LEADER IDENTIFICATION IN SEMI-AUTONOMOUS CONSENSUS PROTOCOLS

CONNECT LAB
Cooperative Networks and Controls

Evyatar Matmon and Daniel Zelazo

Technion - Israel Institute of Technology

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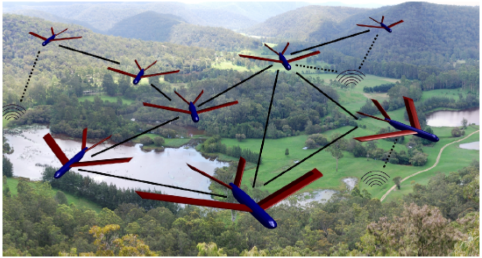
INTRODUCTION

- **Multi-agent systems (MAS)** consist of autonomous agents interacting to achieve a common goal.
- Each agent operates based on local information and decision-making capabilities, yet contributes to the global system behavior.
- MAS are characterized by **decentralized control**, scalability, and robustness to individual agent failures.
- Typical applications of MAS include drone swarms, satellite constellations, robotic fleets in manufacturing and logistics.

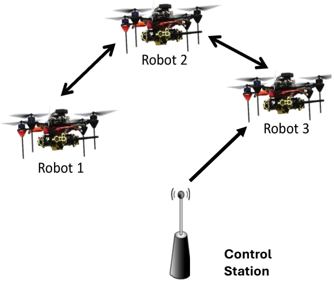


- The security of MAS is vulnerable to cyber-physical attacks, especially through network topology identification.
- **If critical agents are identified**, they become targets for attacks.
- This work explores **identifying leader agents** in networked dynamic systems under a semi-autonomous consensus protocol.

- We are dealing with dynamic agents, and we assume an integrator dynamics for each agent,

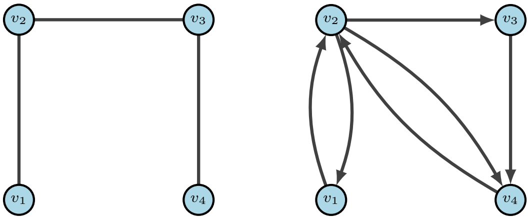
$$\dot{x}_i = u_i, \; i \in \mathcal{V},$$

where $x_i, u_i \in \mathbb{R}$ are the ith agent state and the corresponding input signal, respectively.
- Each agent receives an input signal through information exchange with other agents in the network.



GRAPH THEORY

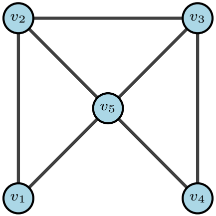
- A **graph** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined by a set of vertices and edges.
- The edge can be **ordered** or **unordered**.
- The graph can be **directed** or **undirected**.



Undirected graph

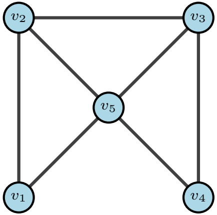
Directed graph

- The **neighbours** of vertex v_i are denoted by $N(v_i)$
- **Degree** of vertex v_i is denoted by by $d(i)$.
- **Degree matrix** $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$. **Adjacency matrix** $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

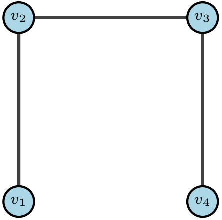


A graph $\mathcal{G}$

- The **induced subgraph** is obtained by removing certain nodes along with their incident edges.



A graph $\mathcal{G}$



An induced subgraph of $\mathcal{G}$

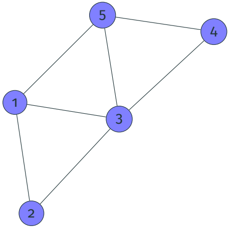
- In the **autonomous consensus** protocol, agents aim to reach agreement via the distributed protocol

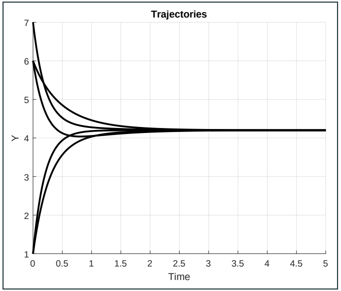
$$\dot{x}_i = \sum_{j \sim i} (x_j - x_i), \quad i \in \mathcal{V}$$

- Under a connectivity assumption of the information exchange graph, the protocol satisfies:

$$\lim_{t \to \infty} x(t) \in \text{span}\{\mathbb{1}_n\}$$

Network Topology $G(\mathcal{V}, \mathcal{E})$





AUTONOMOUS CONSENSUS PROTOCOL CONT.

- The autonomous consensus protocol can be written in a matrix form:

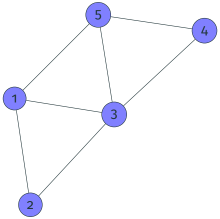
$$\dot{x} = -Lx$$

where L is the **Laplacian matrix**.

- Explicitly expression of the Laplacian matrix: $[L(\mathcal{G})]_{ij} = \begin{cases} d(i), & i = j \\ -1, & i \sim j \\ 0, & \text{otherwise} \end{cases}$.

$$L = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}$$

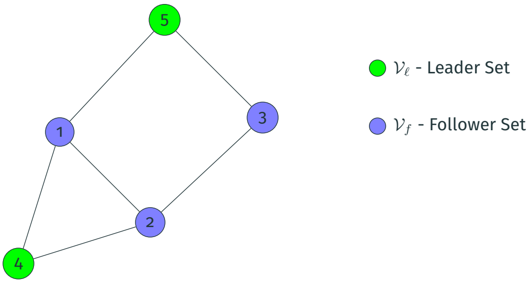
Network Topology $G(\mathcal{V}, \mathcal{E})$

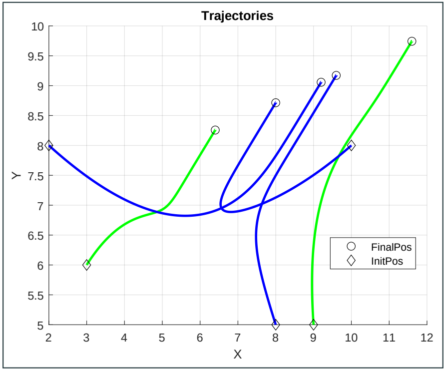


In the **semi-autonomous consensus** protocol, some agents, called leaders, receive an external input:

$$\dot{x}_i = \begin{cases} \sum_{j\sim i}(x_j - x_i) + (u_i^{\text{ex}} - x_i), & i \in \mathcal{V}_\ell, \\ \sum_{j\sim i}(x_j - x_i), & i \in \mathcal{V}_f. \end{cases}$$

Network Topology



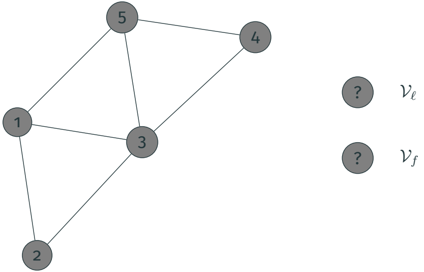


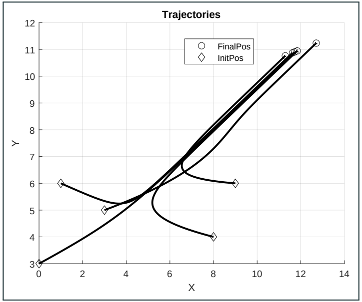
Objective

Identify the leader agents in a semi-autonomous consensus network.

- underlying graph is unknown
- assume constant external inputs
- access to measurements of system state

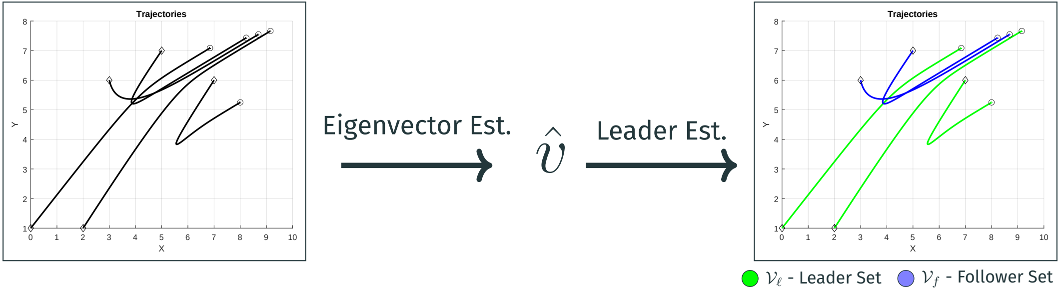
Network Topology





We explore the connection between the Laplacian eigenvectors and leader positions:

- Distributed estimation of Laplacian eigenvectors from system trajectories.
- Identify relationship between eigenvectors of Laplacian with leader positions.



Part 1: Relation Between Velocities and Fiedler Eigenvector

FIEDLER EIGENVECTOR

$$\dot{x}_i = \begin{cases} \sum_{j \sim i}(x_j - x_i) + (u_i^{\text{ex}} - x_i), & i \in \mathcal{V}_\ell, \\ \sum_{j \sim i}(x_j - x_i), & i \in \mathcal{V}_f. \end{cases}$$

- The semi-autonomous protocol can be written as:

$$\dot{x} = -L_B(\mathcal{G})x + \begin{bmatrix} I_{|\mathcal{V}_\ell|} \\ 0_{|\mathcal{V}_f| \times |\mathcal{V}_\ell|} \end{bmatrix} \begin{bmatrix} u_1^{ex} \\ \vdots \\ u_{|\mathcal{V}_\ell|}^{ex} \end{bmatrix}.$$

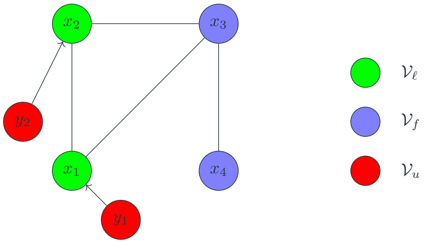
  - $L_B(\mathcal{G})$ is called the grounded Laplacian
  - the eigen-pair (λ_F, v_F) of $L_B(\mathcal{G})$ corresponding to the smallest eigenvalue of are termed the Fiedler eigenvalue and eigenvector

To link the Fiedler vector with node velocities, we transform the semi-autonomous system into an autonomous-like structure:

- Introduce a state variable y representing external control inputs.
- Assume the external input remains constant, giving the dynamics:

$$\dot{y}_i = 0, \quad y_i(0) = u_i^{ex}.$$

Our graph $\bar{\mathcal{G}}$ is directed and consists of three groups: $\mathcal{V}_\ell, \mathcal{V}_f, \mathcal{V}_u$.



- The system dynamics can be expressed as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\bar{L} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} L_B & \bar{L}_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where $\bar{L} = L(\bar{\mathcal{G}})$ is the directed graph Laplacian of $\bar{\mathcal{G}}$. The submatrices are given by:

$$L_B = L(\mathcal{G}) + \begin{bmatrix} I_{|\mathcal{V}_\ell|} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{L}_{12} = \begin{bmatrix} I_{|\mathcal{V}_\ell|} \\ 0 \end{bmatrix}.$$

EIGENVALUES AND EIGENVECTORS OF $\bar{L}$

Let $\{\lambda_i(\bar{L}), v_i(\bar{L})\},\ \{\lambda_i(L_B), v_i(L_B)\}$ be the eigenvalues and corresponding eigenvectors of $\bar{L}$ and L_B respectively. The eigenvalues arranged in ascending order. Then,

$$\lambda_i(\bar{L}) = \begin{cases} 0, & i = 1, \dots, m \\ \lambda_{i-m}(L_B), & i = m+1, \cdots, n+m \end{cases}.$$

and

$$v_i(\bar{L}) = \begin{bmatrix} v_i(L_B)^T & \mathbf{0}_m^T \end{bmatrix}^T, \quad i = m+1, \cdots, m+n.$$

Result

The Fiedler eigenvalue and eigenvector of L_B is also the smallest non-zero eigenvalue for $\bar{L}$.

GROUNDED LAPLACIAN PROPERTIES

Lemma

If $\mathcal{G}$ is connected, then the following properties hold for L_B:

- The **Fiedler Eigenvalue** $\lambda_{\mathbf{F}}$ (smallest eigenvalue) of L_B is positive and simple and satisfies
$$0 < \lambda_F \leq 1$$
- The upper bound of the Fiedler eigenvalue is attained iff all nodes in $\mathcal{G}$ are leaders.
- The **Fiedler Eigenvector** v_F is unique (up to scaling) and is the only positive eigenvector.

- **The relative tempo** is the ratio of velocities of agents,

$$[\bar{\tau}(t)]_i = \frac{\dot{\bar{x}}_i(t)}{\dot{x}_{\text{ref}}(t)}$$

where $\dot{\bar{x}}_{\text{ref}}$ is the velocity of a specific agent chosen as a common divisor for all others.

- The solution for $\dot{\bar{x}} = -\bar{L}\bar{x}$ is given by:

$$\bar{x}(t) = \mathrm{e}^{-\lambda_1(\mathcal{G})t}(\bar{p}_1\bar{x}_0)p_1 + \mathrm{e}^{-\lambda_2(\mathcal{G})t}(\bar{p}_2\bar{x}_0)p_2 + \cdots + \mathrm{e}^{-\lambda_{n+m}(\mathcal{G})t}(\bar{p}_{n+m}\bar{x}_0)p_{n+m}$$

where $\bar{p}_i$ is the ith row of P^{-1}, and p_i is the ith column of P and $\bar{L} = P\Lambda P^{-1}$.

$$[\bar{\tau}(t)]_i = \frac{\dot{\bar{x}}_i(t)}{\dot{x}_{\mathsf{ref}}(t)}$$

- Substituting the solution derivative into the relative tempo expression yields

$$\tau_i = \frac{-\lambda_F(p_{m+1}^T\bar{x}_0)[v_F]_i e^{-\lambda_F t} + \sum_{k=m+2}^{m+n} -\lambda_k(p_k^T\bar{x}_0)[v_k]_i e^{-\lambda_k t}}{-\lambda_F(p_{m+1}^T\bar{x}_0)[v_F]_{ref} e^{-\lambda_F t} + \sum_{k=m+2}^{m+n} -\lambda_k(p_k^T\bar{x}_0)[v_k]_{ref} e^{-\lambda_k t}}, \quad i \in [1, \dots, n].$$

- For sufficient time T:

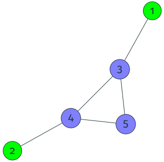
$$[\bar{\tau}(t)]_i \simeq [v_F]_i, \quad t > T$$

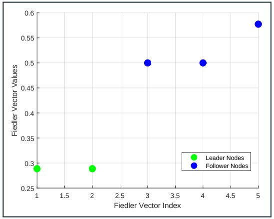
H. Shao and M. Mesbahi, *Degree of relative influence for consensus-type networks*, Portland, OR, USA, 2014, pp. 2676-2681.

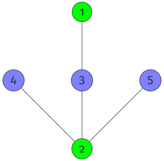
Part 2: Relation Between the Fiedler Eigenvector and Leader Identification

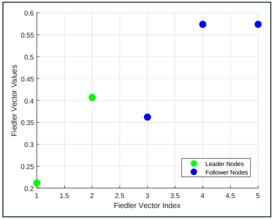
FIEDLER EIGENVECTOR AND SYSTEM TRAJECTORIES

- The Fiedler vector components are associated with the graph structure.

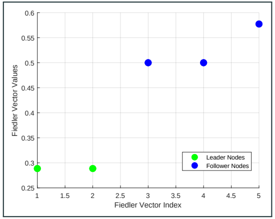


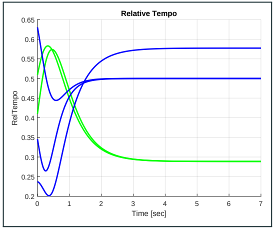






- The velocities of the nodes are linked to the Fiedler vector components.





We will examine a sequence of expanding graphs $\mathcal{G}^{\sigma(i)}$ with some structure constraints:

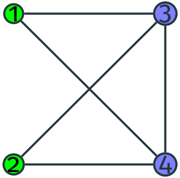
- The leaders set remains constant.
- The leader degree is constant.
- Leader nodes are not connected to each other.

Let $\mathcal{G}_f^{\sigma(i)}$ denote the graph obtained by removing all leader nodes and their incident edges from $\mathcal{G}^{\sigma(i)}$. The additional property in the sequence is as follows:

- The minimum degree in $\mathcal{G}_f^{\sigma(i)}$ is strictly increasing (denoted as $\underline{d}_F$).

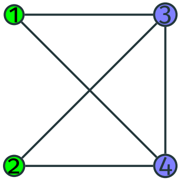
GRAPH SEQUENCE WITH FIEDLER VECTOR SEPARATION

$\underline{d}_F$=1

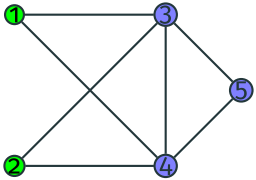


GRAPH SEQUENCE WITH FIEDLER VECTOR SEPARATION

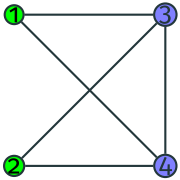
$\underline{d}_F$=1



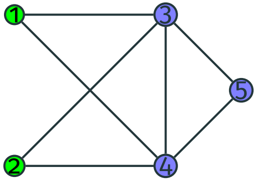
$\underline{d}_F$=2



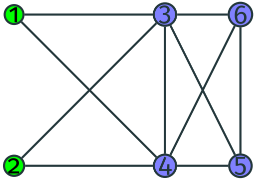
d_F=1



d_F=2

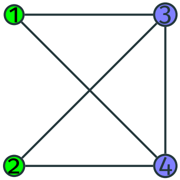


d_F=3

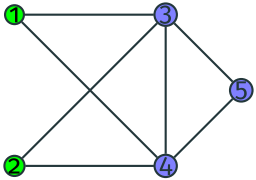


GRAPH SEQUENCE WITH FIEDLER VECTOR SEPARATION

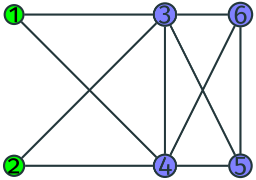
d_F=1



d_F=2

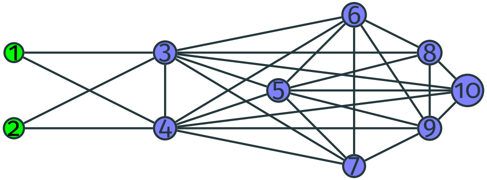


d_F=3



d_F=5

. . .



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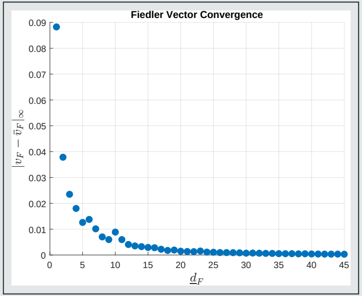
Lemma

The Fidler vector of $\mathcal{G}^{\sigma(i)}$ converges to the following values:

$$\lim_{i\to\infty} [v_F^{\sigma(i)}]_j = [\bar{v}_F]_j \text{ where } [\bar{v}_F]_j = \lim_{i\to\infty} \begin{cases} 1, & j \in \mathcal{V}_f^{\sigma(i)} \\ \frac{\mathbf{d}(j)}{\mathbf{d}(j)+1-\lambda_F^{\sigma(i)}}, & j \in \mathcal{V}_\ell \end{cases}$$

where $d(j)$ is the node degree and λ_F is the Fiedler eigenvalue.



Lemma Proof Sketch.

- Define the **semi-normalized adjacency matrix** as

$$\hat{A} = \hat{D}_{\lambda_F}^{-1} A \in \mathbb{R}^{n \times n},$$

  where $\hat{D}_{\lambda_F} \in \mathbb{R}^{n \times n}$ is given by

$$[\hat{D}_{\lambda_F}]_{ii} = \begin{cases} d(i) + 1 - \lambda_F & i \in \mathcal{V}_\ell \\ d(i) - \lambda_F & i \in \mathcal{V}_F \end{cases}.$$

- Show that v_F is an eigenvector with corresponding eigenvalue $\lambda = 1$.
- Show that

$$\lim_{i \to \infty} [\hat{A}^{\sigma(i)} \bar{v}_F^{\sigma(i)}]_j = [\bar{v}_F]_j$$

Lemma Proof Sketch.

Show that v_F is an eigenvector with corresponding eigenvalue $\lambda = 1$:

- Starting with the relation between L_B and $\hat{D}_{\lambda_F}$

$$L_B = \hat{D}_{\lambda_F} + \lambda_F I - A.$$

- Subtract $\lambda_F I$ from both sides and multiply both side by $(\hat{D}_{\lambda_F})^{-1}$ from left to obtain

$$(\hat{D}_{\lambda_F})^{-1}(L_B - \lambda_F I) = (\hat{D}_{\lambda_F})^{-1}(\hat{D}_{\lambda_F}) - (\hat{D}_{\lambda_F})^{-1}A.$$

- Rearranging above gives

$$(\hat{D}_{\lambda_F})^{-1}A = I - (\hat{D}_{\lambda_F})^{-1}(L_B - \lambda_F I).$$

Lemma Proof Sketch.

$$(\hat{D}_{\lambda_F})^{-1}A = I - (\hat{D}_{\lambda_F})^{-1}(L_B - \lambda_F I).$$

- Note that $(\hat{D}_{\lambda_F})^{-1}A = \hat{A}$, giving

$$\hat{A} = I - (\hat{D}_{\lambda_F})^{-1}(L_B - \lambda_F I).$$

- By multiply both sides by v_F we get:

$$\hat{A}v_F = v_F - (\hat{D}_{\lambda_F})^{-1}(L_B - \lambda_F I)v_F = v_F - (\hat{D}_{\lambda_F})^{-1}(\lambda_F v_F - \lambda_F v_F) = v_F.$$

Therefore, we have $\lambda = 1$ is an eigenvalue of $\hat{A}$ with corresponding eigenvector v_F.

Lemma Proof Sketch.

Show that

$$\lim_{i\to\infty}[\hat{A}^{\sigma(i)}\bar{v}_F^{\sigma(i)}]_j = [\bar{v}_F]_j$$

- Let's looking at the multiplication of $\hat{A}^{\sigma(i)}$ and $\bar{v}_F^{\sigma(i)}$:

$$[\hat{A}^{\sigma(i)}\bar{v}_F^{\sigma(i)}]_j = \begin{cases} (d^{\sigma(i)}(j)-\lambda^{\sigma(i)})^{-1}\left(\sum_{k\in\mathcal{N}^{\sigma(i)}(j)\cap\mathcal{V}_F^{\sigma(i)}}\frac{d^{\sigma(i)}(k)}{d^{\sigma(i)}(k)-\lambda^{\sigma(i)}}+\sum_{k\in\mathcal{N}^{\sigma(i)}(j)\cap\mathcal{V}_\ell}\frac{\mathbf{d}_k}{\mathbf{d}_k+1-\lambda^{\sigma(i)}}\right) \\ (\mathbf{d}_j+1-\lambda^{\sigma(i)})^{-1}\sum_{k\in\mathcal{N}^{\sigma(i)}(j)}\frac{d^{\sigma(i)}(k)}{d^{\sigma(i)}(k)-\lambda^{\sigma(i)}} \end{cases}$$

where the first and second rows corresponding to the follower and leader nodes, respectively. □

Theorem

Let $\mathcal{G}$ be graph where the nodes separated into two groups, leaders $\mathcal{V}_\ell^{\mathcal{G}}$ and followers $\mathcal{V}_f^{\mathcal{G}}$.

If the following conditions are met:

i) $\mathcal{G}$ is connected;
ii) $k \notin \mathcal{N}(j)$ for all $k, j \in \mathcal{V}_\ell$ (leader nodes are not connected to each other);
iii) $\underline{d}_F$ is sufficient large;
iv) $1 - \max_{j \in \mathcal{V}_\ell} \frac{d(j)}{d(j)+1-\lambda_F} > \max_{j,k \in \mathcal{V}_\ell, j>k} |[\bar{v}_{F_s}]_j - [\bar{v}_{F_s}]_k|$ where $v_{F_s} = sort(v_F)$,

then

$$\min_{i \in \mathcal{V}_f} [v_F]_i - \max_{i \in \mathcal{V}_\ell} [v_F]_i > \max_{i,j \in \mathcal{V}_\ell, i>j} |[v_{F_s}]_i - [v_{F_s}]_j|.$$

Theorem Proof Sketch.

Our graph is part of the series defined in the previously stated Lemma:

$$\lim_{i\to\infty} \|c^{\sigma(i)} v_F^{\sigma(i)} - \bar{v}_F^{\sigma(i)}\|^2 = 0,$$

where $[\bar{v}_F^{\sigma(i)}]_j = \begin{cases} 1, & j \in \mathcal{V}_F^{\sigma(i)} \\ \frac{d_j}{d_j+1-\lambda_F^{\sigma(i)}}, & j \in \mathcal{V}_\ell^{\sigma(i)} \end{cases}$ and $c^{\sigma(i)}\|v_F^{\sigma(i)}\| = \|\bar{v}_F^{\sigma(i)}\|$.

For sufficiently large $\underline{d}_F$, the error $S^i = \max_j |c^{\sigma(i)}[v_F^{\sigma(i)}]_j - [\bar{v}_F^{\sigma(i)}]_j|$ satisfies:

$$S^i < \frac{1 - \max_{j\in\mathcal{V}_\ell} \frac{d(j)}{d(j)+1-\lambda_F} - \max_{j\in\mathcal{V}_\ell} \min_{k\in\mathcal{V}_\ell} |\frac{d(j)}{d(j)+1-\lambda_F} - \frac{d(k)}{d(k)+1-\lambda_F}|}{4}$$

Theorem Proof Sketch.

$$S^i < \frac{1 - \max_{j \in \mathcal{V}_\ell} \frac{d(j)}{d(j)+1-\lambda_F} - \max_{j \in \mathcal{V}_\ell} \min_{k \in \mathcal{V}_\ell} |\frac{d(j)}{d(j)+1-\lambda_F} - \frac{d(k)}{d(k)+1-\lambda_F}|}{4}$$

Let's pick i^*, $\underline{d}_F^{\sigma(i^*)} = m$ which lead to the following value $\epsilon < \frac{\epsilon_d}{4}$.
We got:

$$\begin{aligned}
\min_{j \in \mathcal{V}_F} [c^{\sigma(i)} v_F^{\sigma(i)}]_j - \max_{j \in \mathcal{V}_\ell} [c^{\sigma(i)} v_F^{\sigma(i)}]_j &> 1 - \max_{j \in \mathcal{V}_\ell} \frac{d_j}{d_j + 1 - \lambda_F} - 2\epsilon \\
&= \max_{j \in \mathcal{V}_\ell} \min_{k \in \mathcal{V}_\ell} |\frac{d_j}{d_j + 1 - \lambda_F} - \frac{d_k}{d_k + 1 - \lambda_F}| + \epsilon_d - 2\epsilon \\
&> \max_{j \in \mathcal{V}_\ell} \min_{k \in \mathcal{V}_\ell} c^{\sigma(i)} |[v_F^{\sigma(i)}]_j - [v_F^{\sigma(i)}]_k| + \epsilon_d - 4\epsilon \\
&> \max_{j \in \mathcal{V}_\ell} \min_{k \in \mathcal{V}_\ell} c^{\sigma(i)} |[v_F^{\sigma(i)}]_j - [v_F^{\sigma(i)}]_k|
\end{aligned}$$

$\square$

Assuming existence of the result from the theorem, i.e.,

$$\min_{i\in\mathcal{V}_f}[v_F]_i - \max_{i\in\mathcal{V}_\ell}[v_F]_i > \max_{i,j\in\mathcal{V}_\ell,i>j} |[v_{F_s}]_i - [v_{F_s}]_j|.$$

we use the following algorithm to identify the leaders

Algorithm

- Step 1: Measure the agents velocities to an external constant input until steady state.
- Step 2: Calculate the relative tempo and compute the Fiedler vector.
- Step 3: Sort the Fiedler vector $v_{F_s} = \mathrm{sort}(v_F)$ where $[v_{F_s}]_i \leq [v_{F_s}]_{i+1}$.
- Step 4: Calculate the number of leaders n_l with

$$n_l = |\mathcal{V}_\ell| = \arg\max_{j\in\{1,2,3,\cdots,n-1\}} \{[v_{F_s}]_{j+1} - [v_{F_s}]_j\}.$$

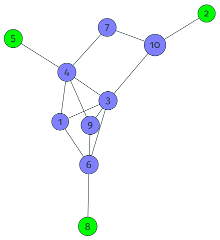
- Step 5: The leaders are corresponding to the smallest n_l components in v_{F_s}.

In this example, we demonstrate a 2D scenario. We consider a system with $n = 10$ agents, where $\{2, 5, 8\} \in \mathcal{V}_L$. Recall the protocol dynamics:

$$\dot{x}_i = \begin{cases} \sum_{j\sim i}(x_j - x_i) + (u_i^{\text{ex}} - x_i), & i \in \mathcal{V}_\ell, \\ \sum_{j\sim i}(x_j - x_i), & i \in \mathcal{V}_f. \end{cases}$$

The external input provided to the leaders is

$$u = \begin{bmatrix} 40 & 35 & 48 & 44 & 16 & 45 \end{bmatrix}^T$$



EXAMPLE CONT.

The grounded Laplacian and the Fiedler vector is given by:

$$L_B = \begin{bmatrix} 3 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 5 & -1 & 0 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 5 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 4 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 3 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 3 \end{bmatrix}, \quad v_F = \begin{bmatrix} 0.37 \\ \mathbf{0.18} \\ 0.37 \\ 0.35 \\ \mathbf{0.19} \\ 0.34 \\ 0.37 \\ \mathbf{0.19} \\ 0.37 \\ 0.32 \end{bmatrix}$$

Next, we verify the conditions outlined in the Theorem:

- Leaders are not connected to each other.
- Degree distribution condition.
- $\underline{d}_F$ is sufficient large.

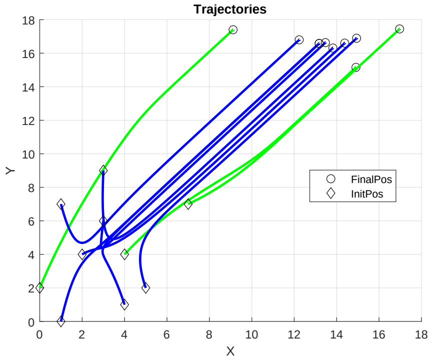
Since all conditions are satisfied, the leaders can be identified using the suggested algorithm.

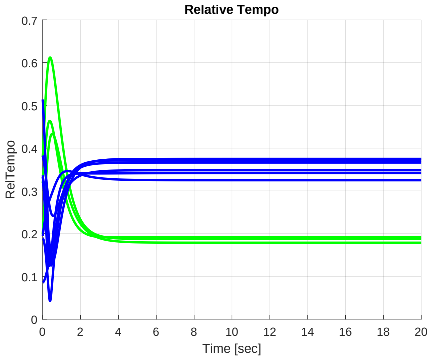
EXAMPLE CONT.

I. Measure the velocities and calculate the relative tempo:

$$\tau = \begin{bmatrix} 0.37 & 0.18 & 0.37 & 0.35 & 0.19 & 0.34 & 0.37 & 0.19 & 0.37 & 0.32 \end{bmatrix}^T$$

We note that this is equal to the Fiedler vector v_F.





II. Identify Leaders

- Sort the Fiedler vector $v_{F_s} = \mathrm{sort}(v_F)$ where $[v_{F_s}]_i \leq [v_{F_s}]_{i+1}$:

$$v_{F_s} = \begin{bmatrix} 0.18 & 0.19 & 0.19 & 0.32 & 0.34 & 0.35 & 0.37 & 0.37 & 0.37 & 0.37 \end{bmatrix}^T$$

Index = $\begin{bmatrix} 2 & 5 & 8 & 10 & 6 & 4 & 7 & 9 & 3 & 1 \end{bmatrix}^T$

- Calculate the number of leaders n_l with

$$n_l = |\mathcal{V}_\ell = \arg \max_{j \in \{1,2,3,\cdots,n-1\}} \{[v_{F_s}]_{j+1} - [v_{F_s}]_j\} = 3.$$

- The leaders correspond to the smallest n_l components in v_{F_s}:

$$\mathcal{V}_\ell = \{2, 5, 8\}$$

- Certain graph structures are more likely to be associated with separation in the components of the Fiedler vector.
- Such graphs can facilitate leader identification through external observation in scenarios with constant external input.

FUTURE WORK

- Investigate scenarios involving non-constant external input signals.
- Develop methods for identifying the complete network structure.
- Explore additional graph topologies related to component separation in the Fiedler vector.