Fekete Points, Formation Control and the Balancing Problem

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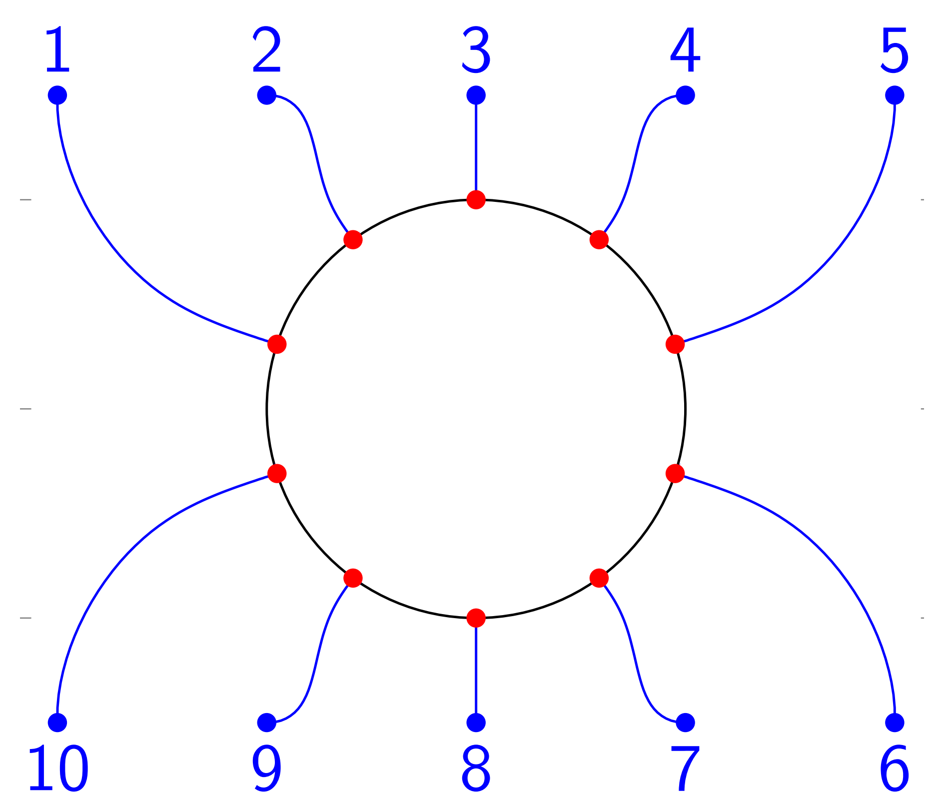

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Symposium on Control Theory & Power Engineering

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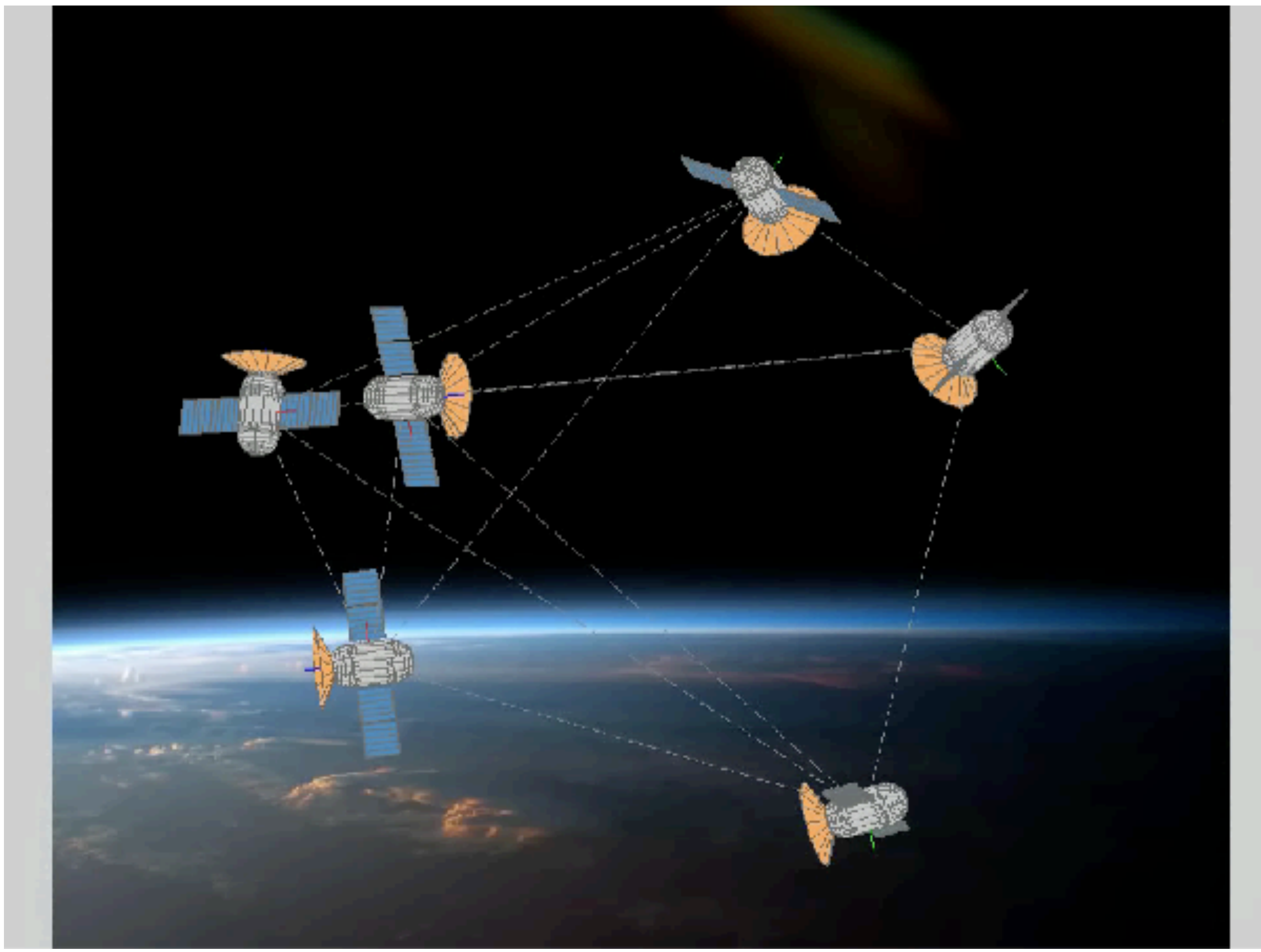
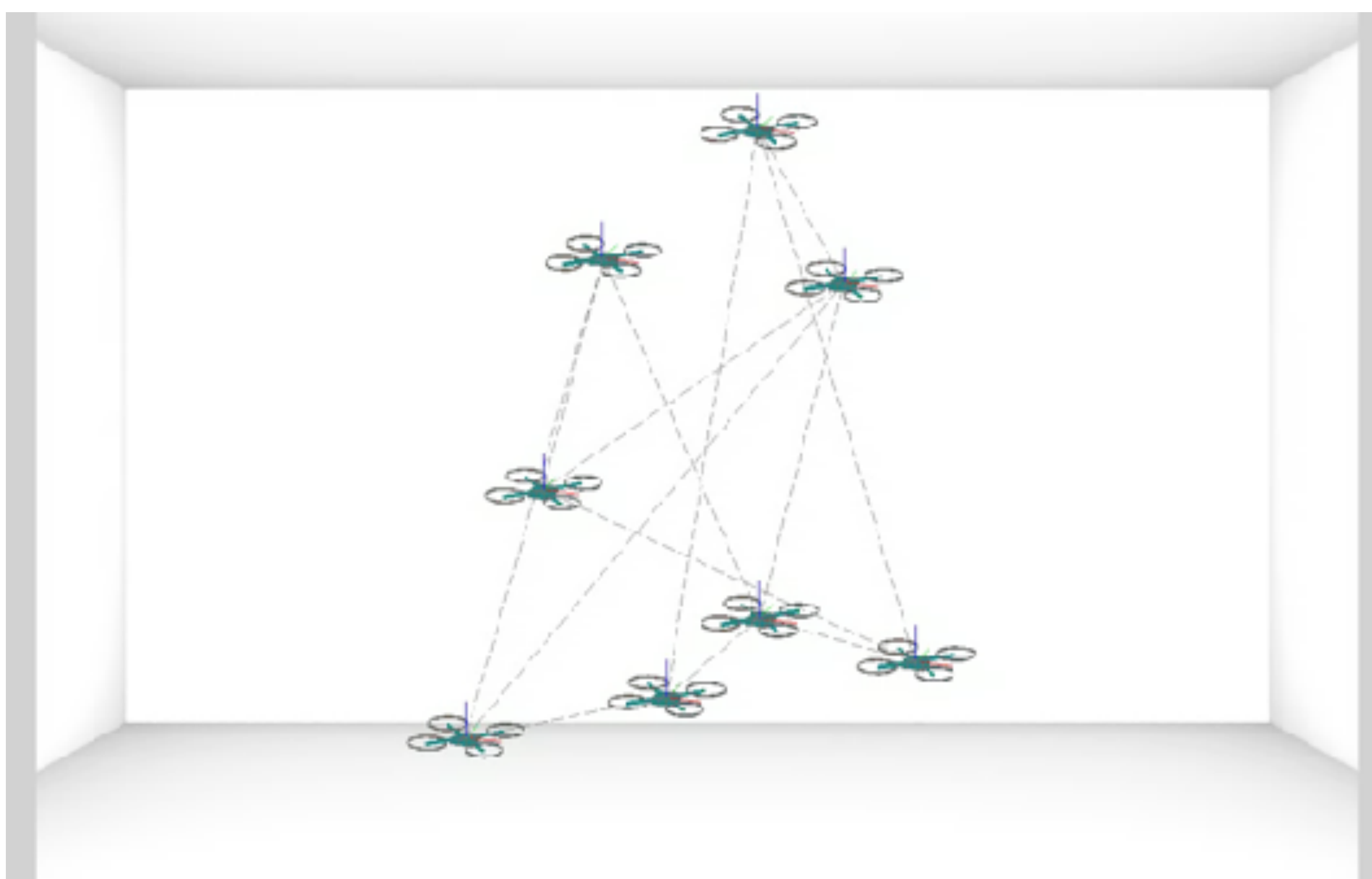
Formation Control is one of the canonical problems in multi-agent and multi-robot coordination

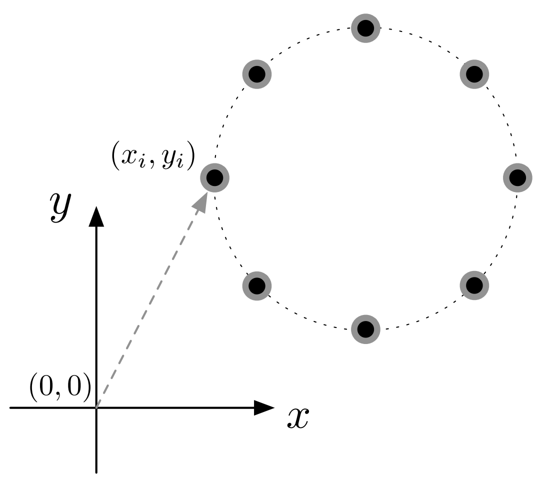


The Formation Control Problem

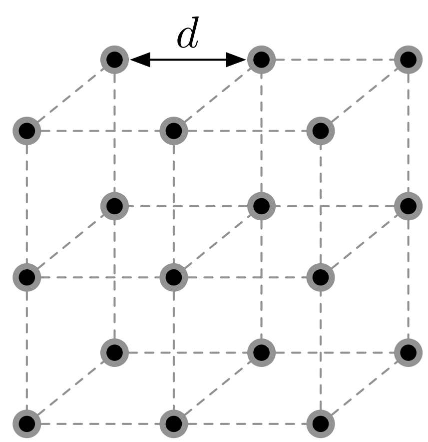
Given a team of robots endowed with the ability to sense relative state information to neighboring robots, design a control for each robot using only *local information* that asymptotically stabilizes the team to a desired formation shape.



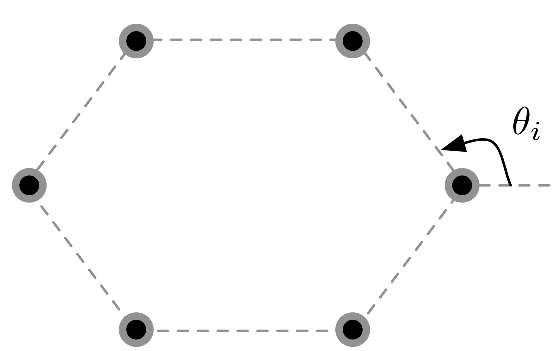
The Formation Control Problem



Formation specified in global coordinates



Formation specified by inter-agent distances

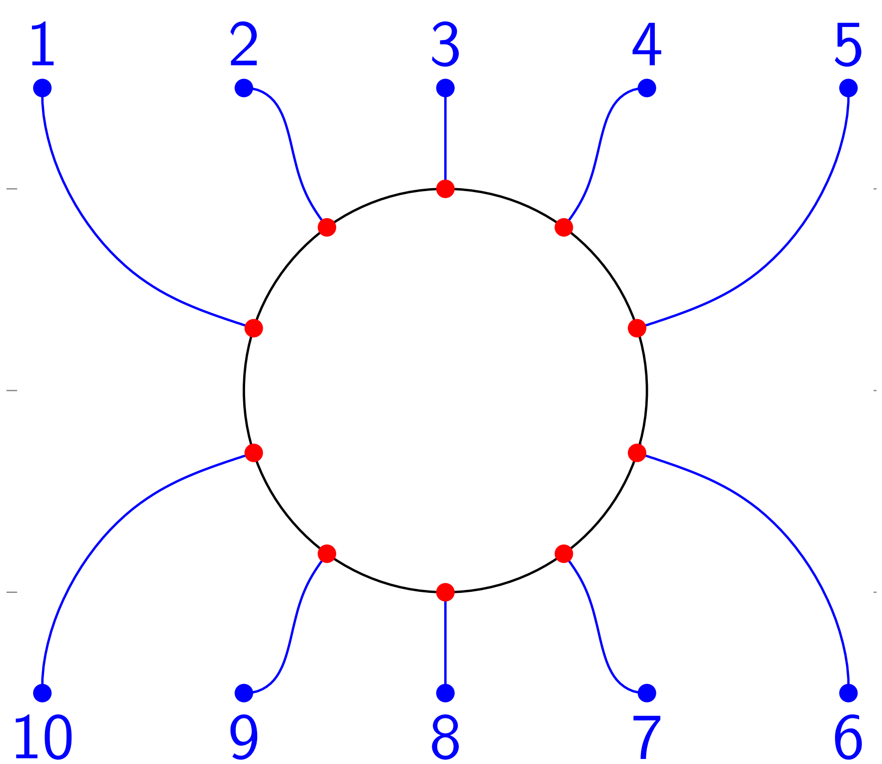


Formation specified by inter-agent bearings

Rigidity Theory

a combinatorial theory for characterizing the "stiffness" or "flexibility" of structures formed by rigid bodies connected by flexible linkages or hinges.

formation control…another approach



formation shape is specified by a compactly embedded submanifold of the ambient Euclidean space

$$M \subset \mathbb{R}^d$$

design a **decentralized** control that drives each agent to the desired submanifold, and a **distributed** control that arranges their configuration on the submanifold in a ***balanced*** fashion

$$\begin{array}{ll} \max & \sum_{j>i} d(x_i, x_j)^2 \\ s.t. & x_i \in M \end{array}$$

formation control…another approach

an example…

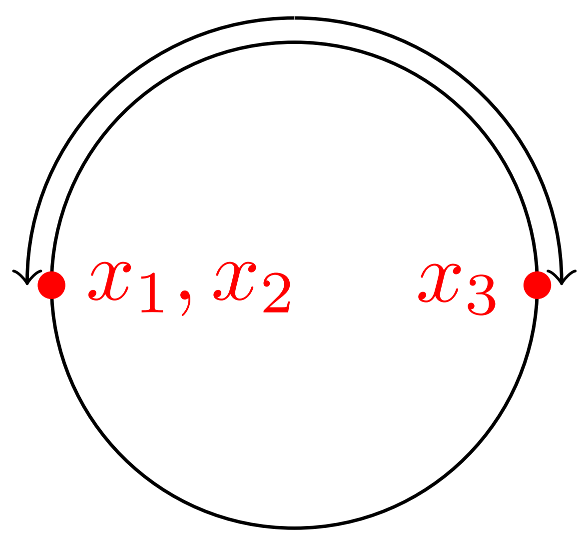
$M \subset \mathbb{R}^2$ is unit circle in the plane

$n = 3$ agents

$$\max \sum_{j>i} d(x_i, x_j)^2$$
$$s.t. \quad x_i \in M$$

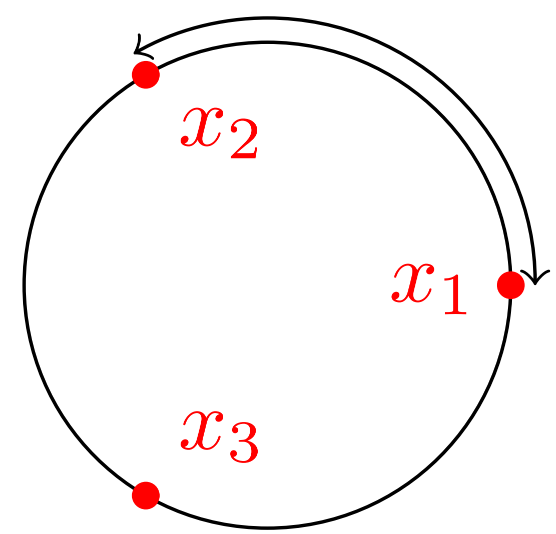
$$d\left(x_1, x_3\right) = \pi$$

$$x_1 = x_2$$
$$x_3 = -x_1$$



$$\sum_{j>i} d(x_i, x_j)^2 = 2\pi^2$$

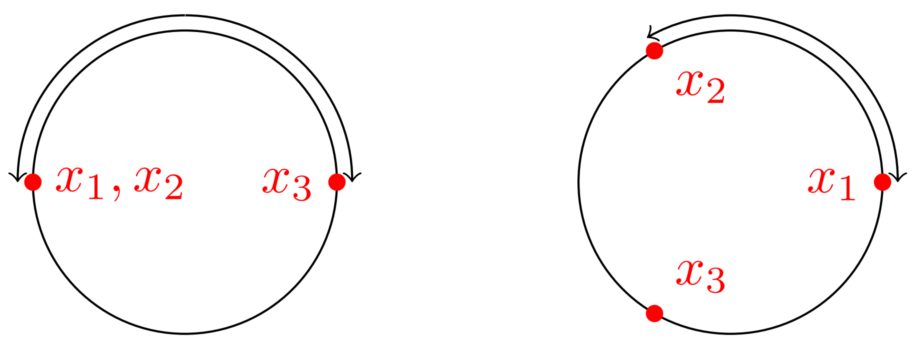
$$d\left(x_1, x_2\right) = 2\pi/3$$



$$x_2 = R\left(\frac{2\pi}{3}\right) x_1$$
$$x_3 = R\left(\frac{2\pi}{3}\right) x_2$$

$$\sum_{j>i} d(x_i, x_j)^2 = 4\pi^2/3$$

formation control…another approach



$$\begin{aligned} \max \quad & \sum_{j>i} d(x_i, x_j)^2 \\ s.t. \quad & x_i \in M \end{aligned}$$

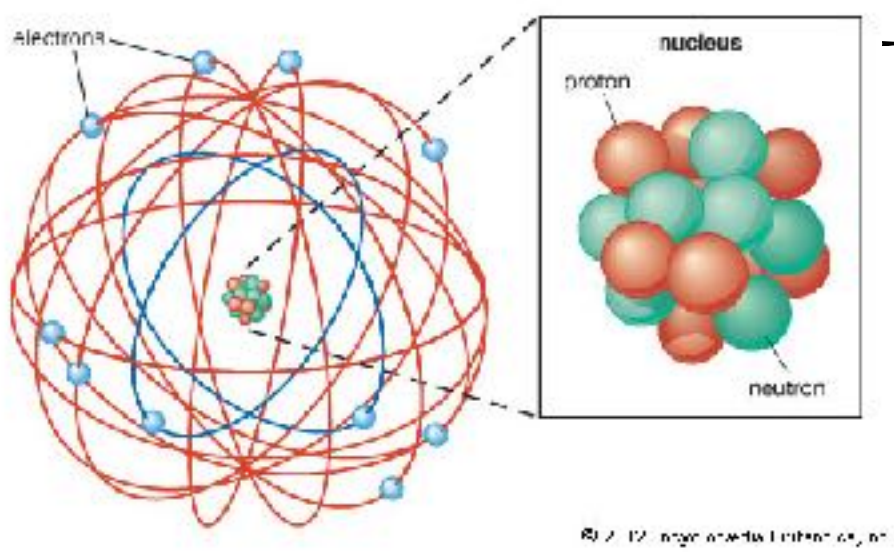
a modification…

chose cost function that is "small"
when agents are close to each other

$$\sum_{j>i} \ln d(x_i, x_j) = \ln \left(\prod_{j>i} d(x_i, x_j) \right)$$

formation control and Fekete points

$$\sum_{j>i} \ln d(x_i, x_j) = \ln \left(\prod_{j>i} d(x_i, x_j) \right)$$



Thomson Atomic Model
(1904)

Föppl
(1912)

Stabile Anordnungen von Elektronen im Atom

$$V_n = \prod_{1 \leq i < j \leq n} (x_j - x_i)^2$$

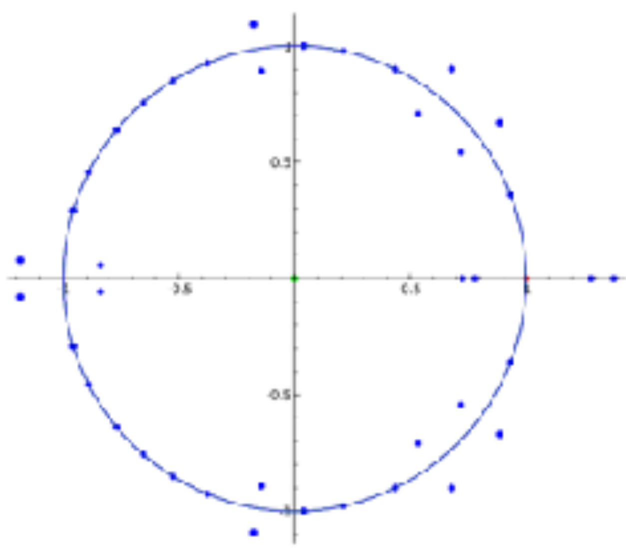
Schur
(1918)

Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten

Vandermode polynomial

formation control and Fekete points

$$\sum_{j>i} \ln d(x_i, x_j) = \ln \left(\prod_{j>i} d(x_i, x_j) \right)$$



roots of Fekete polynomial

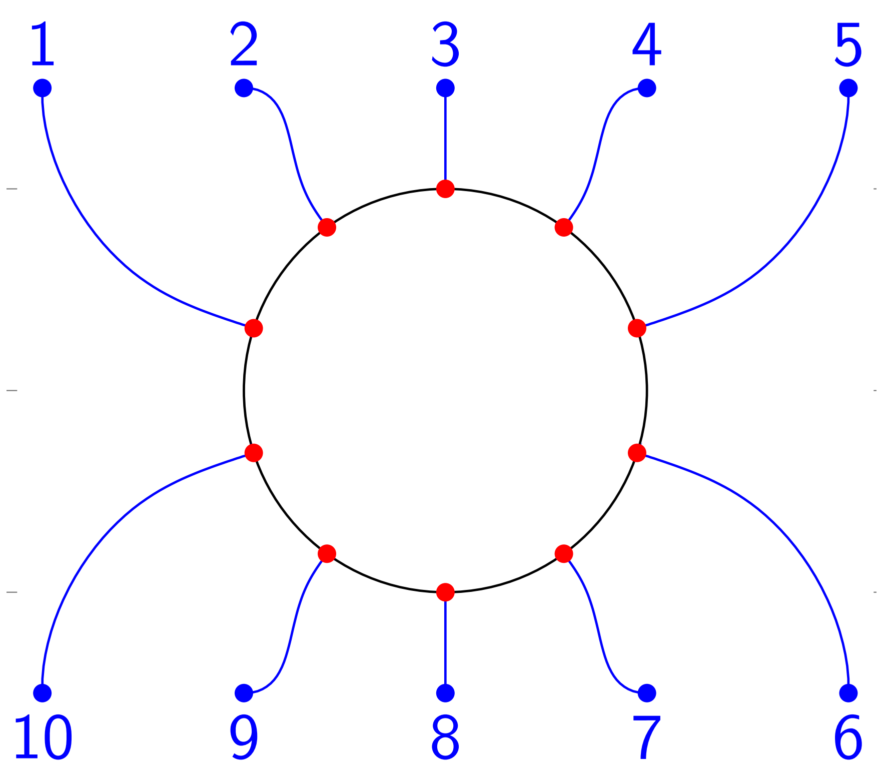
STEVE SMALE

Mathematical Problems for the Next Century[1]

Fekete (1923) — *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*

Smale (1998) — ***Problem 7:*** *Distribution of Points on the 2-Sphere (Fekete points)*

formation control…another approach



Fekete Points

formation shape is specified by a compactly embedded submanifold of the ambient Euclidean space

$$M \subset \mathbb{R}^d$$

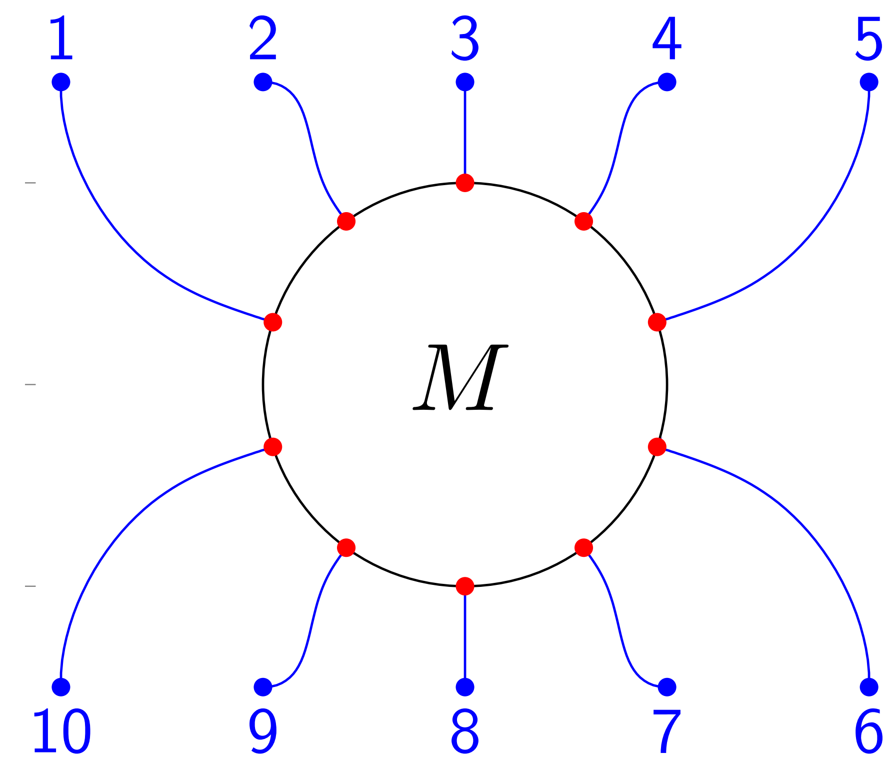
design a **decentralized** control that drives each agent to the desired submanifold, and a **distributed** control that arranges their configuration on the submanifold in a ***balanced*** fashion

$$\begin{aligned} \max \quad & \ln \left(\prod_{j>i} d(x_i, x_j) \right) \\ s.t. \quad & x_i \in M \end{aligned}$$

asymptotic stability of Fekete points

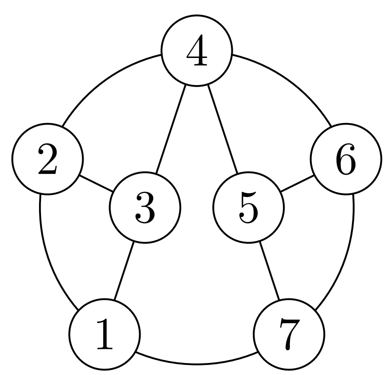
$$\phi : M \to \mathbb{R}$$

$$\phi(x) = \sum_{j>i} W_{ij} \ln(d(x_i, x_j))$$



$x_i \in \mathbb{R}^d$

"information exchange" network



$$W_{ij} = \begin{cases} w_{ij}, & i \sim j \\ 0, & o.w. \end{cases}$$

$$r : \mathbb{R}^d \to M$$

smooth retraction onto the submanifold

asymptotic stability of Fekete points

Technion Israel Institute of Technology

Theorem

The solutions of

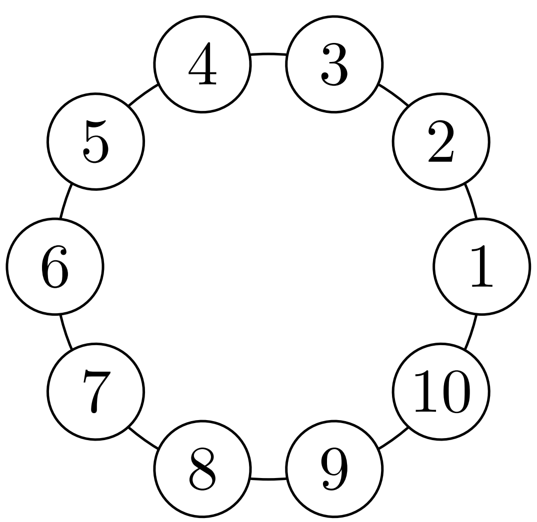
$$\dot{x} = (r(x) - x) + \operatorname{grad} \phi(r(x))$$

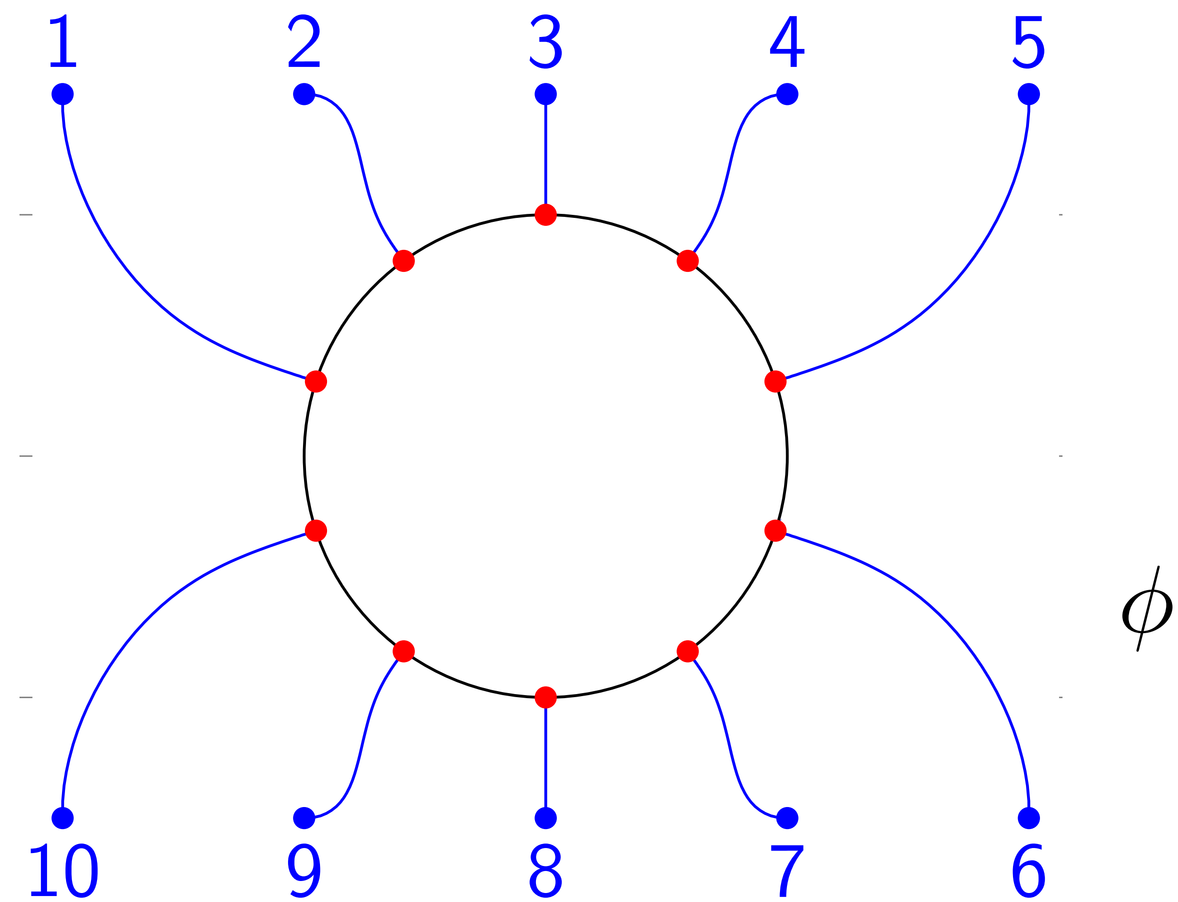
asymptotically approach the maximizers of ϕ in a stable fashion.

$r(x) - x$	a *decentralized control* that asymptotically stabilizes our formation shape
$\operatorname{grad} \phi(r(x))$	a *distributed control* that stabilizes the maximizers of potential function

an example - the unit circle

"information exchange" network



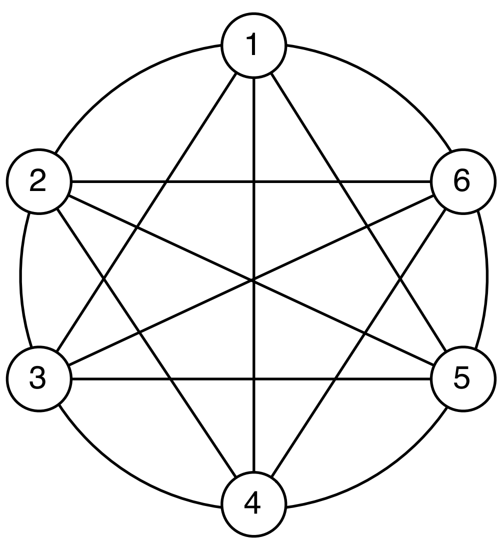


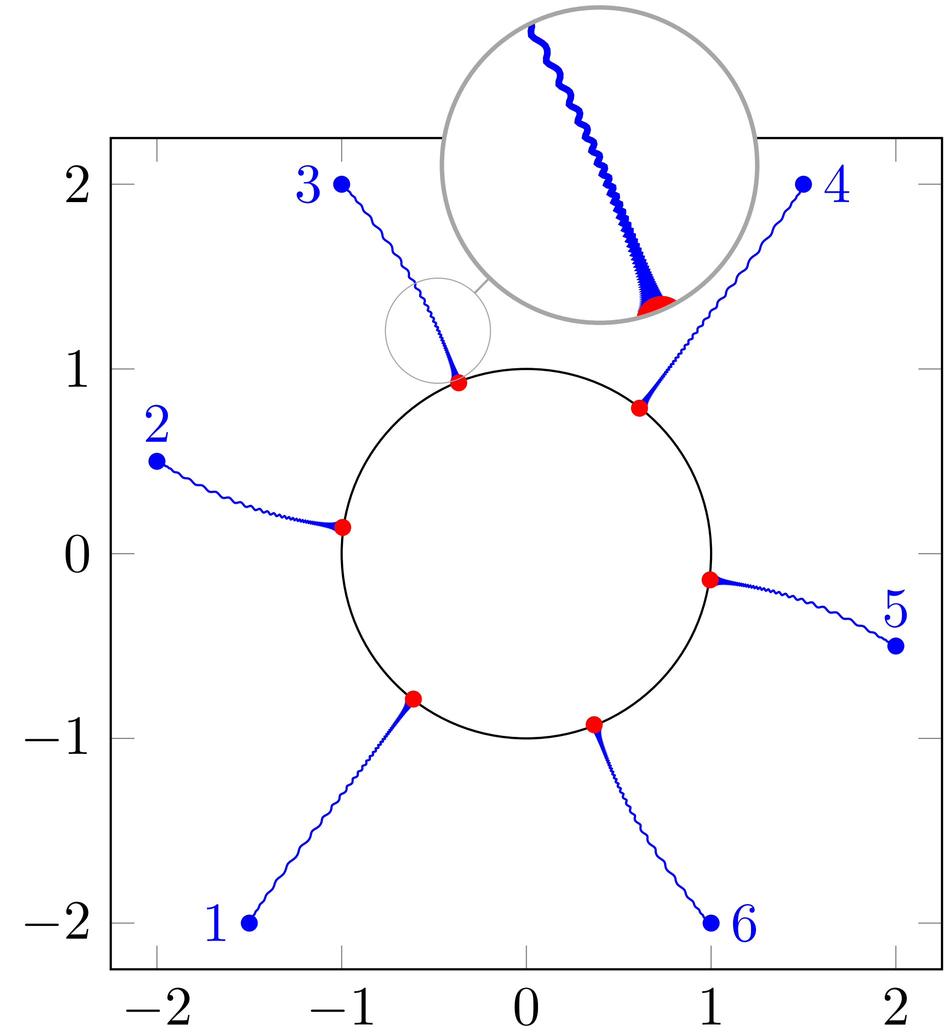
$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right)x_i + \sum_{j=1}^{n}\frac{W_{ij}}{\|x_i\|}\left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i\cdot x_j & x_i\cdot\Omega x_j \\ x_j\cdot\Omega x_i & x_i\cdot x_j\end{bmatrix}\right)\right)^{-1} x_i$$

$$\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

an example - the unit circle

"information exchange" network





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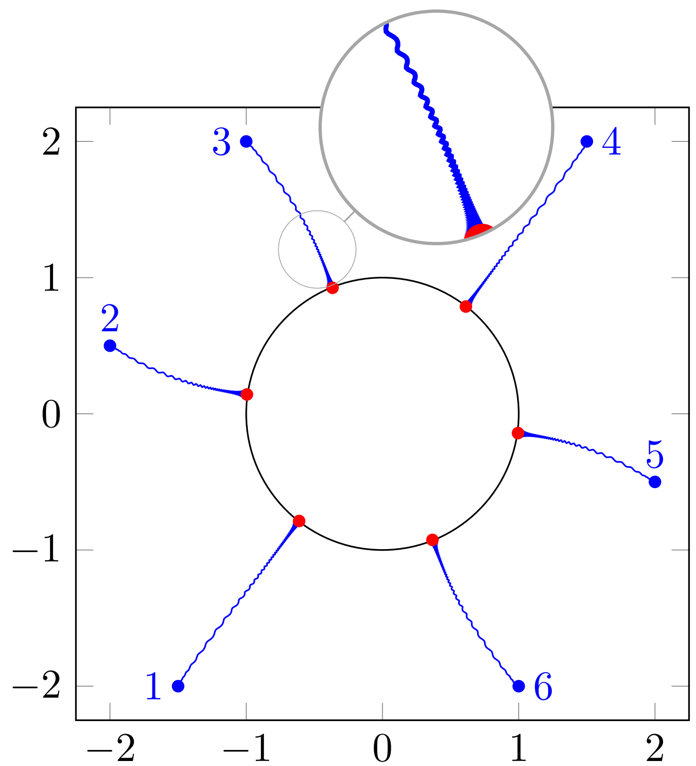

does evenly spaced configuration correspond to equilibrium?

an example - the unit circle

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right) x_i + \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|} \left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} x_i$$

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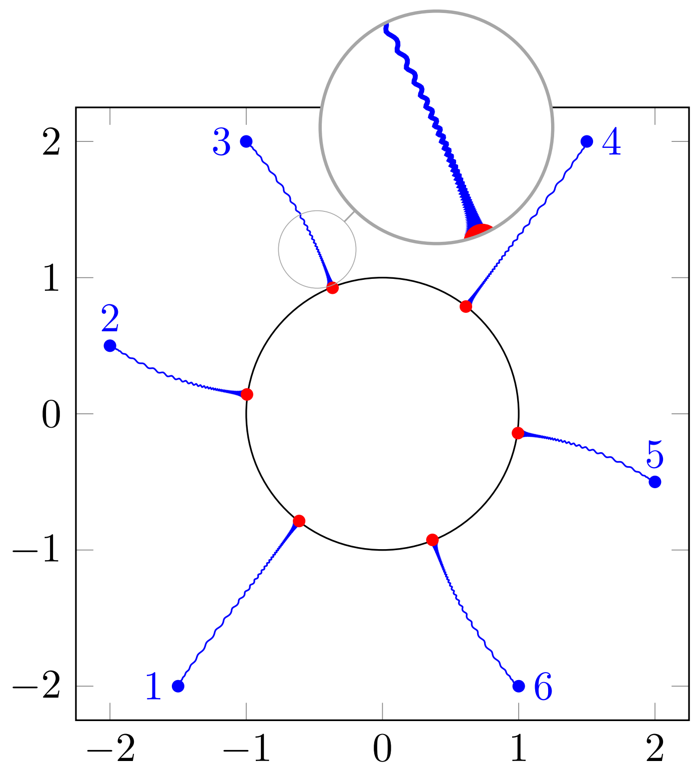

does evenly spaced configuration correspond to equilibrium?

directed angles: $\alpha_{ij}\Omega = \log\left(\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)$

angles between red points

$$\alpha_{12} = -\frac{2\pi}{6},\ \alpha_{13} = -\frac{2\pi}{3},\ \alpha_{14} = \pm\pi,$$
$$\alpha_{15} = \frac{2\pi}{3},\ \alpha_{16} = \frac{2\pi}{6}$$

sum of reciprocals $\frac{1}{\alpha_{12}} + \frac{1}{\alpha_{13}} + \frac{1}{\alpha_{14}} + \frac{1}{\alpha_{15}} + \frac{1}{\alpha_{16}} \neq 0$

an example - the unit circle

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right)x_i + \sum_{j=1}^{n}\frac{W_{ij}}{\|x_i\|}\left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i\cdot x_j & x_i\cdot\Omega x_j \\ x_j\cdot\Omega x_i & x_i\cdot x_j\end{bmatrix}\right)\right)^{-1} x_i$$

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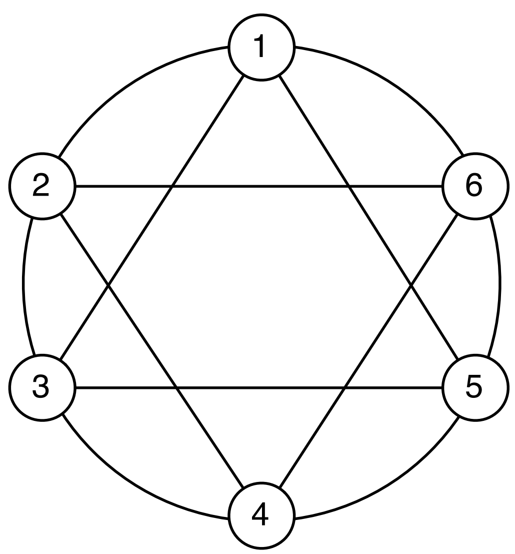

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sum of reciprocals $\frac{1}{\alpha_{12}} + \frac{1}{\alpha_{13}} + \frac{1}{\alpha_{15}} + \frac{1}{\alpha_{16}} = 0$

graph-theoretic characterization of equilibria

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right) x_i + \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|} \left(\log\left(\frac{1}{\|x_i\|\|x_j\|} \begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} x_i$$

equilibrium must satisfy:

$$\begin{bmatrix} 0 & W_{12}/\alpha_{12} & \cdots & \cdots & W_{1n}/\alpha_{1n} \\ W_{21}/\alpha_{21} & 0 & W_{23}/\alpha_{23} & \cdots & W_{2n}/\alpha_{2n} \\ \vdots & W_{32}/\alpha_{32} & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \\ W_{n1}/\alpha_{n1} & W_{n2}/\alpha_{n2} & \cdots & & 0 \end{bmatrix} \mathbf{1} = 0$$

graph-theoretic characterization of equilibria

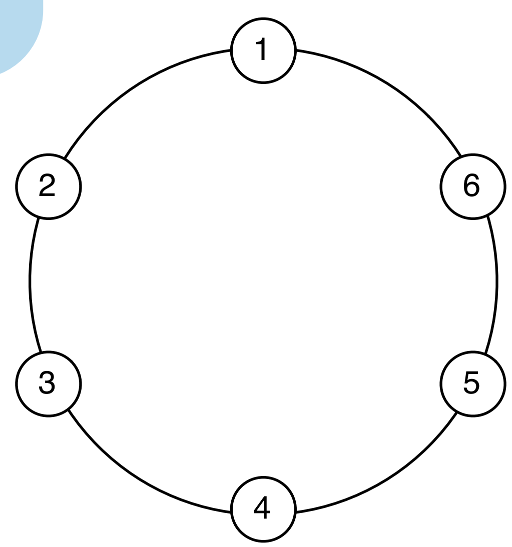
$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right)x_i + \sum_{j=1}^{n}\frac{W_{ij}}{\|x_i\|}\left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i\cdot x_j & x_i\cdot\Omega x_j \\ x_j\cdot\Omega x_i & x_i\cdot x_j\end{bmatrix}\right)\right)^{-1}x_i$$

equivalently…

$$E(\mathcal{G})\begin{bmatrix}\vdots \\ \frac{1}{\alpha_{ij}} \\ \vdots\end{bmatrix} = 0$$

$E(\mathcal{G})$ *incidence matrix* of a graph

null-space characterizes *cycles* in the graph



$$E(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1\end{bmatrix}$$

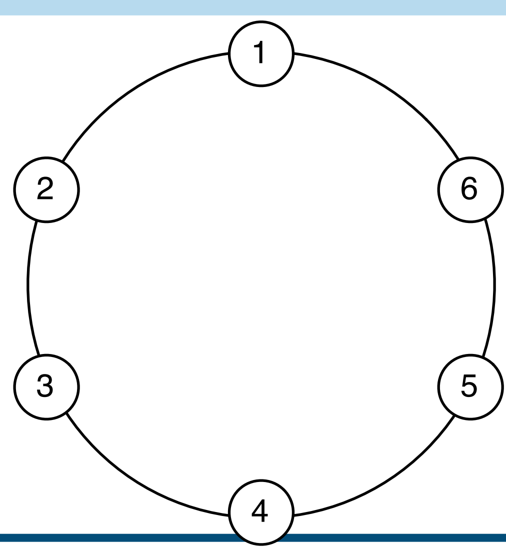
graph-theoretic characterization of equilibria

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right) x_i + \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|} \left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} x_i$$

equivalently...

$$E(\mathcal{G}) \begin{bmatrix} \vdots \\ \frac{1}{\alpha_{ij}} \\ \vdots \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha_{12}} \\ \frac{1}{\alpha_{23}} \\ \frac{1}{\alpha_{34}} \\ \frac{1}{\alpha_{45}} \\ \frac{1}{\alpha_{56}} \\ \frac{1}{\alpha_{61}} \end{bmatrix} = 0$$



one cycle - one equilibria!

$$\frac{1}{\alpha_{12}} = \frac{1}{\alpha_{23}} = \frac{1}{\alpha_{34}} = \frac{1}{\alpha_{45}} = \frac{1}{\alpha_{56}} = \frac{1}{\alpha_{61}}$$

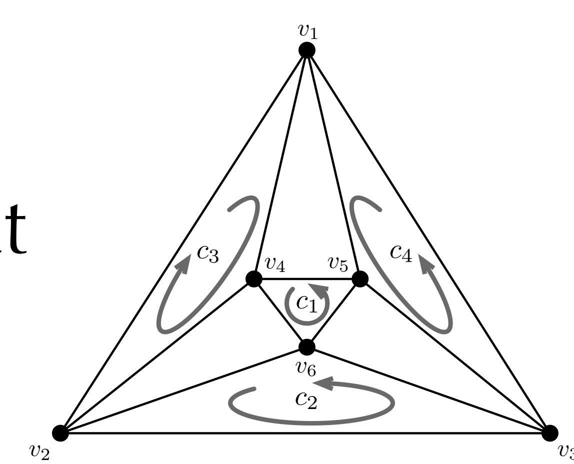
Corollary

The solutions of

$$\dot{x} = (r(x) - x) + \operatorname{grad} \phi(r(x))$$

for M the unit circle, asymptotically converges to a balanced formation if and only if the graph possesses an Eulerian cycle (iff every vertex has even degree)

An *Eulerian Cycle* is a walk on a graph beginning and ending at the same node that traverses each edge only once.



implementation

A Distributed Controller for Formation Balancing and Maneuvering of Multirotor UAVs

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TECHNION Israel Institute of Technology

Conclusions

- Fekete points leads to a novel approach for formation control
- decentralized and distributed implementation
- graph-theoretic interpretations
- extensions:
  - balancing on special Euclidean group
  - time-varying information exchange network
  - formation tracking

Acknowledgements

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University of Stuttgart

Yuyi Liu
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JM Montenbruck, D Zelazo, and F Allgöwer, "*Fekete points, formation control, and the balancing problem,*" IEEE Transactions on Automatic Control, 2016 (*to appear*).

JM Montenbruck, D Zelazo, and F Allgöwer, "*Retraction balancing and formation control,*" 53rd Conference on Decision and Control, Osaka, Japan, 2015.

Y Liu, JM Montenbruck, D Zelazo, F Allgöwer, "*A distributed controller for formation balancing and maneuvering of multirotor UAVs,*" International Conference on Robotics and Automation, Singapore, 2017 (*under review*).