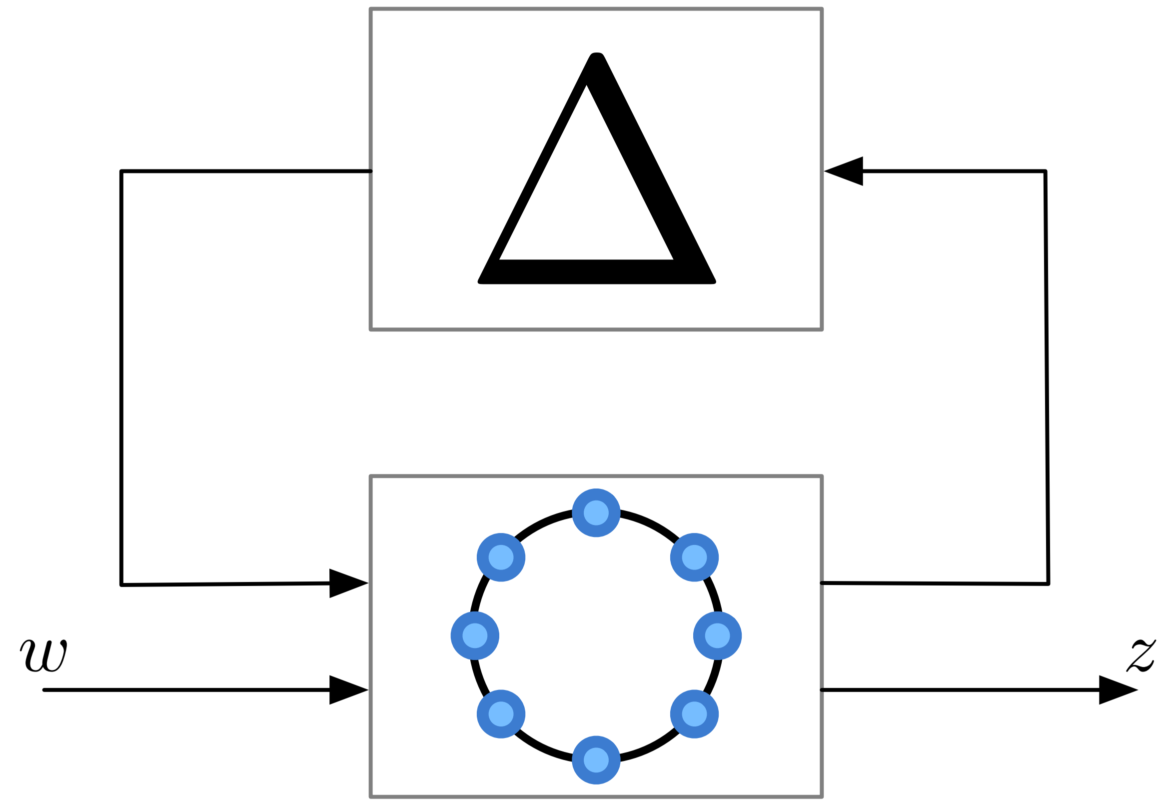
Robust Consensus of Higher Order Agents over Cycle Graphs

Dwaipayan Mukherjee
Daniel Zelazo
Faculty of Aerospace Engineering
Technion-Israel Institute of Technology



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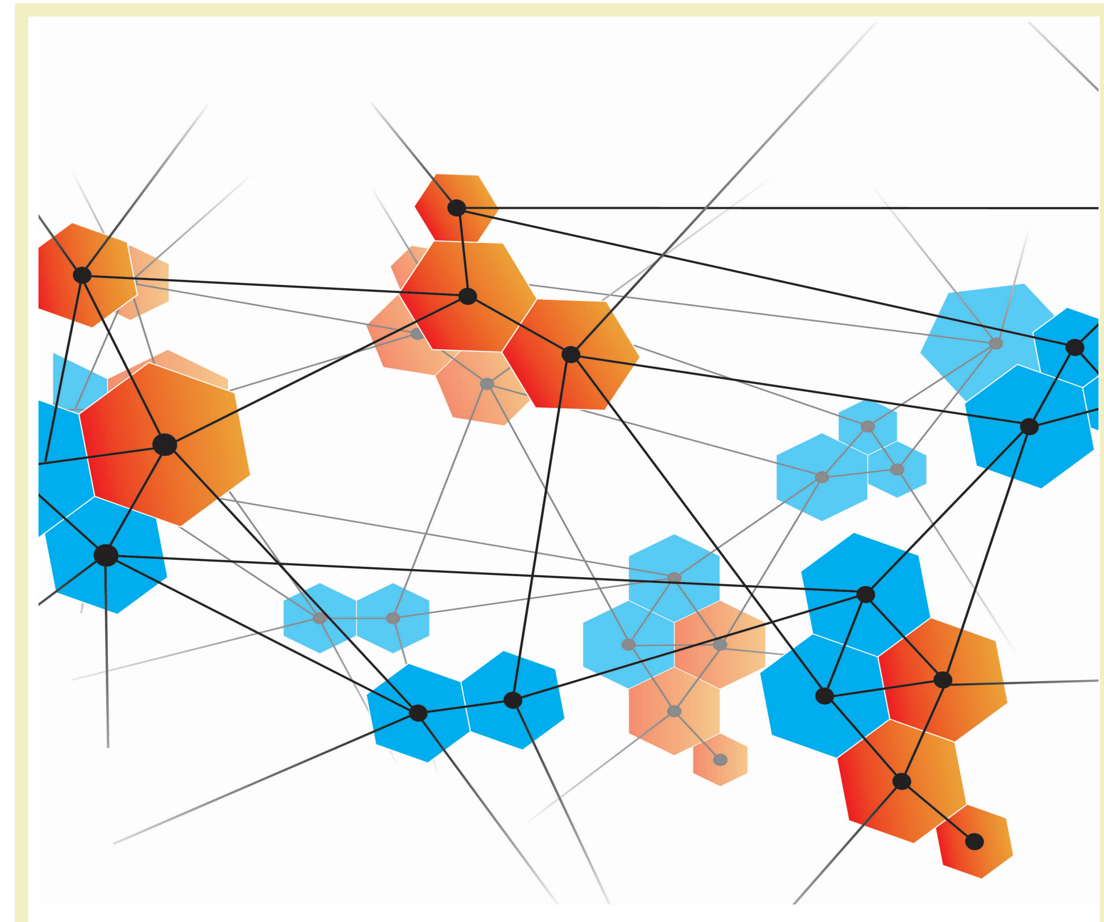
Networked Dynamic Systems

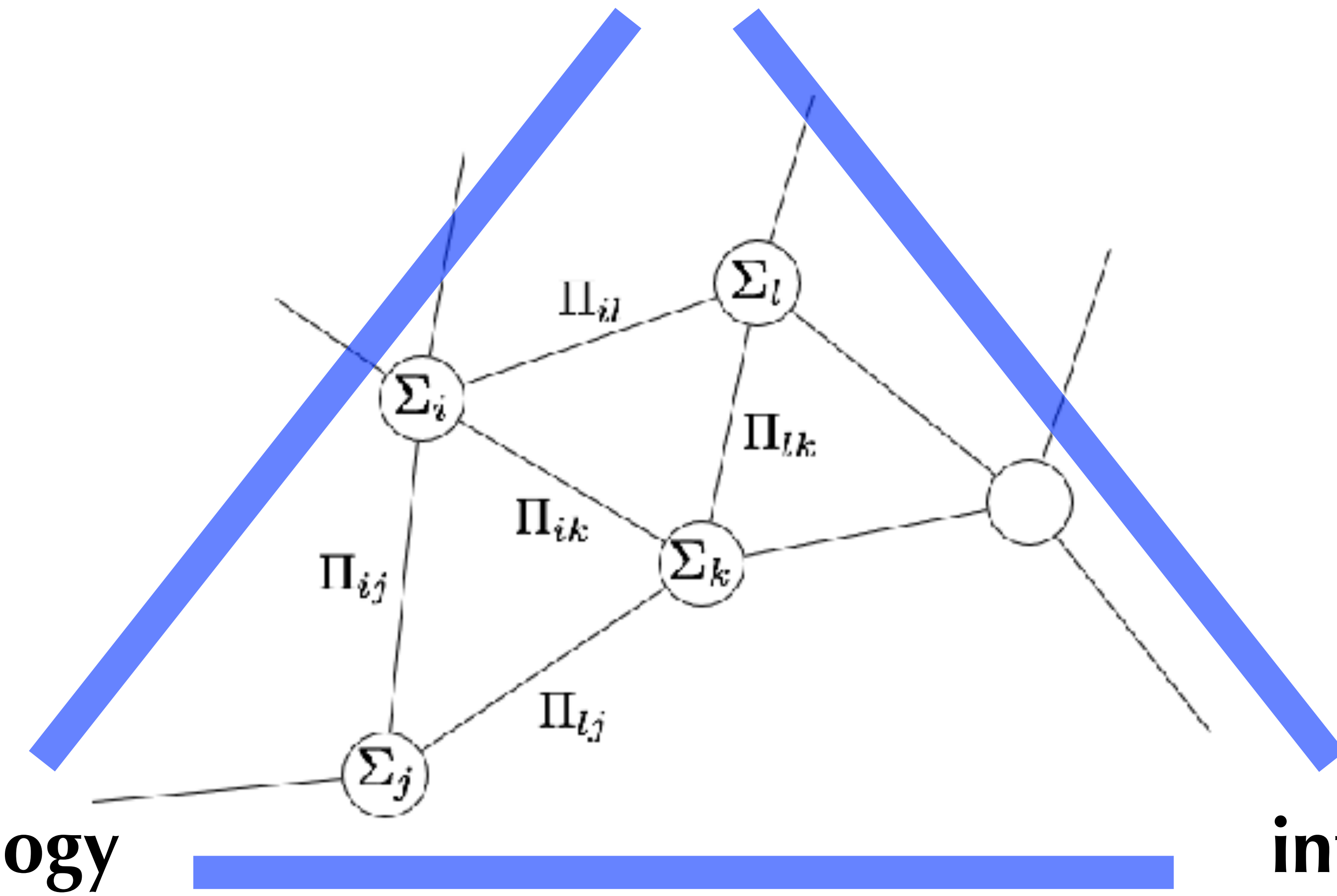


networks of dynamical systems are one of ***the*** enabling technologies of the future

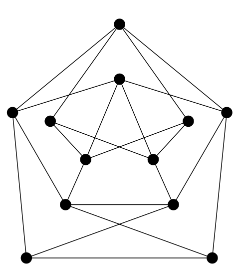
Networked Dynamic Systems

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t))$$

dynamics



topology (graph)

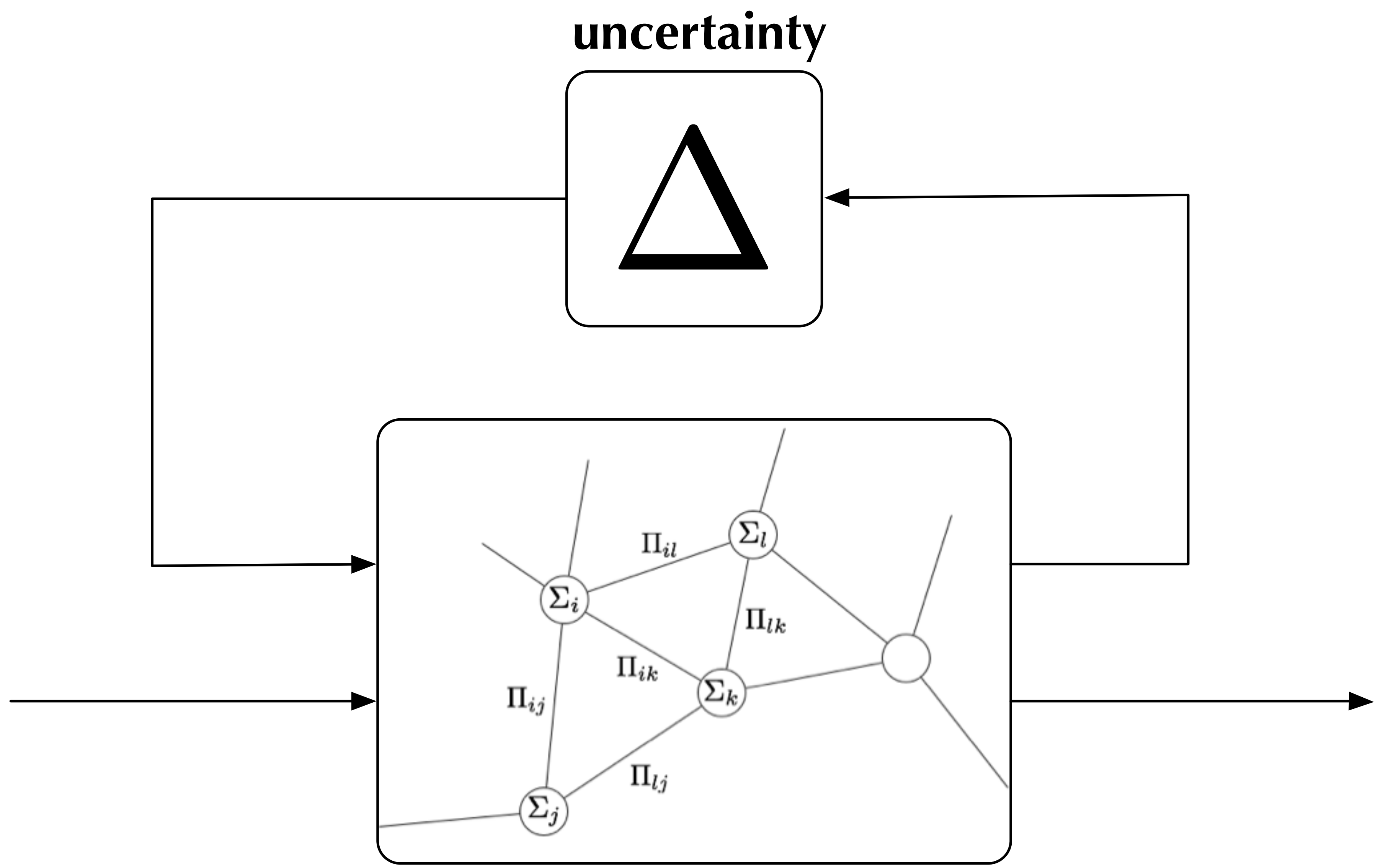


interaction protocol

$$u_i(t) = \Pi_i(x(t), \mathcal{G})$$

Networked Dynamic Systems

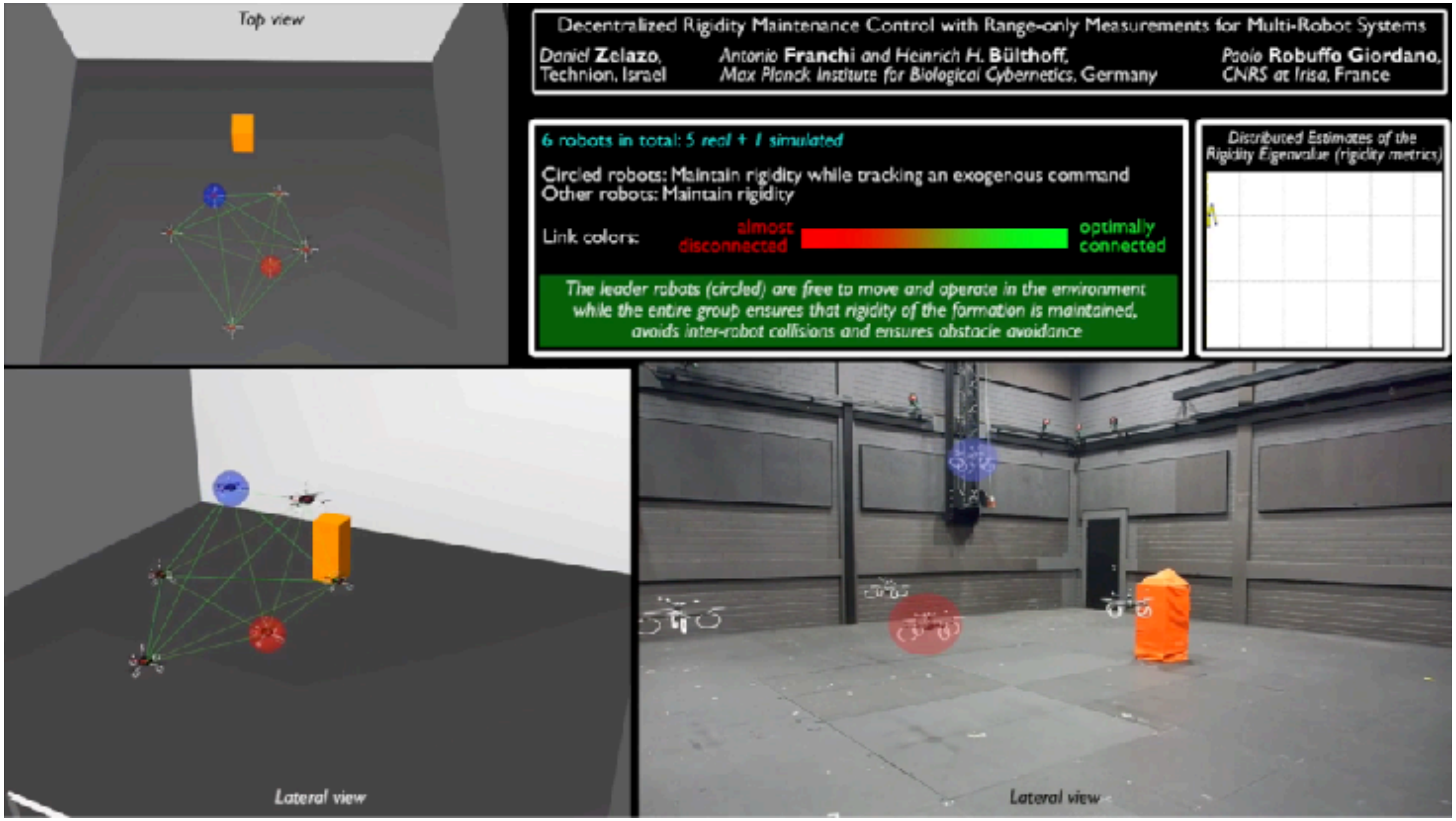
What about robustness?



what is the right way to approach *robustness* of networked dynamic systems?

Linear Consensus

The linear consensus protocol is a *distributed and dynamic protocol* used for computing the average of a set of numbers.

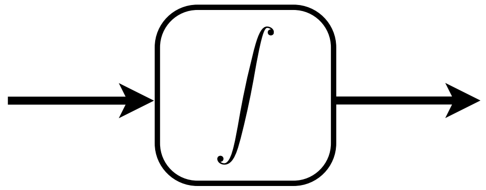


The Consensus Protocol

the classic model…

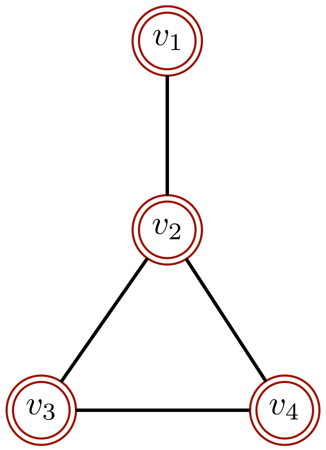
Agent Dynamics

$$\dot{x}_i(t) = u_i(t)$$



Integrator Dynamics

Information Exchange Network



$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

$$\mathcal{W} : \mathcal{E} \to \mathbb{R}$$

Algebraic Representations

Incidence Matrix

- $E(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$
- $E(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Laplacian Matrix

- $L(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$
- $L(\mathcal{G}) = E(\mathcal{G}) W E(\mathcal{G})^T$
- $L(\mathcal{G})\mathbb{1} = 0$

The Consensus Protocol

Consensus Protocol

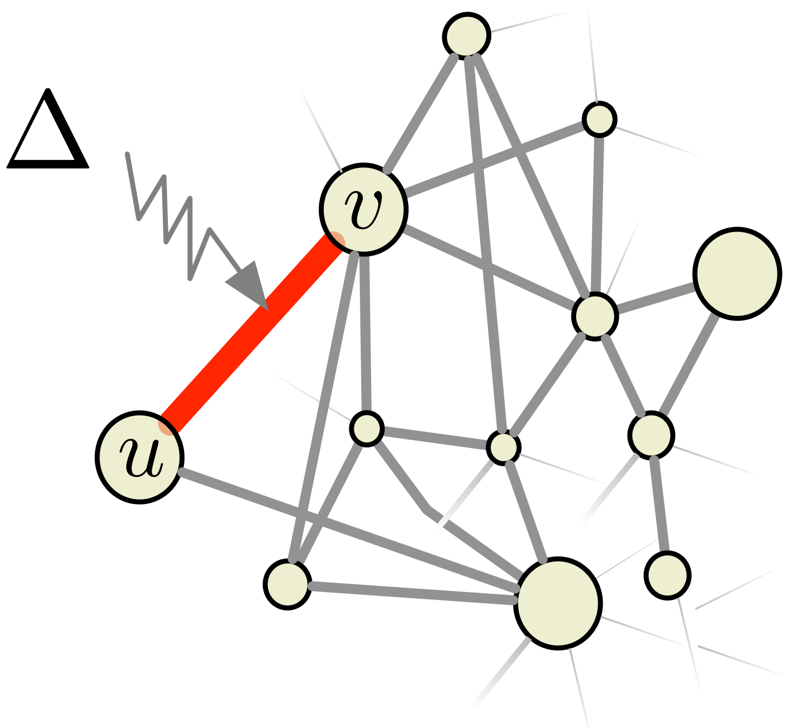
$$u_i(t) = \sum_{i \sim j} w_{ij}(x_j(t) - x_i(t))$$

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

Theorem *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ *be a weighted and connected graph with positive edge weights* $\mathcal{W}(k) > 0$ *for* $k = 1, \ldots, |\mathcal{E}|$*. Then the consensus dynamics synchronizes; i.e.,* $\lim_{t \to \infty} x_i(t) = \beta$ *for* $i = 1, \ldots, |\mathcal{V}|$.

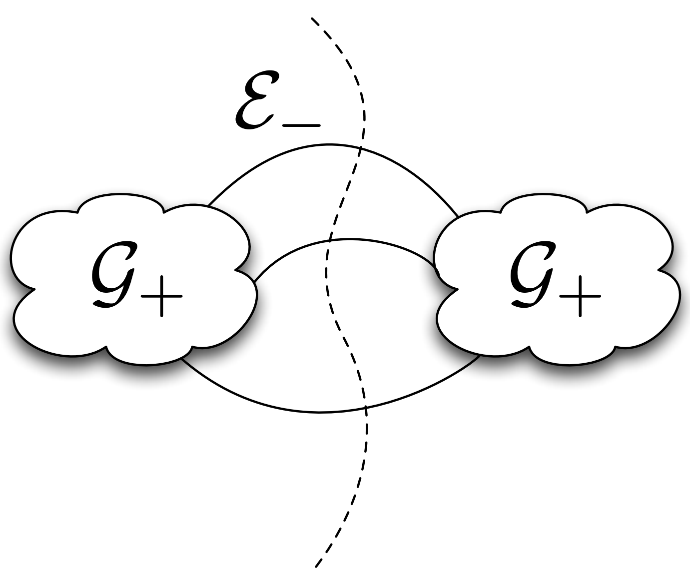
[Mesbahi & Egerstedt, Olfati-Saber, Ren]

Uncertain Consensus Protocol

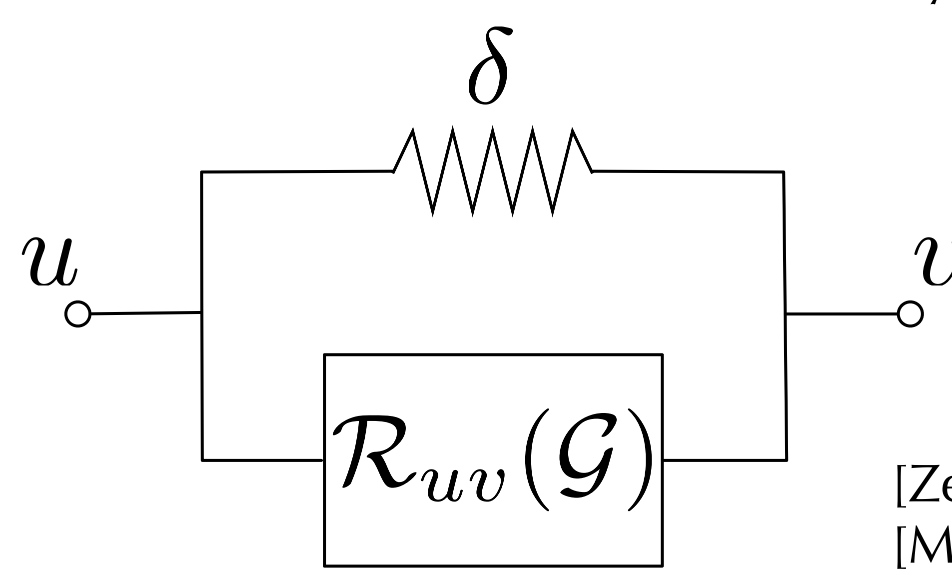


linear consensus with
uncertainties in the edge weights

Graph Cuts and
Combinatorial Interpretations

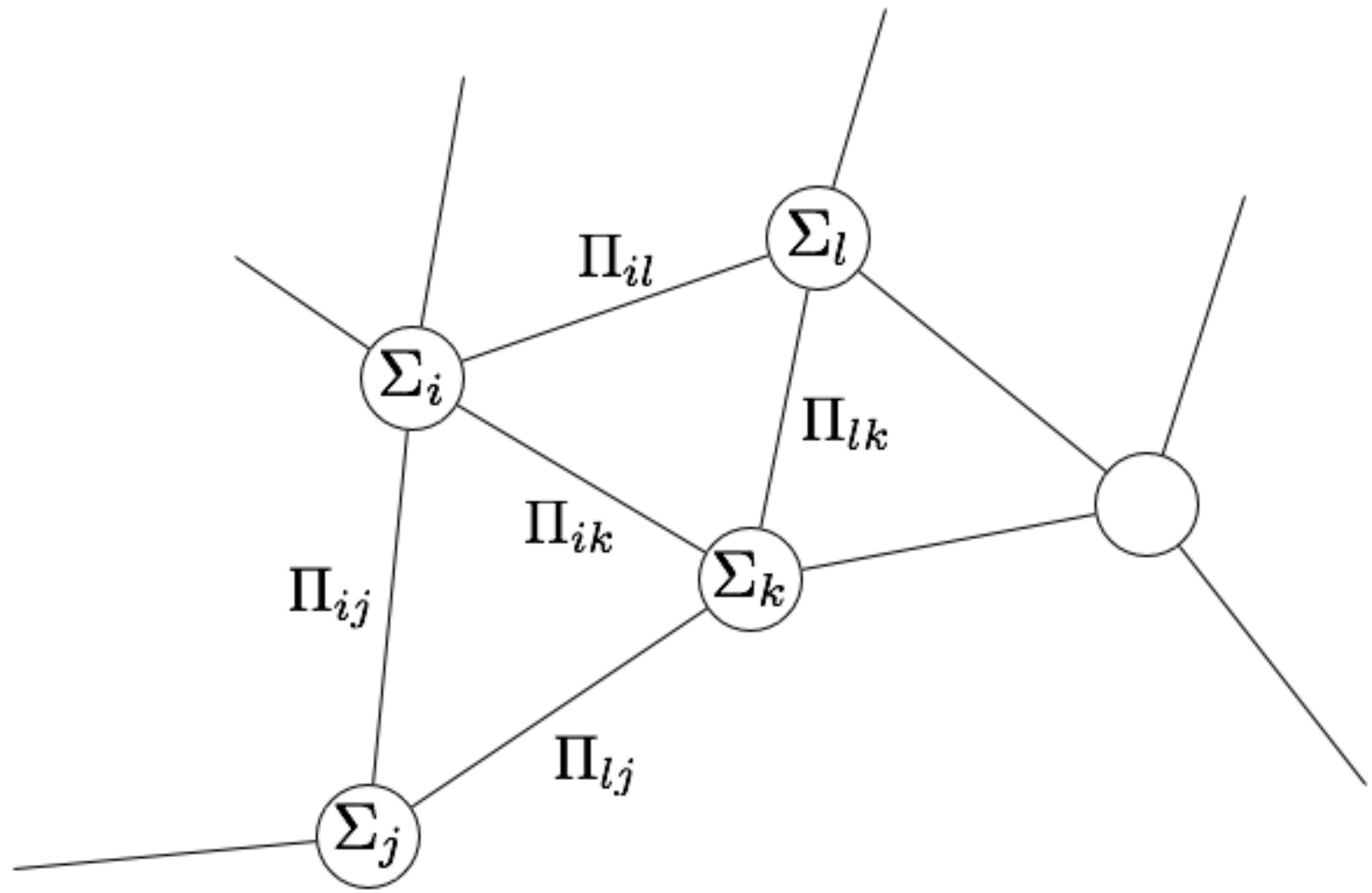


Electrical Network Interpretations:
Effective Resistance and Stability



[Zelazo, Bürger ‘14]
[Mukherjee, Zelazo ’16]

Networked Dynamic Systems



what about more general dynamics?

Interval Plants and Agent Uncertainty

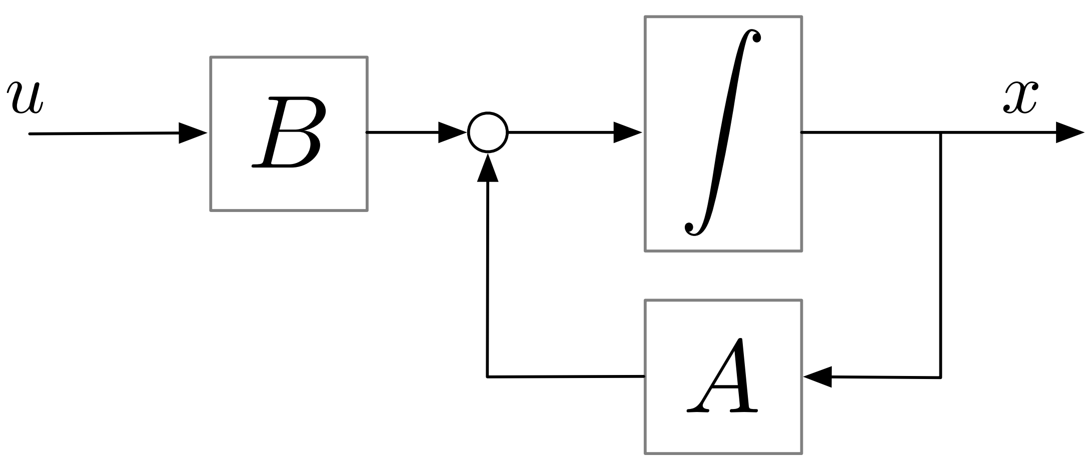
Agent Dynamics

$$x_i^{(m)} + \alpha_{m-1} x_i^{(m-1)} + \cdots + \alpha_0 x_i = u_i$$

Linear, m*th* order

$$\alpha_j \in [\underline{\alpha}_j, \overline{\alpha}_j] \subset \mathbb{R},\ j = 0, 1, \ldots, m-1$$

parameters belong to *interval*



$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

uncertainty of agent dynamics expressed by
interval polynomials describing the dynamics

Interval Plants and Agent Uncertainty

Kharitonov's Theorem

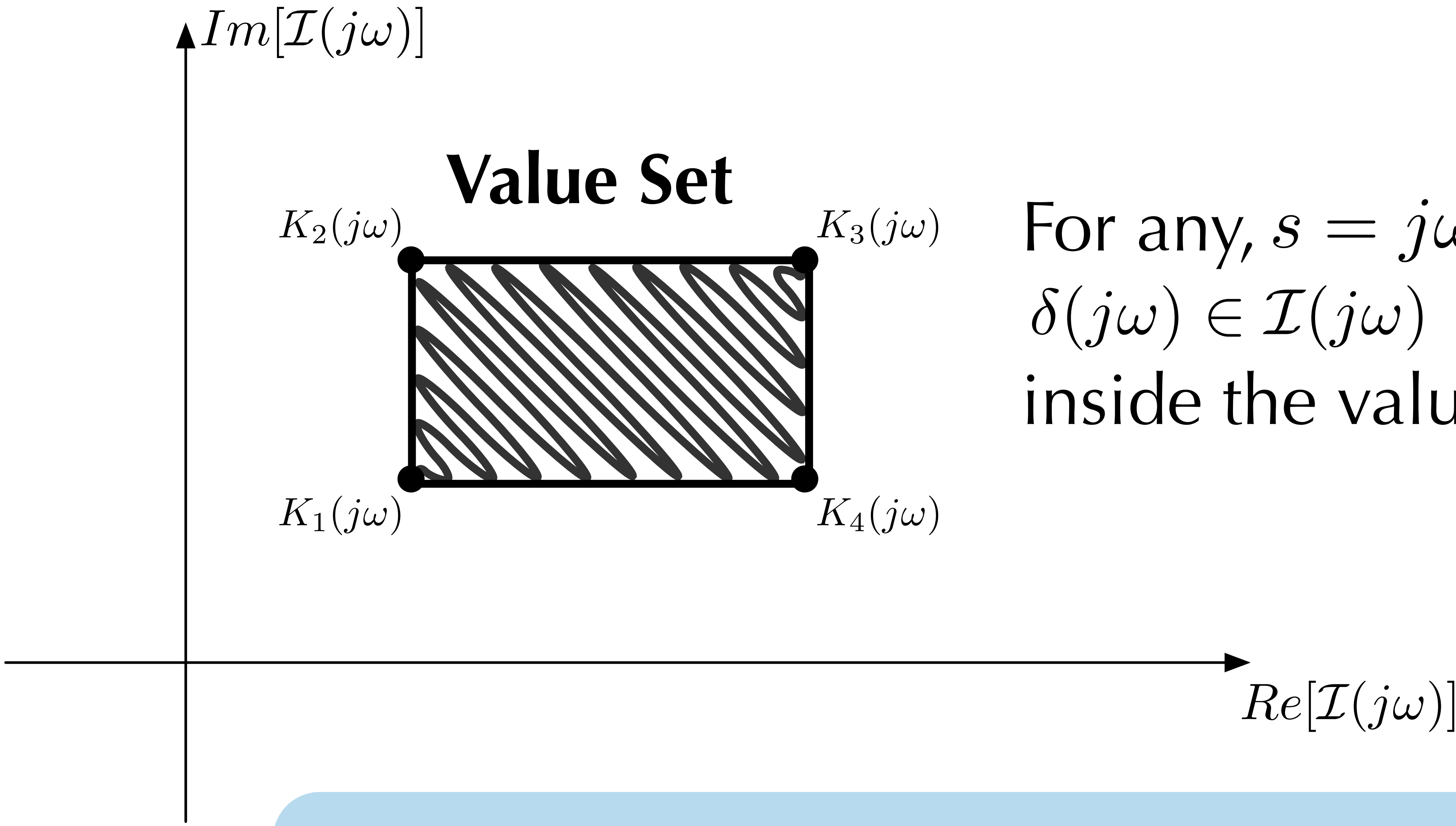
Theorem. *Suppose $\mathcal{I}(s)$ is a set of real polynomials of degree n given by*

$$\delta(s) = \delta_n s^n + \delta_{n-1} s^{n-1} + \cdots + \delta_1 s + \delta_0,$$

where the coefficients lie in the range $\delta_i \in [x_i, y_i]$, $i = 1, 2, \ldots, n$. Every polynomial in the family $\mathcal{I}(s)$ is Hurwitz if and only if the following four extreme polynomials are Hurwitz:

$$K_1(s) = x_0 + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4 + x_5 s^5 + y_6 s^6 + \cdots$$
$$K_2(s) = x_0 + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4 + y_5 s^5 + y_6 s^6 + \cdots$$
$$K_3(s) = y_0 + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4 + y_5 s^5 + x_6 s^6 + \cdots$$
$$K_4(s) = y_0 + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4 + x_5 s^5 + x_6 s^6 + \cdots$$

Interval Plants and Agent Uncertainty



For any, $s = j\omega$, the polynomial $\delta(j\omega) \in \mathcal{I}(j\omega)$ takes a value inside the value set

stability of interval plants can be assessed by checking the stability of the extreme Kharitonov polynomials!

Consensus of Interval Plants on Cycle Graphs

Agent Dynamics

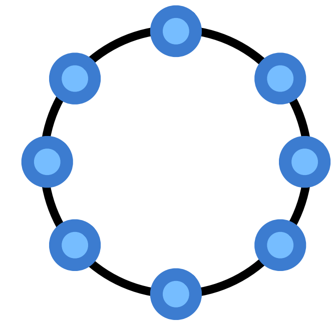
Linear m*th*-order *interval plants*

$$x_i^{(m)} + \alpha_{m-1} x_i^{(m-1)} + \cdots + \alpha_0 x_i = u_i$$

$$\alpha_j \in [\underline{\alpha}_j, \overline{\alpha}_j] \subset \mathbb{R}$$

* assume there exists positive ε such that family of interval polynomials with parameters $\alpha_j \in [\underline{\alpha}_j - \epsilon, \overline{\alpha}_j + \epsilon] \subset \mathbb{R}$ is stable.

Information Exchange Network



$\mathcal{G} = \mathcal{C}_n$

cycle graph

$$L(\mathcal{C}_n) = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$u_i = \sum_{\ell=0}^{m-1} \beta_\ell \left((w_{i,i+1}(x_{i+1}^{(\ell)} - x_i^{(\ell)}) + w_{i,i-1}(x_{i-1}^{(\ell)} - x_i^{(\ell)}) \right)$$

Design gains β_l and weights w to ensure synchronization of agents is achieved for any realization of the interval plant.

Higher-Order Consensus

$$\begin{cases} x_i^{(m)} + \alpha_{m-1} x_i^{(m-1)} + \cdots + \alpha_0 x_i = u_i \\ u_i = \sum_{\ell=0}^{m-1} \beta_\ell \left((w_{i,i+1}(x_{i+1}^{(\ell)} - x_i^{(\ell)}) + w_{i,i-1}(x_{i-1}^{(\ell)} - x_i^{(\ell)}) \right) \end{cases}$$

$$x^{(k)} = \begin{bmatrix} x_1^{(k)} & \cdots & x_n^{(k)} \end{bmatrix}^T$$

$$\begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(m)} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & I_n & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & I_n \\ \Lambda_0 & \Lambda_1 & \cdots & \Lambda_{m-1} \end{bmatrix}}_{\overline{A}} \begin{bmatrix} x^{(0)} \\ \vdots \\ x^{(m-1)} \end{bmatrix} \quad , \ \Lambda_j = -\alpha_j I_n - \beta_j L(\mathcal{C}_n)$$

stability depends on eigenvalues of $\overline{A}$

Higher-Order Consensus

Characteristic polynomial of $\overline{A}$

$$0 = P(s) \quad = \det\left(s^m I_n + \textstyle\sum_{j=0}^{m-1}(\alpha_j I_n + \beta_j L(\mathcal{C}_n))s^j\right)$$

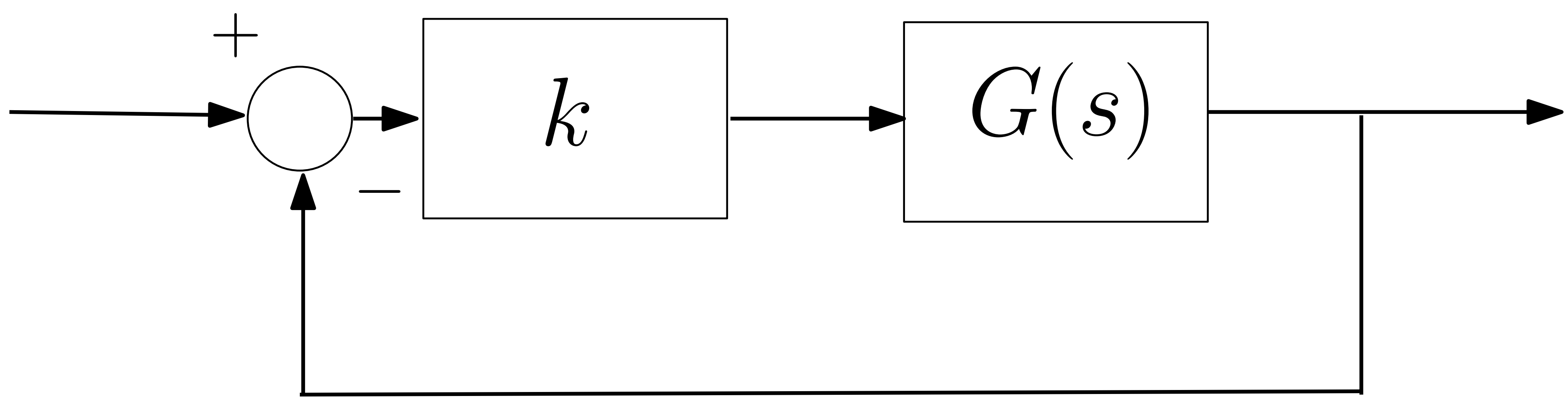
$$= \Pi_{i=1}^{n}(s^m + \textstyle\sum_{j=0}^{m-1}(\alpha_j + \beta_j\lambda_i(L(\mathcal{C}_n))s^j)$$

$\lambda_i(L(\mathcal{C}_n))$ eigenvalues of the Laplacian matrix

consensus achievable when polynomial is stable!

$$\overline{P}(s) = \Pi_{i=2}^{n}\left(s^m + \sum_{j=0}^{m-1}(\alpha_j + \beta_j\lambda_i(L(\mathcal{C}_n))s^j\right)$$

A Feedback Interpretation



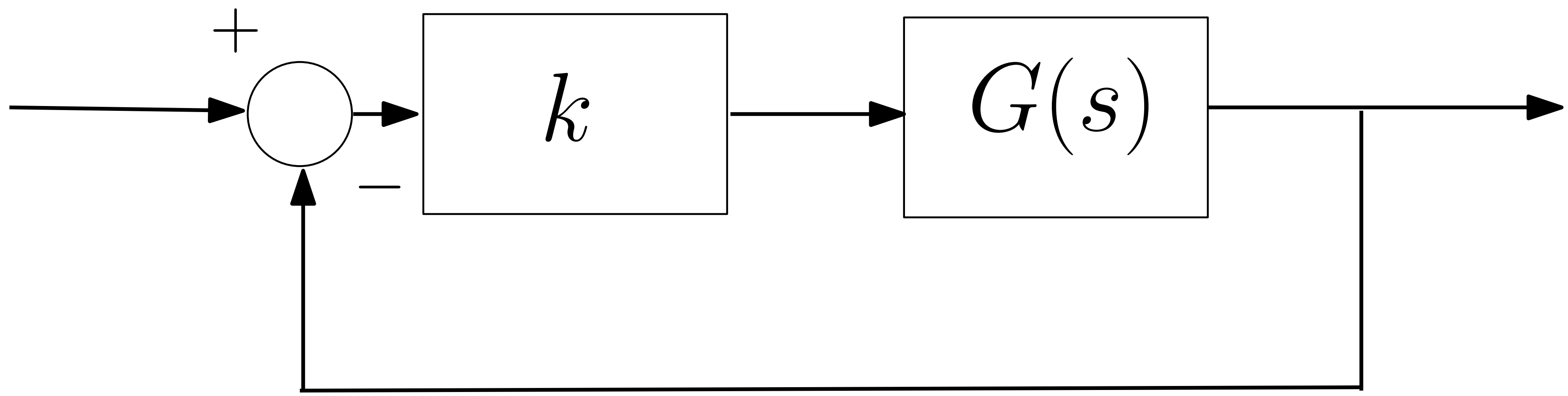
$$G(s) = \frac{\beta_{m-1}s^{m-1} + \beta_{m-2}s^{m-2} + \cdots + \beta_0}{s^m + \alpha_{m-1}s^{m-1} + \cdots + \alpha_0}$$

the closed-loop:

$$H(s) = k\frac{\beta_{m-1}s^{m-1} + \beta_{m-2}s^{m-2} + \cdots + \beta_0}{\left(s^m + \sum_{j=0}^{m-1}(\alpha_j + k\beta_j s^j\right)}$$

Laplacian eigenvalues plays the role of the feedback gain!

A Feedback Interpretation



closed-loop is an interval polynomial!

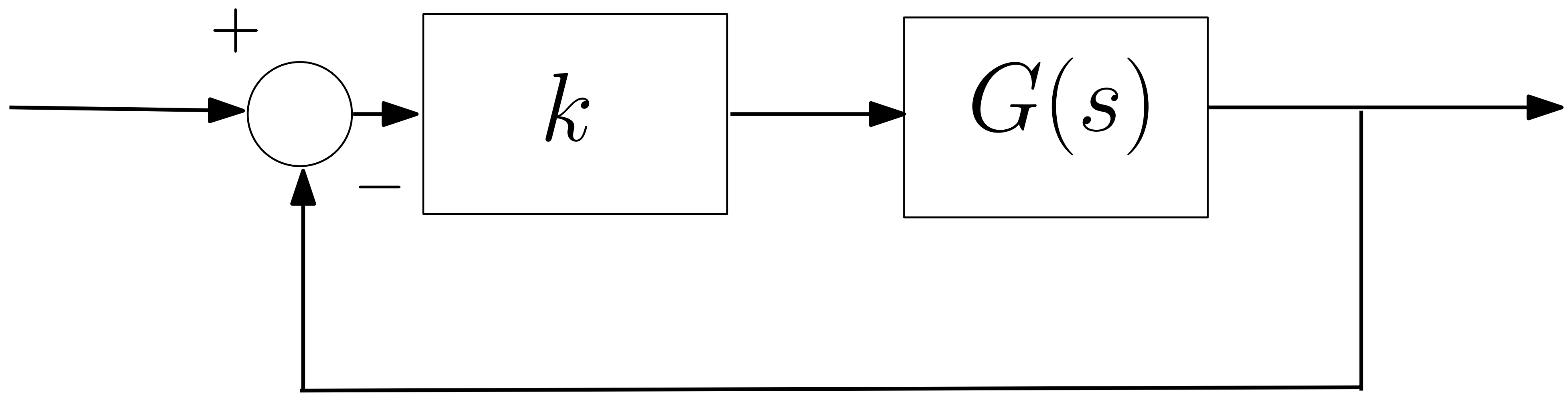
$$H(s) = k\frac{\beta_{m-1}s^{m-1} + \beta_{m-2}s^{m-2} + \cdots + \beta_0}{\left(s^m + \sum_{j=0}^{m-1}(\alpha_j + k\beta_j s^j\right)}$$

intervals:

$$[\underline{\alpha}_j + k\beta_j, \, \overline{\alpha}_j + k\beta_j] \qquad j = 0, \ldots, m-1$$

Kharitonov's Theorem can be used to determine a range for the control gains that ensure stability of the interval polynomial!

A Feedback Interpretation



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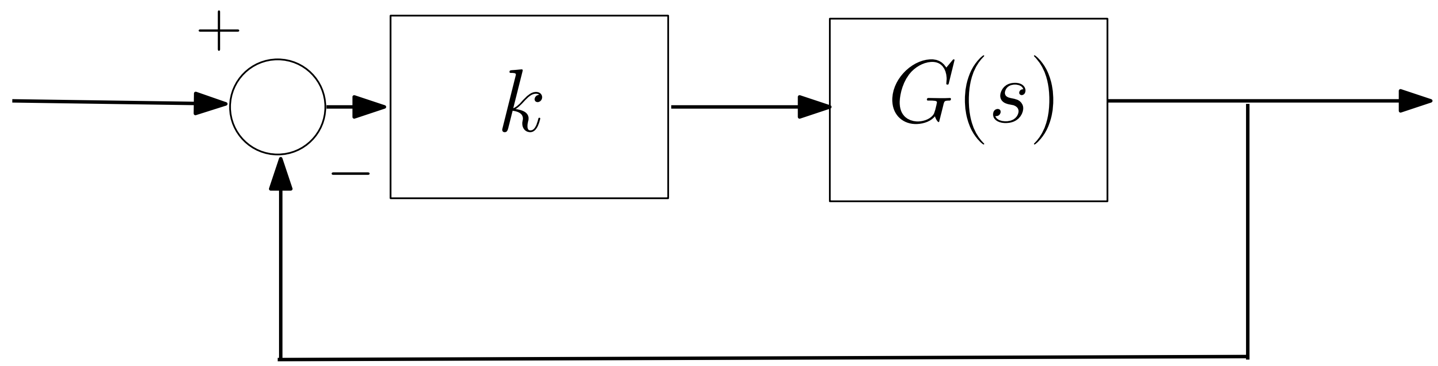
$$H(s) = k\frac{\beta_{m-1}s^{m-1} + \beta_{m-2}s^{m-2} + \cdots + \beta_0}{\left(s^m + \sum_{j=0}^{m-1}(\alpha_j + k\beta_j s^j\right)}$$

intervals:

$$[\underline{\alpha}_j + k\beta_j,\, \overline{\alpha}_j + k\beta_j] \qquad j = 0, \ldots, m-1$$

$$k \in (\underline{k},\, \overline{k})$$

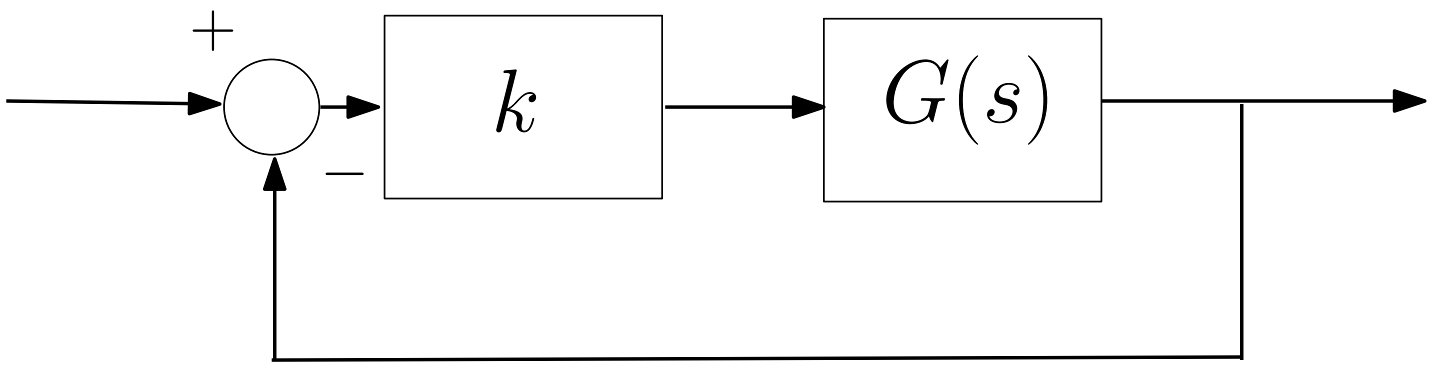
Consensus of Interval Plants on Cycle Graphs



Theorem. *Let $k \in (\underline{k}, \overline{k})$ be the range of stabilizing gains for the plant $G(s)$ and assume the gains β_ℓ are given. If the non-zero eigenvalues of the Laplacian matrix $L(\mathcal{C}_n)$ belong to the interval $(\underline{k}, \overline{k})$, then the interval plants achieve consensus, and are robustly stable for any realization of the system.*

Stability Margins of Interval Plants ⟷ **Spectrum of Cycle Graph Laplacian**

Consensus of Interval Plants on Cycle Graphs



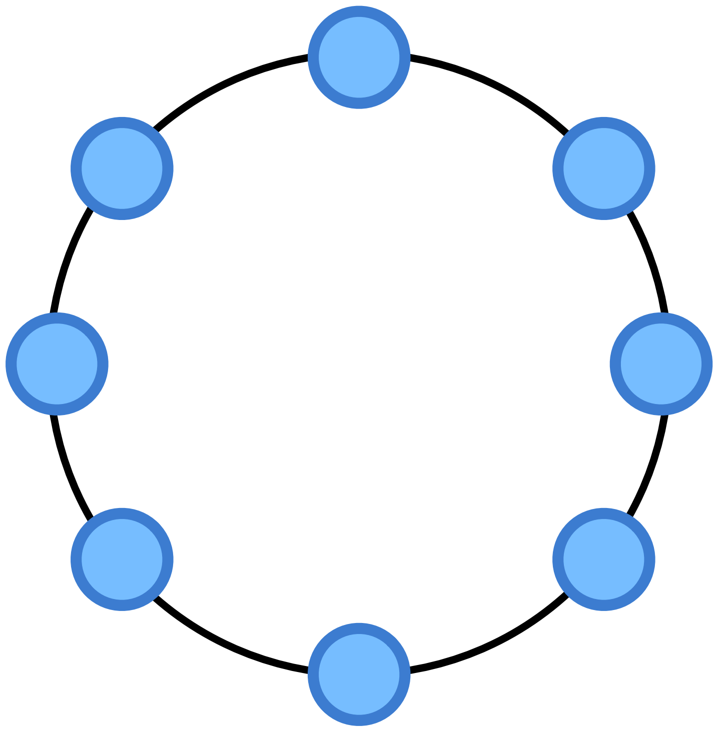
Corollary. *Let* $k \in (\underline{k}, \overline{k})$ *be the range of stabilizing gains for the plant* $G(s)$ *and assume the gains* β_ℓ *are given. The interval plants achieve consensus, and are robustly stable for any realization of the system if*

$$\frac{\underline{k}}{\overline{k}} < 4\gamma \sin^2(\pi/2n),$$

where $\gamma = 1$ *if* n *is odd, or* $\gamma = \cos^2(\pi/2n)$ *if* n *is even.*

Spectrum of cycle graphs are known!

Graph Weight Uncertainty



Assume edge weights are also uncertain

$$w_{ij} = \mu + \Delta$$

Theorem. *Let $k \in (\underline{k}, \overline{k})$ be the range of stabilizing gains for the plant $G(s)$. The interval plants achieve consensus, and are robustly stable for any realization of the systems in the presence of edge weight perturbations for all Δ satisfying*

$$\frac{\underline{k}}{4\sin^2(\pi/n)} < \Delta < \frac{\overline{k}}{4\phi} - \mu,$$

where $\phi = 1$ if n is even, or $\phi = \cos^2(\pi/2n)$ if n is odd.

Numerical Example

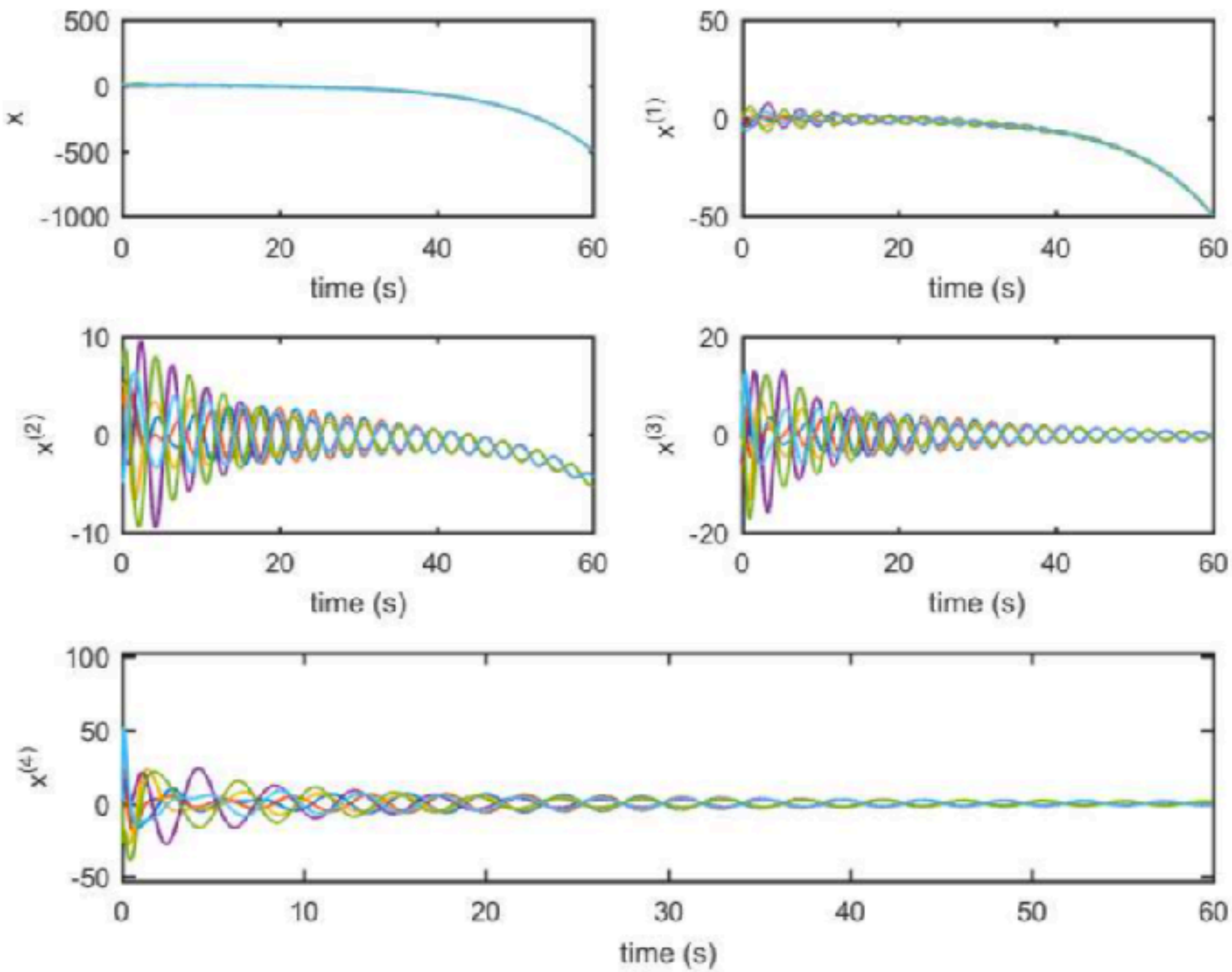


Figure: Consensus in states of 5^{th} order uncertain agents over C_6.

Numerical Example

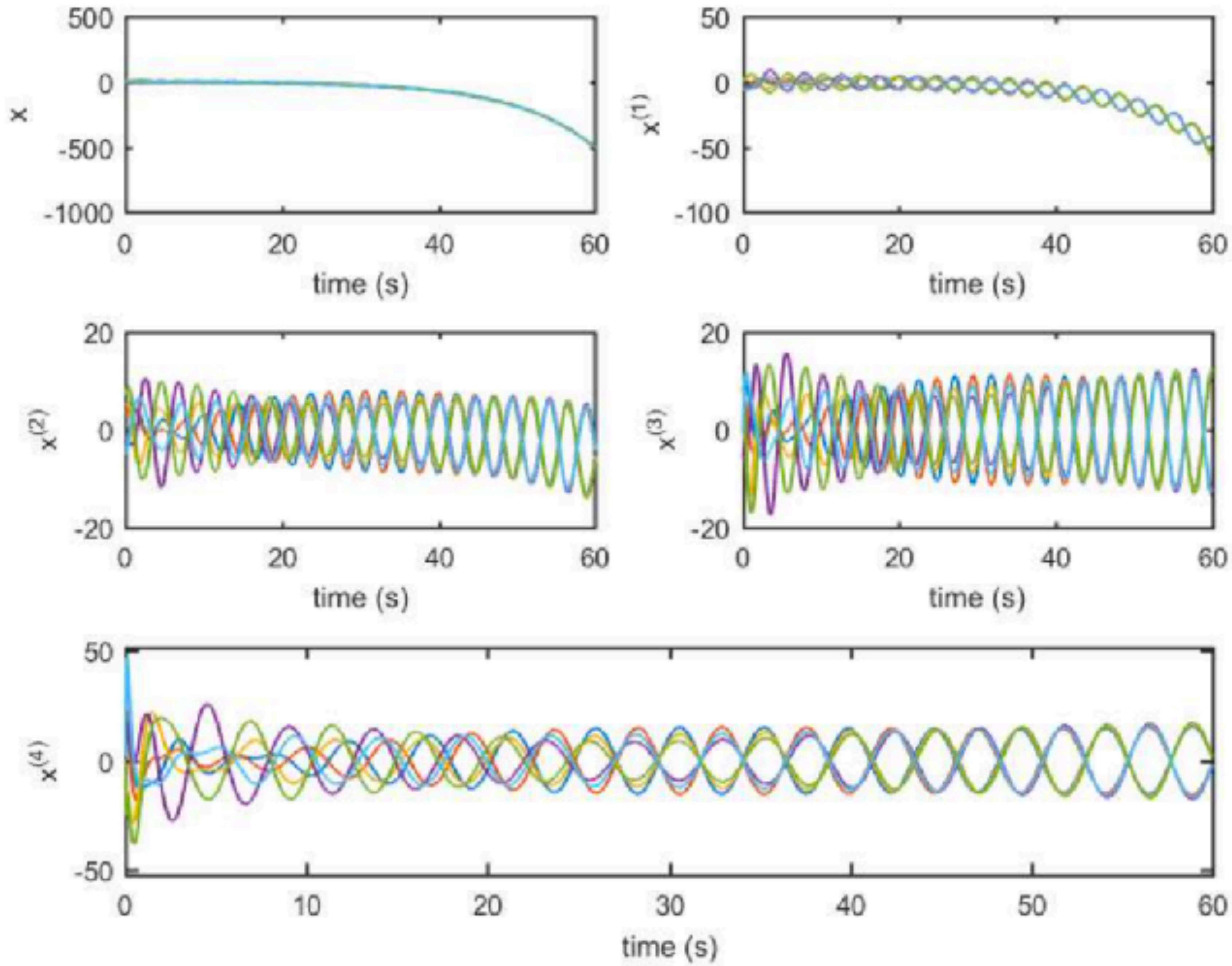
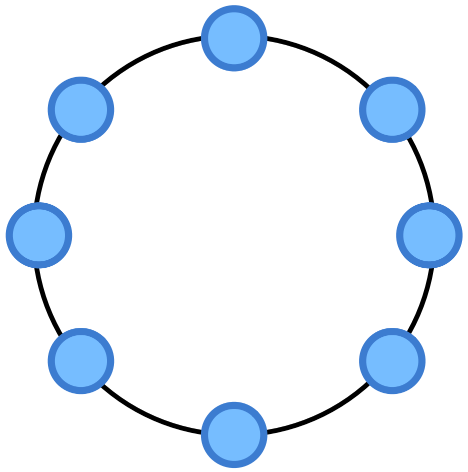
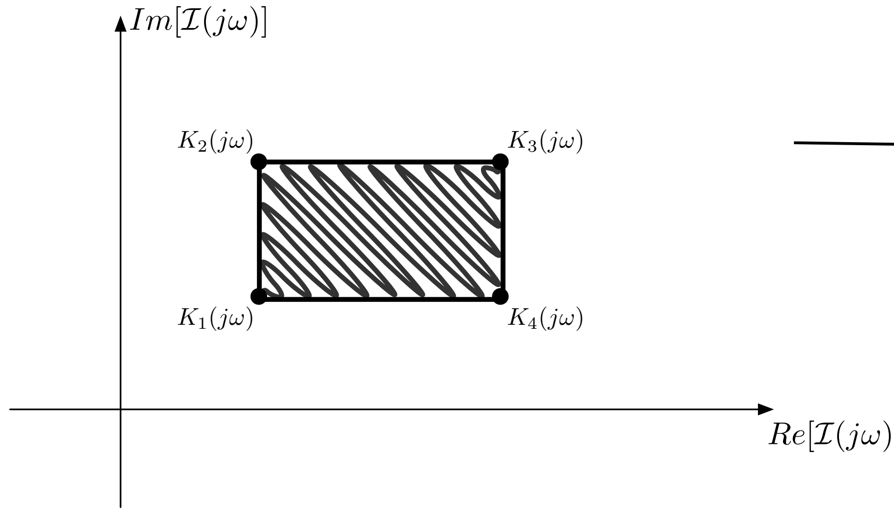
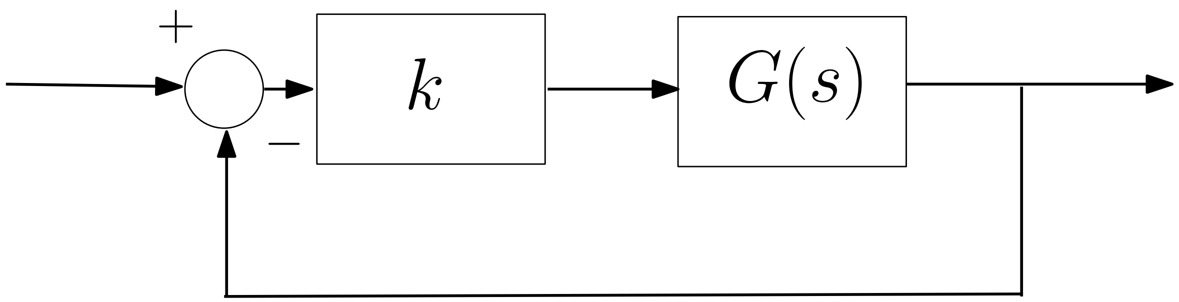


Figure: Consensus breaks down with a perturbation in edge weight that violates Theorem 4.

Concluding Remark







consensus of interval plants

- Kharitonov's Theorem
- uncertainty bounds and spectral properties of Cycle graph
- design methods and robustnesss analysis

future work

- consider more general graph structures
- extend to directed graphs

D. Mukherjee and D. Zelazo, "*Consensus of Higher Order Agents: Robustness and Heterogeneity*," IEEE Transactions on the Control of Network Systems, (under review).

Acknowledgements

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