# A PASSIVITY ANALYSIS FOR NONLINEAR CONSENSUS ON DIGRAPHS

2025 IAAC CONTROL CONFERENCE

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CoNeCt Cooperative Networks and Controls Lab

**Applications**:

- Formation flying
- Power grid
- Automated transportation networks...

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- Formation flying
- Power grid
- Automated transportation networks...
- Fundamental problem: Output consensus

**Multi-agent networks**: A group of SISO agents $\Sigma_i$ interact over a graph $\mathcal{G}$ with SISO edge controllers $\Pi_k$:

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i) \end{cases}, i \in [1, n] \quad \Pi_k : \begin{cases} \dot{\eta}_k = \phi_k(\eta_k, \zeta_k) \\ \mu_k = \psi_k(\eta_k, \zeta_k) \end{cases}, k \in [1, m]$$

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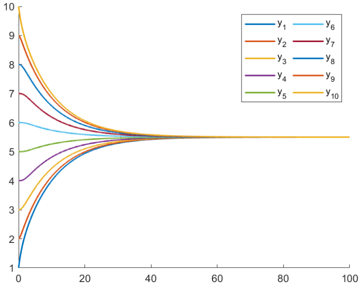
**Output consensus problem**:
Design distributed $\Pi_k$s, such that

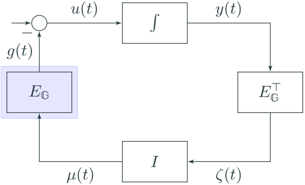
$$\lim_{t \to \infty} (y_i(t) - y_j(t)) = 0, \ \forall i, j$$

$$\Leftrightarrow \lim_{t \to \infty} y(t) \in S$$

where $S = \text{span}(\mathbb{1})$ denotes the agreement space.



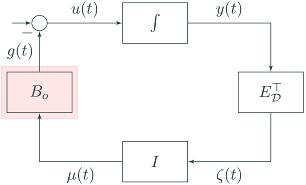
## Graph Topologies Matter!




**Undirected Networks**

$$\dot{x}(t) = -E_{\mathbb{G}} E_{\mathbb{G}}^{\top} x(t)$$

$$y(t) = x(t)$$

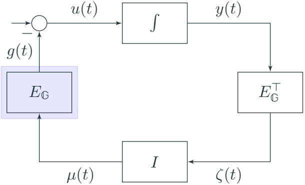


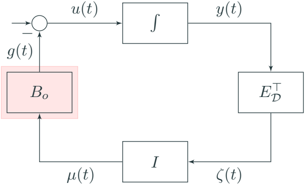

**Directed Networks**

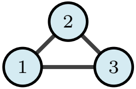
$$\dot{x}(t) = -B_o E_{\mathcal{D}}^{\top} x(t)$$

$$y(t) = x(t)$$

## Graph Topologies Matter!



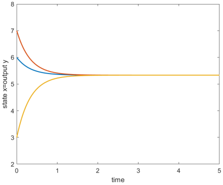




Undirected $\mathbb{G}$

$$\lim_{t\to\infty} y(t) = \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top x(0)$$

Average consensus



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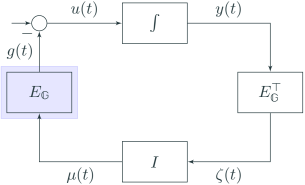


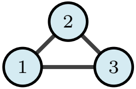


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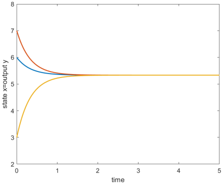
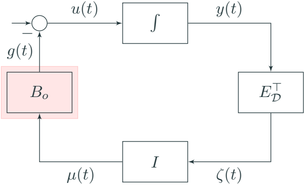
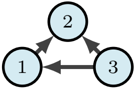


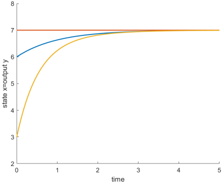



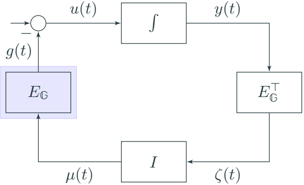

Directed $\mathcal{D}$

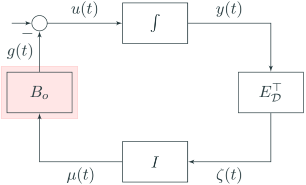
$$\lim_{t\to\infty} y(t) = (q_1^\top x(0))\mathbb{1}_n$$

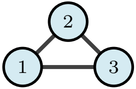
Regular consensus



## Graph Topologies Matter!



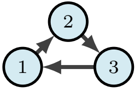




Undirected $\mathbb{G}$

$$\lim_{t\to\infty} y(t) = \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^{\top}x(0)$$

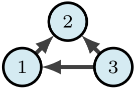
Average consensus




Balanced $\mathcal{D}_b$

$$\lim_{t\to\infty} y(t) = \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^{\top}x(0)$$

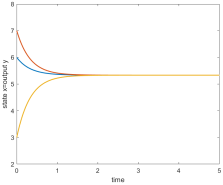
Average consensus

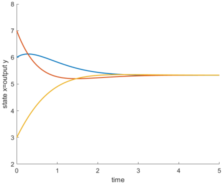


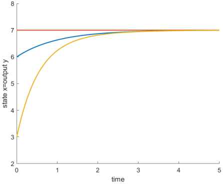

Directed $\mathcal{D}$

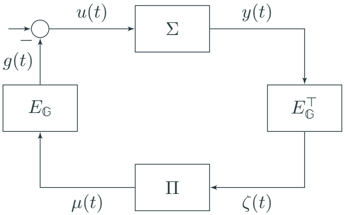
$$\lim_{t\to\infty} y(t) = (q_1^{\top}x(0))\mathbb{1}_n$$

Regular consensus

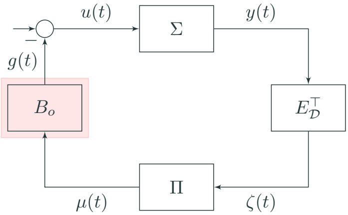


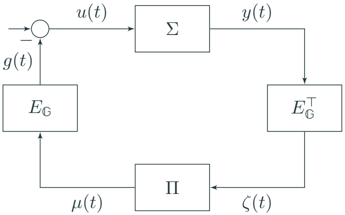



**Undirected** $(\Sigma, \Pi, \mathbb{G})_E$




**Directed** $(\Sigma, \Pi, \mathcal{D})_{B_o}$

# NETWORK SYSTEMS AND PASSIVITY

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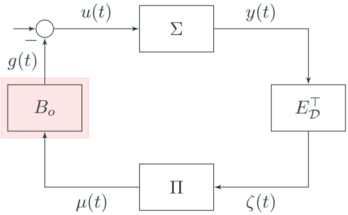
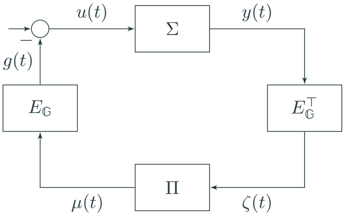



**Directed** $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Symmetric operator** $E_{\mathbb{G}} \Pi E_{\mathbb{G}}^{\top}$

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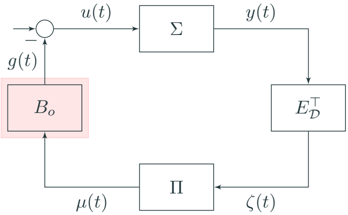
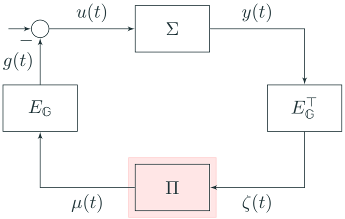



**Directed** $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Symmetric operator** $E_{\mathbb{G}} \Pi E_{\mathbb{G}}^{\top}$
- **Passivity Analysis** ✓

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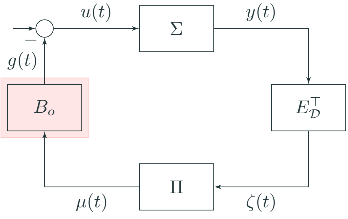
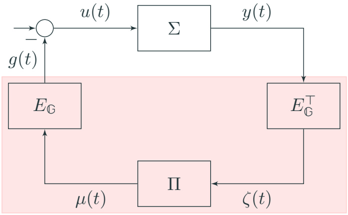



**Directed** $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Symmetric operator** $E_{\mathbb{G}} \Pi E_{\mathbb{G}}^{\top}$
- **Passivity Analysis** $\checkmark$

Passive $\Pi$ $\quad \mu^{\top}(t)\zeta(t) \geq \dot{V}(\eta(t))$

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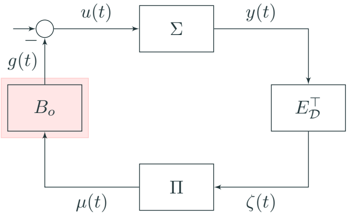


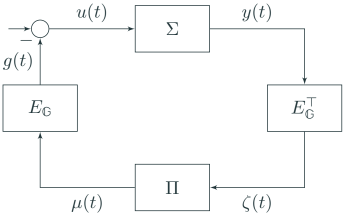

**Directed** $(\Sigma, \Pi, \mathcal{D})_{B_o}$

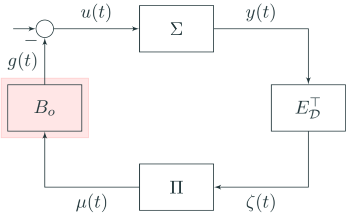
- **Symmetric operator** $E_{\mathbb{G}}\Pi E_{\mathbb{G}}^\top$
- **Passivity Analysis** ✓

Passive $\Pi$ $\quad \mu^\top(t)\zeta(t) \geq \dot{V}(\eta(t))$

$\Rightarrow$ Passive $E_{\mathbb{G}}\Pi E_{\mathbb{G}}^\top$

$$g^\top(t)y(t) = \mu^\top(t)E_{\mathbb{G}}^\top y(t) = \mu^\top(t)\zeta(t)$$




**Undirected** $(\Sigma, \Pi, \mathbb{G})_E$

**Directed** $(\Sigma, \Pi, \mathcal{D})_{B_o}$

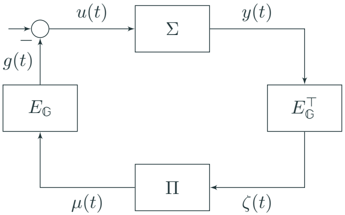
- **Symmetric operator** $E_{\mathbb{G}} \Pi E_{\mathbb{G}}^{\top}$
- **Passivity Analysis** ✓

Passive $\Pi$ $\quad \mu^{\top}(t)\zeta(t) \geq \dot{V}(\eta(t))$

$\Rightarrow$ Passive $E_{\mathbb{G}} \Pi E_{\mathbb{G}}^{\top}$

$$g^{\top}(t)y(t) = \mu^{\top}(t)E_{\mathbb{G}}^{\top}y(t) = \mu^{\top}(t)\zeta(t)$$

- A decoupled analysis
- Convergence, stability

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- **Symmetric operator** $E_{\mathbb{G}} \Pi E_{\mathbb{G}}^{\top}$
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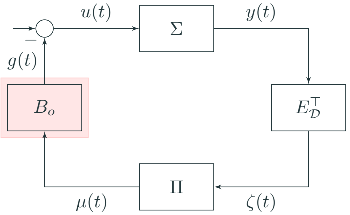
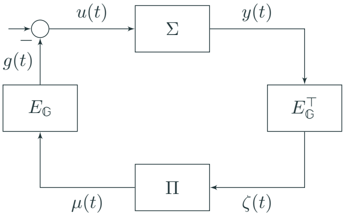
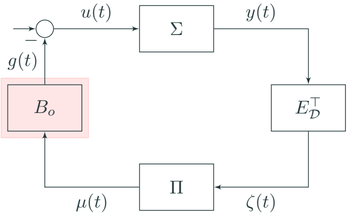



**Directed** $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Asymmetric** operator $B_o \Pi E_{\mathcal{D}}^{\top}$




**Undirected** $(\Sigma, \Pi, \mathbb{G})_E$

- **Symmetric operator** $E_{\mathbb{G}} \Pi E_{\mathbb{G}}^\top$
- **Passivity Analysis** ✓

Passive $\Pi$ $\quad \mu^\top(t)\zeta(t) \geq \dot{V}(\eta(t))$

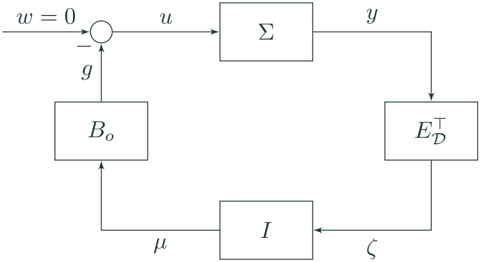
$\Rightarrow$ Passive $E_{\mathbb{G}} \Pi E_{\mathbb{G}}^\top$

$$g^\top(t)y(t) = \mu^\top(t)E_{\mathbb{G}}^\top y(t) = \mu^\top(t)\zeta(t)$$

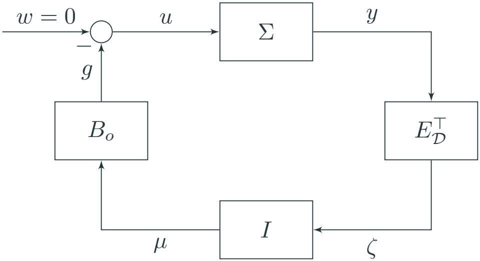
- A decoupled analysis
- Convergence, stability

**Directed** $(\Sigma, \Pi, \mathcal{D})_{B_o}$

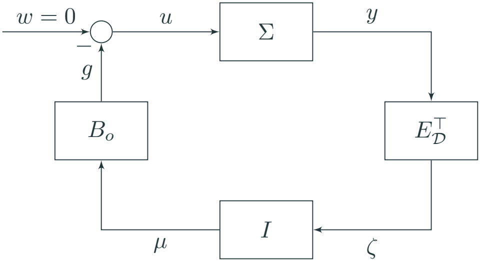
- **Asymmetric** operator $B_o \Pi E_{\mathcal{D}}^\top$
- **Passivity Analysis?**




- General agents $\Sigma$, Linear controllers $\Pi = I : \mu = \zeta$



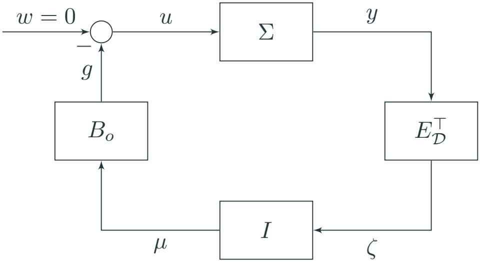

- General agents $\Sigma$, Linear controllers $\Pi = I : \mu = \zeta$
- Is the operator $B_o I E_{\mathcal{D}}^{\top}$ passive?




- General agents $\Sigma$, Linear controllers $\Pi = I : \mu = \zeta$
- Is the operator $B_o I E_{\mathcal{D}}^\top$ passive?
  - **Balanced** $\mathcal{D}$: Passive

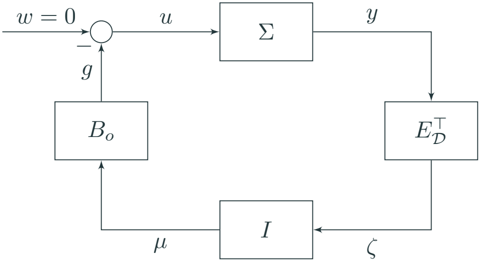
[1] F.-Y. Yue and D. Zelazo, "A passivity analysis for nonlinear consensus on balanced digraphs," ECC2025.

# CONTRIBUTION 1: PASSIVITY ANALYSIS FOR $(\Sigma, \Pi, \mathcal{D})_{B_o}$




- General agents $\Sigma$, Linear controllers $\Pi = I : \mu = \zeta$
- Is the operator $B_o I E_{\mathcal{D}}^{\top}$ passive?
  - **Balanced** $\mathcal{D}$: Passive $\overset{\text{passive } \Sigma}{\longrightarrow}$ passivity analysis ✓


[1] F.-Y. Yue and D. Zelazo, "A passivity analysis for nonlinear consensus on balanced digraphs," ECC2025.




- General agents $\Sigma$, Linear controllers $\Pi = I : \mu = \zeta$
- Is the operator $B_o I E_{\mathcal{D}}^\top$ passive?
  - **Balanced** $\mathcal{D}$: Passive $\overset{\text{passive } \Sigma}{\rightarrow}$ passivity analysis ✓
  - **General** $\mathcal{D}$: Not Passive

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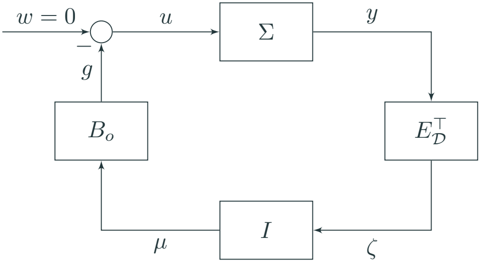
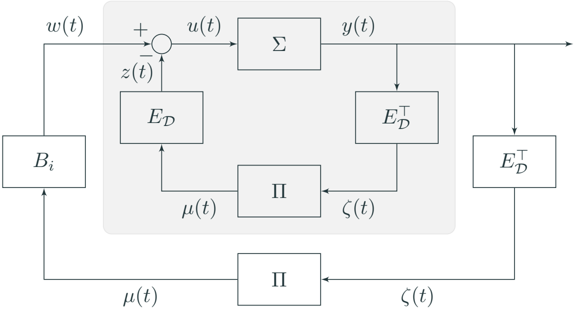
# CONTRIBUTION 1: PASSIVITY ANALYSIS FOR $(\Sigma, \Pi, \mathcal{D})_{B_o}$




- General agents $\Sigma$, Linear controllers $\Pi = I : \mu = \zeta$
- Is the operator $B_o I E_{\mathcal{D}}^{\top}$ passive?
  - **Balanced** $\mathcal{D}$: Passive $\overset{\text{passive } \Sigma}{\rightarrow}$ passivity analysis ✓
  - **General** $\mathcal{D}$: Not Passive
- For general controller dynamics $\Pi$: the operator $B_o \Pi E_{\mathcal{D}}^{\top}$ may not be passive.

[1] F.-Y. Yue and D. Zelazo, "A passivity analysis for nonlinear consensus on balanced digraphs," ECC2025.

# CONTRIBUTION 1: A GENERAL APPROACH FOR DIRECTED COUPLING

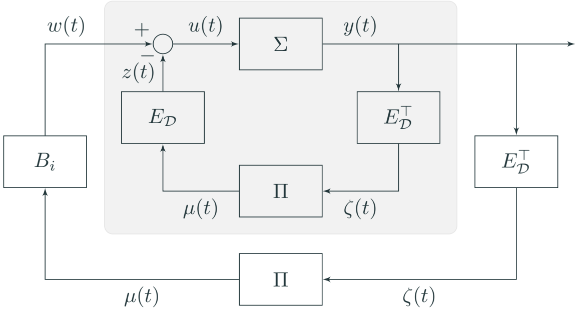



$$(\Sigma, \Pi, \mathcal{D}, w)$$

- Loop decomposition of $(\Sigma, \Pi, \mathcal{D})_{B_o}$: $E_{\mathcal{D}} = B_o + B_i$: equivalence
- First branch ($y \to z$): $E_{\mathcal{D}} \Pi E_{\mathcal{D}}^{\top}$ is passive, given passive $\Pi$.
- Second branch ($y \to w$): external input with directed information

[1] F.-Y. Yue and D. Zelazo, "A passivity analysis for nonlinear consensus on balanced digraphs," ECC2025.

# CONTRIBUTION 1: A GENERAL APPROACH FOR DIRECTED COUPLING

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- Loop decomposition of $(\Sigma, \Pi, \mathcal{D})_{B_o}$: $E_{\mathcal{D}} = B_o + B_i$: equivalence
- First branch ($y \to z$): $E_{\mathcal{D}} \Pi E_{\mathcal{D}}^{\top}$ is passive, given passive $\Pi$.
- Second branch ($y \to w$): external input with directed information
- $(\Sigma, \Pi, \mathcal{D})_{B_o} \Leftrightarrow (\Sigma, \Pi, \mathcal{D}, w)$

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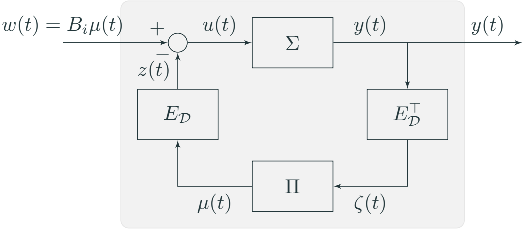
# OUTPUT CONSENSUS AND CONVERGENCE TO A SUBMANIFOLD




- Output consensus: $\lim_{t\to\infty} y(t) \in \mathrm{span}(\mathbb{1}) = S$ submanifold!

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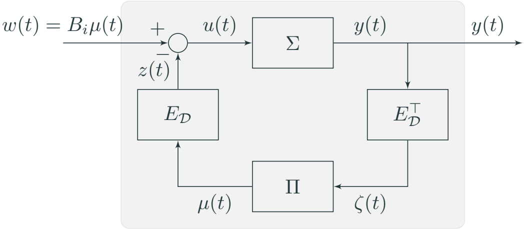
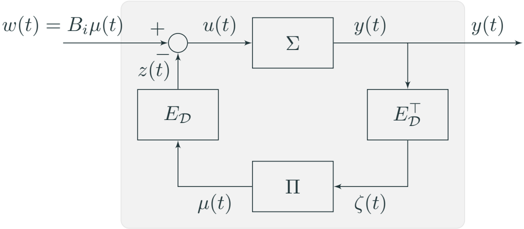
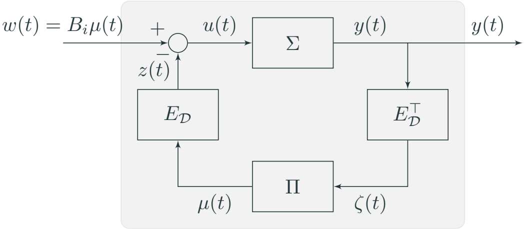



- Output consensus: $\lim_{t\to\infty} y(t) \in \mathrm{span}(\mathbb{1}) = S$ submanifold!
- Converge to the agreement submanifold $S$: $\lim_{t\to\infty} \mathrm{Proj}_{S^\perp}(y(t)) = 0$

# OUTPUT CONSENSUS AND CONVERGENCE TO A SUBMANIFOLD




- Output consensus: $\lim_{t\to\infty} y(t) \in \mathrm{span}(\mathbb{1}) = S$ submanifold!
- Converge to the agreement submanifold $S$: $\lim_{t\to\infty} \mathrm{Proj}_{S^\perp}(y(t)) = 0$
- **Connection to Passivity?**

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- **Connection to Passivity?**

  Passivity relations[1] (point-wise)

$$u(t)^\top \operatorname{Proj}_{S^\perp}(y(t)) \geq l\|u(t)\|^2 + e\|\operatorname{Proj}_{S^\perp}(y(t))\|^2$$

$$z(t)^\top \operatorname{Proj}_{S^\perp}(y(t)) \geq l\|z(t)\|^2 + e\|\operatorname{Proj}_{S^\perp}(y(t))\|^2$$

[1] J. M. Montenbruck, M. Arcak, and F. Allgöwer, "An input-output framework for submanifold stabilization," IEEE TAC, 2017.

# CONTRIBUTION 2: PASSIVITY W.R.T. SUBMANIFOLD

$$\Lambda: \ \dot{x}(t) = f(x(t), u(t)), y(t) = h(x(t)), \quad f: (\mathbb{R}^n, \mathbb{R}^p) \to \mathbb{R}^n, \ h: \mathbb{R}^n \to \mathbb{R}^p$$

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Recall **Classical Passivity**

- Storage Function $V: \mathbb{R}^n \to \mathbb{R}$
  $(1)\ V(x) \geq 0$; $(2)\ V(0) = 0$
- Passivity indices $\exists \delta, \varepsilon \geq 0$

**Passive:** $u^\top(t)y(t) \geq \dot{V}(x(t)) + \delta\|u(t)\|_2^2 + \varepsilon\|y(t)\|_2^2, \ \forall t$

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**Passivity w.r.t. Submanifold** $S$ [1]**:**

[1] F.-Y. Yue and D. Zelazo, “A passivity analysis for nonlinear consensus on digraphs.” (Submitted)

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**Passivity w.r.t. Submanifold $S$ [1]:**

- Constrained Storage Function $Q : \mathbb{R}^n \to \mathbb{R}$
  $(1)\ Q(x) \geq 0$; $(2)\ Q(x) = 0, \forall h(x) \in S$
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[1] F.-Y. Yue and D. Zelazo, “A passivity analysis for nonlinear consensus on digraphs.” (Submitted)

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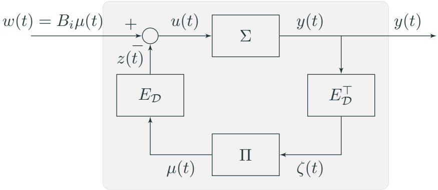
**S-Passive:** $u^\top(t)\,\mathrm{Proj}_{S^\perp}(y(t)) \geq \dot{Q}(x(t)) + \varepsilon\|\mathrm{Proj}_{S^\perp}(y(t))\|_2^2 + \delta\|u(t)\|_2^2, \ \forall t$

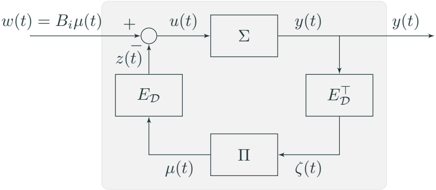
[1] F.-Y. Yue and D. Zelazo, “A passivity analysis for nonlinear consensus on digraphs.” (Submitted)




$(\Sigma^o, \Pi, \mathcal{D}, w)$

Integrator-like agents:

$$\Sigma_i^o : \begin{cases} \dot{x}_i(t) = u_i(t), \\ y_i(t) = h_i(x_i(t)), \end{cases} \qquad i \in [1, n]$$




$(\Sigma^o, \Pi, \mathcal{D}, w)$

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Under what passivity conditions on $\Sigma$ and $\Pi$ does the output of the system converge to the agreement submanifold $S$?

## CONTRIBUTION 3: A PASSIVITY-BASED ANALYSIS

Directed networks$(\Sigma^o, \Pi, \mathcal{D}, w)$

Integrator-like agents $\Sigma_i^o : \dot{x}_i(t) = u_i(t),\ y_i(t) = h_i(x_i(t)),\ i \in [1, n]$

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**Conditions**

1. $h_i$s are continuously differentiable and monotone passive
2. $h_i$s have bounded derivatives, i.e., $\frac{\partial h_i(x)}{\partial x} \leq m$

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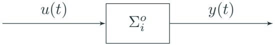


**Passivity of agents $\Sigma_i^o$**

- $u_i y_i \geq \dot{V}_i(x_i)$, $V_i(x_i) = \int_0^{x_i} h(\sigma) d\sigma$ [1]

[1] H.K. Khalil, "Nonlinear systems (3rd ed)", Upper Saddle River, NJ: Prentice hall, 2002.

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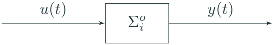


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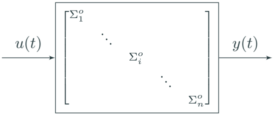


## $S$-**Passivity** of $\Sigma$

- Constrained $Q(x) = \frac{1}{2} h^\top(x)(I - \frac{1}{|\mathcal{V}|}\mathbb{1}\mathbb{1}^\top)h(x)$
- $M = \max(1, |1 - m|)$

$$u^\top \operatorname{Proj}_{S^\perp}(y) \geq \dot{Q}(x) - \frac{M}{2}\|u\|_2^2 - \frac{M}{2}\|\operatorname{Proj}_{S^\perp}(y)\|_2^2$$

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Directed networks $(\Sigma^o, \Pi, \mathcal{D}, w)$

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3. Controllers $\Pi_k$ are input-output passive

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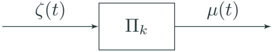


**Passivity of controllers $\Pi_k$**

- $\zeta_k \mu_k \geq \dot{W}_k(\eta_k) + \alpha_k \mu_k^2 + \gamma_k \zeta_k^2,\ \alpha_k, \gamma_k > 0$
- $\alpha_k \gamma_k < 1/4$, $\alpha = \min(\alpha_k)$, $\gamma = \min(\gamma_k)$

# CONTRIBUTION 3: A PASSIVITY-BASED ANALYSIS

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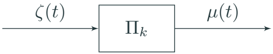


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### **Passivity** of $E_{\mathcal{D}} \Pi E_{\mathcal{D}}^\top$

- $z^\top \operatorname{Proj}_{S^\perp}(y) \geq \sum_{k=1}^{p} \dot{W}_k + \alpha \|\mu\|_2^2 + \gamma \lambda_2 \| \operatorname{Proj}_{S^\perp}(y) \|_2^2$
- Passive $(z, \operatorname{Proj}_{S^\perp}(y))$

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Integrator-like agents $\Sigma_i^o : \dot{x}_i(t) = u_i(t),\ y_i(t) = h_i(x_i(t)),\ i \in [1, n]$

**Theorem**

1. $h_i$ are continuously differentiable and monotone passive
2. $h_i$ have bounded derivatives, i.e., $\frac{\partial h_i(x)}{\partial x} \leq m,\ M = \max(1, |1-m|)$
3. Controllers $\Pi$: $\zeta^\top \mu \geq \sum \dot{W}_k(\eta_k) + \alpha\|\mu\|_2^2 + \gamma\|\zeta\|_2^2,\ \alpha\gamma < \frac{1}{4}$

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4. $\alpha \geq \max(D_o)\frac{M}{2}$ and $\gamma\lambda_2 > \frac{M}{2}$

where $\max(D_o)$ and $\lambda_2$ denote the maximal out-degree and the algebraic connectivity of graph $\mathcal{D}$.

Directed networks$(\Sigma^o, \Pi, \mathcal{D}, w)$

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where $\max(D_o)$ and $\lambda_2$ denote the maximal out-degree and the algebraic connectivity of graph $\mathcal{D}$.

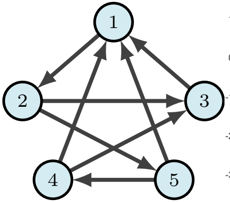
Then, the network $(\Sigma^o, \Pi, \mathcal{D}, w)$ achieves output consensus.

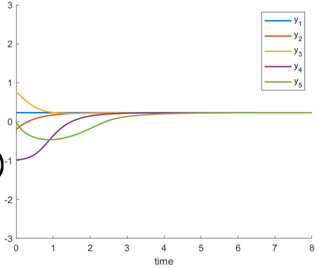
# CASE STUDY: A HETEROGENEOUS NETWORK SYSTEM

- Systems

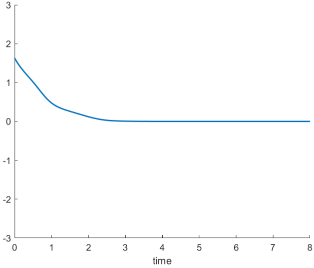
  $\Sigma^o : \dot{x}(t) = u(t), \; y(t) = [x_1(t), x_2(t), \tanh(x_3(t)), \tanh(x_4(t)), \frac{x_5(t)}{1+|x_5(t)|}]^\top$

- Parameters
  - Constrained storage function: $Q(x) = \frac{1}{2}h^\top(x)(I - \frac{1}{|\mathcal{V}|}\mathbb{1}\mathbb{1}^\top)h(x)$
  - Algebraic connectivity: $\lambda_2 = 3$
  - Maximal out-degree: $\max(D_o) = 2$




Outputs of agents




Evolution of $Q(x(t))$

# CASE STUDY: A HETEROGENEOUS NETWORK SYSTEM

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  $\Pi : \mu(t) = 2\zeta(t)$

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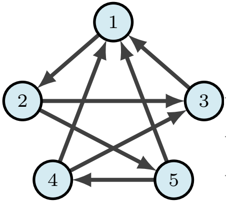
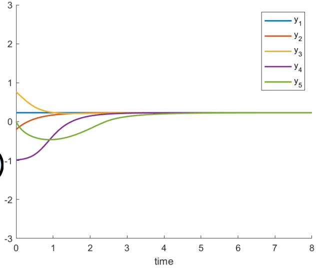
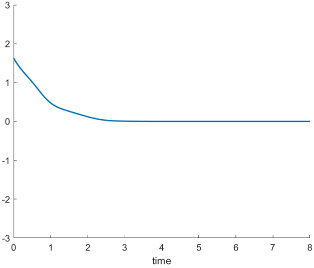



Outputs of agents




Evolution of $Q(x(t))$

# CASE STUDY: A HETEROGENEOUS NETWORK SYSTEM

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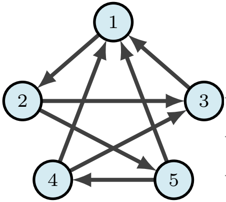
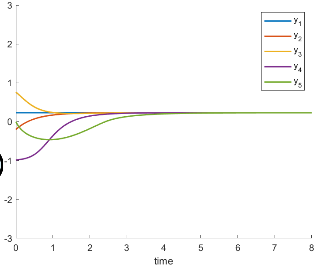
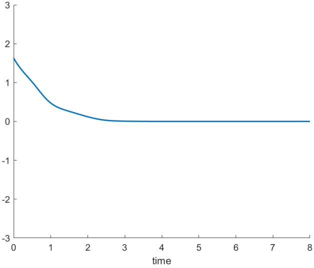
  - Algebraic connectivity: $\lambda_2 = 3$
  - Maximal out-degree: $\max(D_o) = 2$
- **A sufficient condition**




Outputs of agents

Evolution of $Q(x(t))$

Contributions:

- A general approach that enables a passivity analysis for the network systems with directed coupling.
- Constrained storage functions, Passivity w.r.t. submanifolds.
- A passivity-based analysis for integrator-like agents that interact over digraphs.

Future work:

- complex dynamics, other passivity properties.
- A sufficient and necessary condition.

# Thank-You!

DEPARTMENT OF AEROSPACE ENGINEERING

TECHNION
Israel Institute of Technology

CoNeCt
Cooperative Networks and Controls Lab