ON STRUCTURAL RANK AND NETWORK RESILIENCE

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July 5, 2021
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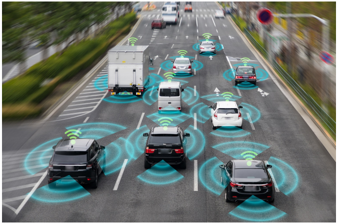
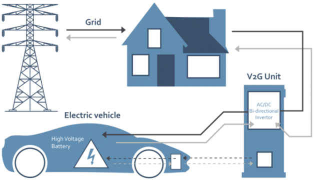
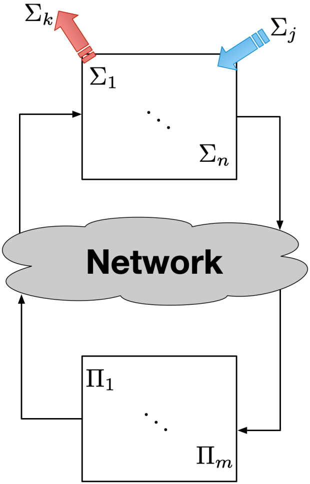
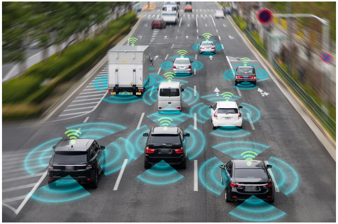


network of self-driving cars



smart-grid with EV integration





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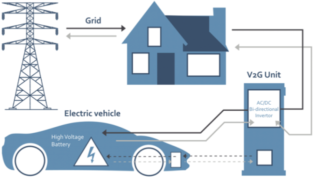

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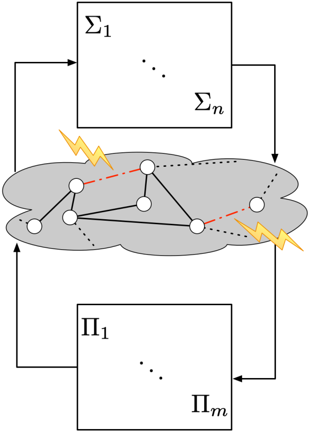
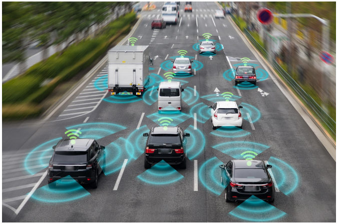


OPEN MULTI-AGENT SYSTEMS



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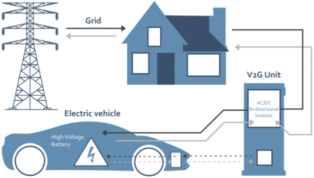

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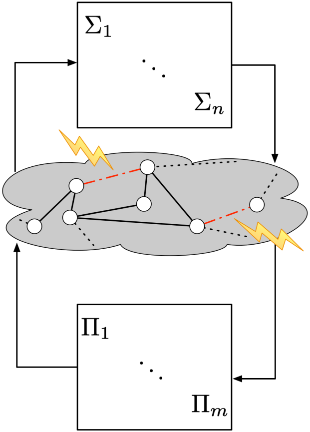
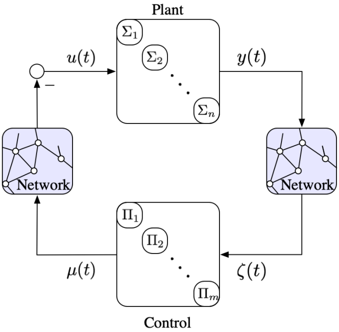


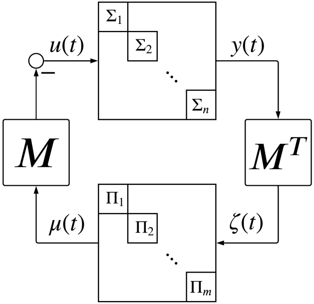
Resillience and robustness of network systems required for safe operations



$(\Sigma, \Pi, \mathcal{G})$

Components of a networked system

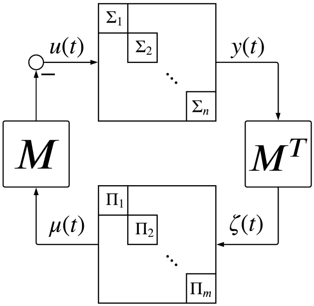
- agents - dynamical systems that should interact with eachother to achieve some goal
- network - communication and sensing infrastructure for sharing of information
- controllers - computational nodes that process information from the network to make decisions for each agent



(Σ, Π, M)

Network Interconnection

- Network is encoded by a matrix $M \in \mathbb{R}^{n\times m}$
- $[M]_{ij} = \begin{cases} \star, & \text{controller } j \text{ access to agent } i \\ 0, & \text{otherwise} \end{cases}$



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A Stability Result

The stability of the dynamic network (Σ, Π, M) can be guaranteed for output-strictly passive agent dynamics Σ_i and passive controller dynamics Π_e.

[Corollary of B&Z 2014]

PASSIVATION BY THE NETWORK

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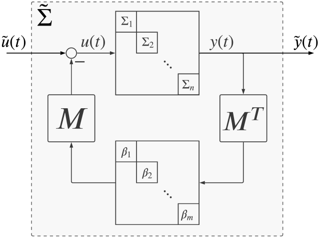
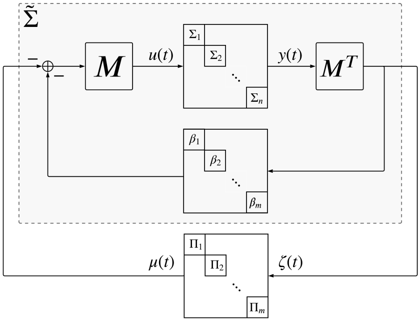


- ρ_i is passivity index of each agent
  - $\rho_i = 0$: passive
  - $\rho_i > 0$: strictly output-passive
  - $\rho_i < 0$: output passive short
- $R = \mathrm{diag}(\rho_1, \dots, \rho_n)$

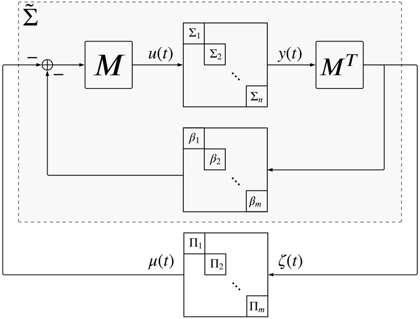
Lemma

Assume that $\rho_i < 0$ for at least one agent. If $R + M\mathrm{diag}(\beta)M^T$ is positive definite, then $\tilde{\Sigma} : \tilde{u}(t) \mapsto \tilde{y}(t)$, is output-strict passive with respect to any steady-state input-output pair. Furthermore, there exists scalars β_i, $i = 1, \dots, m$ such that $R + M\mathrm{diag}(\beta)M^T > 0$ if and only if $x^T R x > 0$ for any $x \in \ker(M^T)$.

PASSIVATION BY THE NETWORK



- if $M^T M$ is full-rank, we can always passivy the systems with a constant network gain β
- stability of network is guaranteed for any passive controllers and correct gain β



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A Question

For a given network matrix M, how many of its entries can be changed to a 0 before MM^T loses rank?

A sparsity pattern $\mathcal{S}(n,m) \subset \mathbb{R}^{n\times m}$ is a vector subspace that admits a basis containing only standard basis basis matrices E_{ij}

- $E(\mathcal{S}) = \{(i,j) \mid S \in \mathcal{S}(n,m) \text{ has } [S]_{ij} = \star\}$
- $\dim(\mathcal{S}(n,m)) = |E(\mathcal{S})|$

$$\begin{bmatrix} \star & 0 & 0 \\ 0 & \star & \star \\ 0 & 0 & 0 \end{bmatrix} = \star E_{11} + \star E_{22} + \star E_{23}$$

RANK OF SPARSITY PATTERNS

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The rank of a sparsity pattern $\mathcal{S}(n,m)$, denoted $\operatorname{rk}\mathcal{S}(n,m)$, is the maximal value of the ranks of matrices in $\mathcal{S}(n,m)$.

- Given patterns $\mathcal{S}(n,m)$ and $\mathcal{S}'(n,m)$, we write $\mathcal{S}(n,m) \succeq \mathcal{S}'(n,m)$ if $E(\mathcal{S}) \subseteq E(\mathcal{S}')$

Rank Resilience

Given positive integers n and m with $m \geq n$, a sparsity pattern $\mathcal{S}(n,m)$ is k-resilient, for $0 \leq k \leq |E(\mathcal{S})|$, if the following hold:

i) All patterns $\mathcal{S}' \preceq \mathcal{S}$ with $|E(\mathcal{S}')| \geq |E(\mathcal{S})| - k$ are of rank n

ii) There exists a $\mathcal{S}' \preceq \mathcal{S}$ with $|E(\mathcal{S}')| = |E(\mathcal{S})| - k - 1$ whose rank is less than n.

We say that $\mathcal{S}$ is strongly k-resilient if it contains a direct sum of $(k+1)$, but not $(k+2)$ patterns each of which is 0-resilient.

- $\mathrm{rsl}(\mathcal{S})$ denotes degree of resilience of $\mathcal{S}$
- $\mathrm{s-rsl}(\mathcal{S})$ denotes degree of strong resilience

P0: Given a sparsity pattern $\mathcal{S}(n,m)$, what is its degree of (strong) resilience?

P1: Given a sparsity pattern $\mathcal{S}(n,m)$, what is the least number of $\star$-entries one should add to obtain a degree of (strong) resilience k? This can be expressed as a solution to the following problem,

$$\min |E(\mathcal{S}')| \text{ s.t. } \mathcal{S}' \succeq \mathcal{S} \text{ with } \text{(s-)rsl}(\mathcal{S}') = k. \tag{1}$$

P2: Given a sparsity pattern $\mathcal{S}$, what is the largest degree of (strong) resilience we can achieve by adding p $\star$-entries? This is equivalent to the following problem,

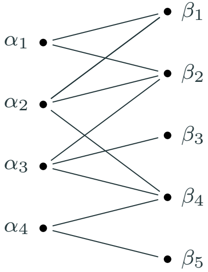
$$\max \text{(s-)rsl}(\mathcal{S}') \text{ s.t. } \mathcal{S}' \succeq \mathcal{S} \text{ with } |E(\mathcal{S}')| = |E(\mathcal{S})| + p. \tag{2}$$

BIPARTITE GRAPHS AND SPARSITY PATTERNS

Every sparsity pattern can be associated with a bipartite graph

$$\begin{pmatrix} \star & \star & 0 & 0 & 0 \\ \star & \star & 0 & \star & 0 \\ 0 & \star & \star & \star & 0 \\ 0 & 0 & 0 & \star & \star \end{pmatrix}$$

(a)

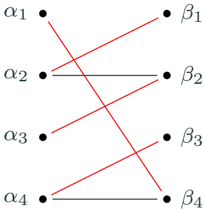


(b)

- if ijth entry of $\mathcal{S}$ is a $\star$, then (α_i, β_j) is an edge
- $G(n, m) = (V_\alpha, V_\beta, E)$
  - $\alpha_1, \ldots, \alpha_n$ are left-nodes
  - $\beta_1, \ldots, \beta_m$ are right-nodes

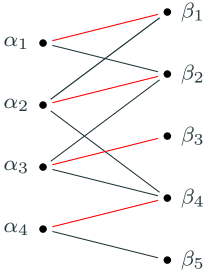
Perfect Matchings

If $n = m$, a perfect matching P in $G(n, n)$ is a set of n edges such that each node of $G(n, n)$ is incident to exactly one of these n edges.



Left- and Right-Perfect Matchings

A bipartite graph $G(n,m) = (V_\alpha \cup V_\beta, E)$, with $m \geq n$, admits a *left-perfect matching* if there exist n distinct right nodes $\beta_{i_1}, \ldots, \beta_{i_n}$ such that the subgraph $G'(n,n)$ induced by $V_\alpha \cup \{\beta_{i_1}, \ldots, \beta_{i_n}\}$ has a perfect matching. Similarly, one can define *right-perfect matching* for the case $n \geq m$.

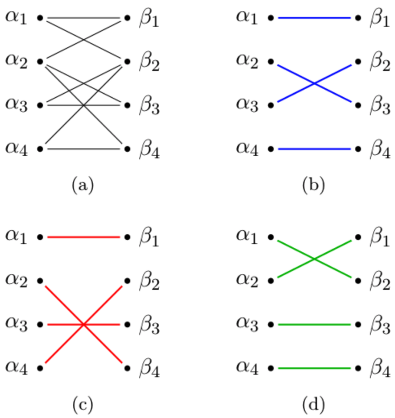


Lemma

i) A sparsity pattern $\mathcal{S}(n, m)$ is of rank n if and only if its associated bipartite graph $G(n, m)$ admits a left-perfect matching.

ii) A bipartite graph $G(n, m)$ is k-resilient if and only if for *any* subset $E' \subset E$ with $|E'| = k$, $G'(n, m) = (V_\alpha \cup V_\beta, E - E')$ contains a left-perfect matching.

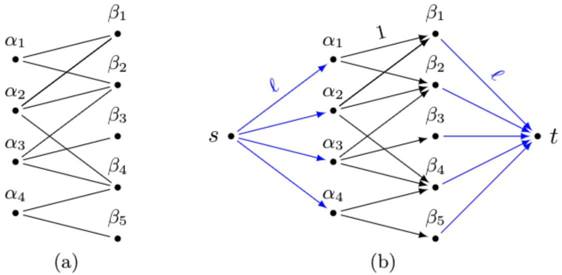
iii) A bipartite graph $G(n, m)$ is strongly k-resilient if and only if it has exactly $(k + 1)$ *disjoint* left-perfect matchings.



- graph in (a) contains 3 distinct perfect matchings (b,c,d)
- there is no common edge in the perfect matchings
- since $\deg(\alpha_i) = \deg(\beta_i) = 2$ for $i = 1, 3, 4$, the graph in (a) is 1-resilient
- since the pairwise intersections of the matchings is not empty, it is not strongly 1-resilient

HOW TO DETERMINE THE RANK RESILIENCE (PO)

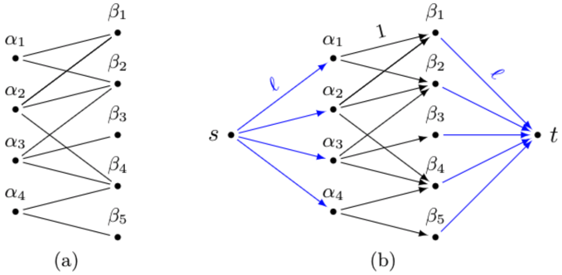
equivalent to a corresponding max-flow problem



- add two new nodes to $G(n,m)$ - source s and sink t
- add edges $\{s\alpha_i, \beta_j t \mid 1 \leq i \leq n, 1 \leq j \leq m\}$
- assign capacities to each edge (ℓ to new edges, 1 to edges in $G(n,m)$)
- F_ℓ are flows on $\bar{G}(n,m)$: for $f \in F_\ell$, $|f| = \sum_{\beta_j \in V_\beta} f(\beta_j t)$

HOW TO DETERMINE THE RANK RESILIENCE (PO)

equivalent to a corresponding max-flow problem

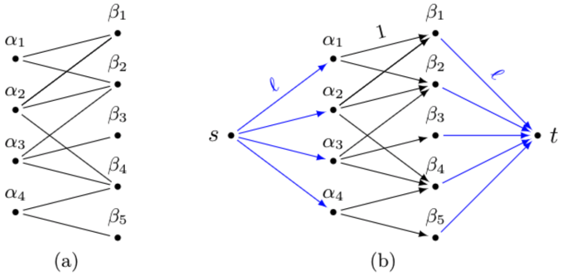


Definition

Given the digraph $\bar{G}(n, m)$ and a nonnegative integer ℓ, we say that a flow $f \in F_\ell$ on $\bar{G}(n, m)$ is *saturated* if $|f| = n\ell$. We denote by $\bar{F}_\ell$ the set of saturated flows on $\bar{G}(n, m)$.

HOW TO DETERMINE THE RANK RESILIENCE (PO)

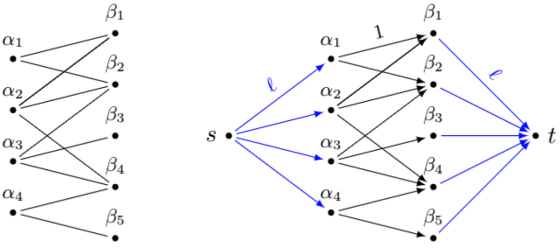
equivalent to a corresponding max-flow problem



- saturated flows are always max flows
- saturated flows are always integer (inegrality theorem)
- Given a flow $f \in \bar{F}_\ell$ on $\bar{G}$, we define *the subgraph of* $G(n, m)$ *induced by the flow* f as follows:

$$G_f(n, m) := (V_\alpha \cup V_\beta, E_f) \text{ with } E_f := \{(\alpha_i, \beta_j) \in E \mid f(\alpha_i\beta_j) \neq 0\}.$$

equivalent to a corresponding max-flow problem



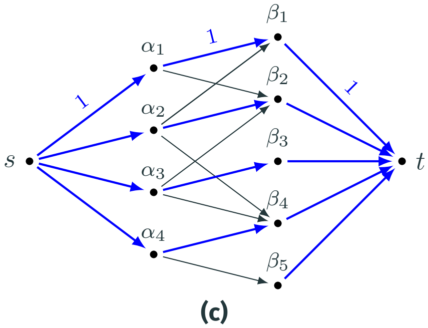
(a) (b)

Theorem

1. If $\bar{F}_1 = \varnothing$, then there does not exist a left-perfect matching in $G(n, m)$.
2. If $\bar{F}_1 \neq \varnothing$, then, the degree of strong resilience of $G(n, m)$ is given by
$$\text{s-rsl}\, G(n,m) = \max\left\{\ell \geq 1 \mid \bar{F}_\ell \neq \varnothing\right\} - 1.$$

EXAMPLE

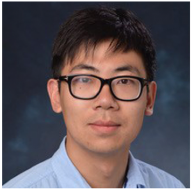
$G(4,5)$ is strongly 1-resilient



- shows saturated flows f_ℓ for $\ell = 1$ in (a) and $\ell = 2$ in (b)
- edges with non-zero values are in blue
- for $\ell = 2$, two disjoint left-perfect matchings are $\{(\alpha_1, \beta_1), (\alpha_2, \beta_4), (\alpha_3, \beta_2), (\alpha_4, \beta_5)\}$ and $\{(\alpha_1, \beta_2), (\alpha_2, \beta_1), (\alpha_3, \beta_3), (\alpha_4, \beta_4)\}$

- structural rank and rank resilience provides a new way to think of network robustness
- structural rank intimately related to perfect matchings in bipartite graphs
- max-flow algorithms provide constructive approach to determine rank resilient properties
- a contribution to the canon of structural control theory (controlability, stabilty)

ACKNOWLEDGEMENTS



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