every $j \in \Gamma_i$, there is a $\gamma_j \in \Gamma$ such that $\tau(\gamma_j)p_j = p_i$ for all $j \in \Gamma_i$

isometries of configuration p coincide with symmetries of the automorphisms of $\mathcal{G}$

- in $\tau(\Gamma)$-symmetric frameworks, the configurations p are in a special geometric position (not necessarily generic)
- symmetry can lead to unexpected infinitesimal flexibility/rigidity

Symmetric Formation Control Objective

Consider a group of n integrator agents that interact over the Γ-symmetric sensing graph $\mathcal{G}$. Let $p^\star \in \mathbb{R}^{dn}$ be a configuration such that $(\mathcal{G}, p^\star)$ is $\tau(\Gamma)$-symmetric for some desired point group $\tau(\Gamma)$, and let $\mathcal{V}_0$ be a set of representatives of the vertex orbits of $\mathcal{G}$ under Γ. Design a control $u_i(t)$ for each agent i such that

(i) $\lim_{t \to \infty} \|p_i(t) - p_j(t)\| = \|p_i^\star - p_j^\star\| = d_{ij}^\star$ for all $ij \in \mathcal{E}$; (distance constraints)

(ii) for each $i \in \mathcal{V}_0$, $\lim_{t \to \infty} \|p_i(t) - \tau(\gamma_j) p_j(t)\| = 0 \;\forall\; j \in \Gamma_i$, $\gamma_j \in \Gamma$. (symmetry constraints)

- the formation potential

$$F_f(p(t)) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d^\star_{ij})^2 \right)^2$$

- the symmetry potential

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{j \in \Gamma_i} \|p_i(t) - \tau(\gamma_j) p_j(t)\|^2$$

- the symmetric formation potential

$$F(p(t)) = F_f(p(t)) + F_s(p(t))$$

A GRADIENT APPROACH

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- closed-loop dynamics

$$\dot{p}(t) = -R(p(t))^T \left(R(p(t))p(t) - (d^\star)^2\right) - Qp(t)$$

where Q is symmetric and a block matrix with

$$Q_{ij} = \begin{cases} (|\Gamma_i| - 1)I, & i = j,\ i \in \mathcal{V}_0 \\ -\tau(\gamma_j), & i \in \mathcal{V}_0, j \in \Gamma_i \\ I, & i = j,\ j \notin \mathcal{V}_0,\ j \in \Gamma_i \\ -\tau(\gamma_j)^{-1}, & j \in \mathcal{V}_0,\ i \in \Gamma_j \\ 0, & \text{o.w.} \end{cases}.$$

- $Q_{ij} \in \mathbb{R}^{|\Gamma_i|d \times |\Gamma_i|d}$
- $Q_{ij} \in O(\mathbb{R}^d)$ (orthogonal group)
- $\tau(\gamma_j)^{-1} = \tau(\gamma_j)^T$

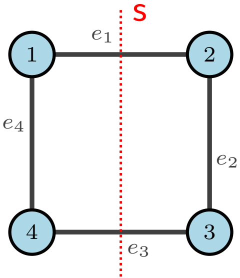
- structure of Q requires information exchange between $i \in \mathcal{V}_0 \cap \Gamma_i$ and $j \in \Gamma_i$

“NICE” GRAPHS

- symmetric formation potential makes no assumption on relation between the graph $\mathcal{G}$ and the point group $\tau(\Gamma)$
- we restrict our study to graphs where communication required by symmetric potential use same edges as $\mathcal{G}$

Assumption 1

For each $i \in \mathcal{V}_0$ and $j \in \Gamma_i \setminus \{i\}$, the edge ij is in $\mathcal{E}$.



- $\Gamma = \{\mathrm{Id}, \psi_4\} \subseteq \mathrm{Aut}(\mathcal{G})$
- $\Gamma_1 = \Gamma_2 = \{1, 2\},\ \Gamma_3 = \Gamma_4 = \{3, 4\}$
- $\mathcal{V}_0 = \{1, 4\}$
- isometry $\tau(\gamma) : (a, b) \mapsto (-a, b)$

satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$ and for each $i \in \mathcal{V}_0$ and $j \in \Gamma_i \setminus \{i\}$, the edge ij is in $\mathcal{E}$

Theorem

Consider a team of n integrator agents interacting over a Γ-symmetric graph $\mathcal{G}$ satisfying Assumption 1 that can be drawn with maximum point group symmetry $\mathcal{S}$ in $\mathbb{R}^d$, and let

$$\mathcal{F}_f = \{p \in \mathbb{R}^{dn} \,|\, \|p_i - p_j\| = d^\star_{ij}\; ij \in \mathcal{E}\}, \text{ and } \mathcal{F}_s = \{p \in \mathbb{R}^{dn} \,|\, \tau(\gamma)(p_i) = p_{\gamma(i)}\; \forall \gamma \in \Gamma,\; i \in \mathcal{V}\}.$$

Then for initial conditions $p_i(0)$ satisfying

$$\sum_{ij \in \mathcal{E}} (\|p_i(0) - p_j(0)\| - d^\star_{ij})^2 \leq \epsilon_1, \text{ and } \|p_i(0) - \tau(\gamma_j)p_j(0)\|^2 \leq \epsilon_2$$

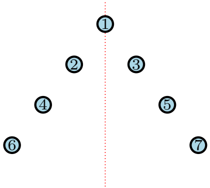
for all $i \in \mathcal{V}_0$ and $j \in \Gamma_i$, for a sufficiently small and positive constant ϵ_1 and ϵ_2, the control

$$u = -\nabla F(p(t)),$$

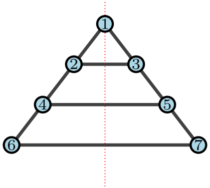
renders the set $\mathcal{F}_f \cap \mathcal{F}_s$ exponentially stable, i.e.

$$\lim_{t \to \infty} \|p_i(t) - p_j(t)\| = d^\star_{ij} \text{ and } \lim_{t \to \infty} \tau(\gamma)(p_i(t)) = \lim_{t \to \infty} p_{\gamma(i)}(t) \quad \text{for all } \gamma \in \Gamma, i \in \mathcal{V}.$$

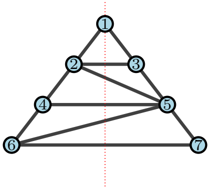
- formation flight for aircraft originated in WWI
- Vic formation used by pilots to improve visual communication and defensive advantages



Vic formation with symmetry mirror

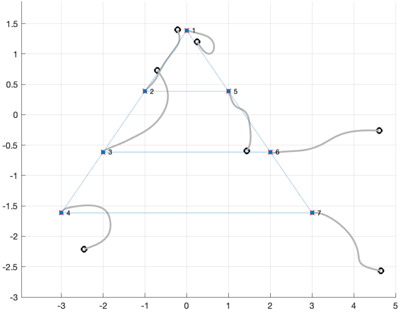


Flexible framework (9 edges; satisfies Assumption 1)

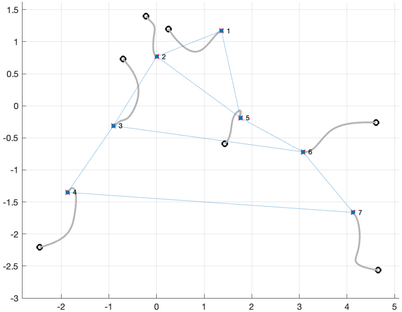


Minimally Rigid framework (11 edges)

EXAMPLE: THE VIC FORMATION



- symmetry constraints force agents to correct formation
- requires less agent communication than standard formation control with MIR requirement



- with flexible framework and only formation potential can not guarantee convergence to correct shape

CONCLUDING REMARKS

Summary

- exploit notions of symmetry in formation control
- $\tau(\Gamma)$-symmetric graphs captures symmetry of configurations and graphs
- symmetric formation potential used to design distributed control law with less edges compared to "traditional" formation control strategies

Future Work

- formation maneuvering requires time-varying point group symmetries
- relax requirement of edges between all nodes in vertex orbit
- richer trajectories including symmetry-preserving flexes

Questions?