Cyclically-Monotone Relations and their use in Passivity-Based Cooperative Control

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University of Groningen 16.02.2017

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Duality in Cooperative Control Problems

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networks of dynamical systems are one of *the* enabling technologies of the future

dynamics

 $\dot{x}_i(t) = f_i(x_i(t), u_i(t))$

Analysis

- steady-state behavior
- interplay between dynamics and graph
- equilibrium configurations

Synthesis

- design of distributed protocols
- design of "good" network structures
- robust

can we reveal *deep* **results describing the underlying behavior of these systems?**

in this talk…

Network Optimization

- optimal flow/optimal potential problems
- monotone/cyclically monotone relations and convex functions

Passivity-based Cooperative Control

- \bullet equilibrium independent passivity
- steady-state input/output maps

Duality Theory for Cooperative Control

Network Optimization

Network Flows and Monotropic Optimization

"...in fact, the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity." R. Tyrrell Rockafellar SIAM Review, 1993

Shortest Path Problem

Max-Flow Problem

Minimum Cost Flow Problem

Network Definitions

A **network (graph)** is a mathematical structure used to model pairwise $\sqrt{2}$ relations between objects. $\frac{1}{\sqrt{2}}$

$$
\mathcal{G} = (\mathbf{V}, \mathbf{E})
$$

Passivity-based Cooperative Control χ¨ⁱ = fi(˙χi) + ! k=(i,j)∈E ψk(χ^j − χi), i ∈ V *Graph*: G = (V, E) (undirected)

Incidence Matrix

$$
E(\mathcal{G}) = \mathbb{R}^{|\mathbf{V}| \times |\mathbf{E}|}
$$

\n
$$
[E]_{ik} = \begin{cases} +1 & \text{if } i \text{ is positive end of } k \\ -1 & \text{if } i \text{ is negative end of } k \\ 0 & \text{otherwise} \end{cases}
$$

Network Optimization

$$
u+E(\mathcal{G})\mu=0
$$

$$
\zeta = E^T(\mathcal{G})\mathbf{y}
$$

"flow networks" "potential networks" "flow networks" "potential networks"

 $\mu' \zeta = -y' u.$ theory broadly connects elements of graph theory to a family of convex optimization problems. The beauty of this problems. The beauty of this problem is a family of this problem is a family of this problem is a family of *conversion formula*

Network Optimization Optimal Flow Problem Optimal Flow Problem Optimal Potential Problem $\sum_{i=1}^{\infty}$

Optimal Flow Problem O or time al Γ la explicit ρ *|*E*|* **Optimal** Optimal Flow Problem

$$
\min_{\mathbf{u}, \mu} \quad \sum_{i=1}^{|\mathbf{V}|} C_i^{div}(\mathbf{u}_i) + \sum_{k=1}^{|\mathbf{E}|} C_k^{flux}(\mathbf{\mu}_k)
$$

s.t. $u + E\mu = 0$.

- u_i : *divergence* (in/out-flow) \parallel at a node node
Contractor Contractor Contractor Contractor Contractor Contractor Contractor Contractor Contractor Contractor
Contractor Contractor Contractor Contractor Contractor Contractor Contractor Contractor Contractor Contract u*i*: *divergence* (in/out-flow) at a u*i*: *divergence* (in/out-flow) at a
- \mathfrak{u}_k : *flow* on an edg µ*k*: *flow* on an edge µ*k*: *flow* on an edge *flow* on an edge μ_k : *flow* on an e

Optimal Flow Problem Optimal Potential Problem Optimal Potential Problem Optimal Potential Problem *|*V*| |*E*|* **II** Optimal Potential Pro Optimal Potential Problem

$$
\min_{\mathbf{y}, \zeta} \sum_{i=1}^{|\mathbf{V}|} C_i^{pot}(\mathbf{y}_i) + \sum_{k=1}^{|\mathbf{E}|} C_k^{ten}(\zeta_k)
$$

s.t. $\zeta = E^{\top} \mathbf{y}$.

y*i*: *potential* at a node y*i*: *potential* at a node y*i*: *potential* at a node *potential* at a node y*i*: *potential* at a node y*i*: *potential* at a node y*i*: *potential* at a node

tension (potential difference) across an potential difference) n edge ⇣*k*: *tension* (potential di↵erence) $\frac{1}{2}$ an edge *tension* (potential difference) across an edge *k*: *tension* (potential difference) S^{κ} across a ζ_k : $\frac{1}{2}$ ζ_k : ζ_k ζ_k ζ_k ζ_k ζ_k ζ_k ζ_k acios₃ *tension* (potential difference) across a

Dual Optimization Problems defined over the "same" network Both problems are *duals*: Both problems are *duals*: Both problems are *duals*: Both problems are *duals*:

$$
C_i^{pot}(\mathbf{y}_i) := C_i^{div,*} = -\inf_{\tilde{\mathbf{u}}_i} \left\{ C_i^{div}(\tilde{\mathbf{u}}_i) - \mathbf{y}_i \tilde{\mathbf{u}}_i \right\}
$$

Cpot

Network Optimization

Max Flow-Min Cut Theorem

The maximum value of an S-T flow is equal to the minimum capacity over all s-t cuts.

Elegant illustration of *Duality Theory*

Diffusively Coupled Networks

Diffusively Coupled Networks

Kumamoto Model

$$
\dot{\theta}_i = -k \sum_{i \sim j} \sin(\theta_i - \theta_j)
$$

Traffic Dynamics Model

$$
\dot{v}_i = \kappa_i \left(V_i^0 - v_i + V_i^1 \sum_{i \sim j} \tanh(p_j - p_i) \right)
$$

Neural Network $C\dot{V}_i = f(V_i, h_i) + \sum_{i \sim j} g_{ij}(V_j - V_i)$ \dot{h}_i = $g(V_i, \dot{h}_i)$

Duality and Cooperative Control

Optimal Flow Problem Optimal Flow Problem

$$
\min_{\mathbf{u}, \mu} \sum_{i=1}^{|\mathbf{V}|} C_i^{div}(\mathbf{u}_i) + \sum_{k=1}^{|\mathbf{E}|} C_k^{flux}(\mu_k)
$$

s.t.
$$
\mathbf{u} + E\mu \neq 0.
$$

Optimal Potential Problem \overrightarrow{t} Optimal Potential P

$$
\min_{\mathbf{y}, \zeta} \sum_{i=1}^{|\mathbf{V}|} C_i^{pot}(\mathbf{y}_i) + \sum_{k=1}^{|\mathbf{E}|} C_k^{ten}(\zeta_k)
$$

s.t. $\zeta = \boxed{E^{\top} \mathbf{y}}$.

λ Ω iterated systems **The Output Agreement Problem**

Plant: Dynamics on nodes *<u>Iante Dynamics</u>*

$$
\Sigma_i: \quad \dot{x}_i(t) = f_i(x_i(t), u_i(t), \mathbf{w}_i)
$$

$$
y_i(t) = h_i(x_i(t), u_i(t), \mathbf{w}_i)
$$

Controllers: Dynamics on edges

$$
\Pi_k: \quad \dot{\eta}_k(t) = \zeta_k(t) \n\mu_k(t) = \psi_k(\eta_k(t), \zeta_k(t))
$$

tion via gra Interconnection via graph incidence matrix

$$
\begin{cases} \n\zeta(t) & = E(\mathcal{G})^T y(t) \\ \nu(t) & = E(\mathcal{G})\mu(t) \n\end{cases}
$$

Plant

1see, e.g., M. Arcak: Passivity as a Design Tool for Group Coordination, TAC, 2007 **Control Objective** lim $t \rightarrow \infty$ $\zeta(t)=0$

Controller

Necessary Conditions for Output Agreement

Lemma

If the networked system has a steady-state solution, u, y , then the solution must satisfy $u \in \mathcal{R}(E(\mathcal{G})), y \in \mathcal{N}(E^T(\mathcal{G})) = \text{span} \{1\}$

steady-state input **u** • controller must be able to generate the

 $\mathbf{y} = \beta \mathbf{1}$ • output agreement means output of each agent is identical, i.e.,

Passivity for Cooperative Control

a "classic" result…

- assume there exists constant signals $\mathbf{u}, \mathbf{y}, \boldsymbol{\mu}, \boldsymbol{\zeta}$ s.t. $\mathbf{u} = -E\boldsymbol{\mu}, \boldsymbol{\zeta} = E^T\mathbf{y}$
- each dynamic system is output strictly passive with respect to u*i,* y*ⁱ*

$$
\frac{d}{dt}S_i(x_i(t)) \le (y_i(t) - y_i)(u_i(t) - u_i) - \rho_i ||y_i(t) - y_i||^2
$$

• each controller is passive with respect to ζ_k , μ_k *iii*
 e each controller is passive with respect to ℓ , II, *function Wk*(⌘*k*(*t*)) *such that*

 $\mathcal{L}(\mathcal{X})$ and $\mathcal{L}(\mathcal{X})$

i i X i X i i X i X i i X i i X i i X i i i i i

S˙

$$
\frac{d}{dt}W_k(\eta_k(t)) \leq (\mu_k(t) - \mu_k)(\zeta_k(t) - \zeta_k)
$$

 \mathbf{T} heorem \mathbf{T} **NOW, THE BASIC CONVERGENCE RESULTS OF A REPORT OF A RESULTS OF A** The basic convergence result for the basic convergence re **Theorem** [Arcak 2007]

 $X \sim \mathbb{R}^{n \times n}$

S˙

Theorem 3.4 (Convergence of Passive Networks) *Consider the dynamical network* (6)*,* (7)*,* (8)*,* (9) *and suppose As*network output converges to the constant value y, i.e, Suppose the above assumptions are satisfied. Then the

$$
\lim_{t \to \infty} y(t) = \mathbf{y}
$$

i i X i X i i X i X i i X i i X i i X i i i i i

⇢*i*k*yi*(*t*) ^y*i*k² + (*y*(*t*) ^y)

Passivity Shortcomings

a critical assumption is the existence of constant signals

$$
\mathbf{u} = -E\mu, \, \zeta = E^T \mathbf{y}
$$

equilibrium depends on all properties "globally"

can not be verified "locally"

Equilibrium Independent Passivity

Definition [Hines et. al. Automatica 2011]

A control system

$$
\begin{array}{rcl}\n\dot{x} & = & f(x, u) \\
y & = & h(x, u)\n\end{array}
$$

is *equilibrium independent passive* (EIP) if

- i) \exists a set *U* and *function* $k_x(u)$ s.t. $f(k_x(u), u) = 0 \,\forall u \in \mathcal{U}$
- ii) the system is passive with respect to the equilibrium input-output pair $u, y = h(k_x(u), u)$

Equilibrium Independent Passivity

Lemma [Hines et. al. Automatica 2011]

If Σ is EIP, then $k_y(u)$ is monotonically increasing.

Equilibrium input-output maps are *monotone functions!*

Equilibrium Independent Passivity

Lemma [Hines et. al. Automatica 2011]

If Σ is EIP, then $k_y(u)$ is monotonically increasing.

Passivity-based Cooperative Control Equilibrium input-output maps are *monotone functions!*

but…

the integrator is *passive w.r.t.*
$$
U = \{0\}
$$

and any output $y \in \mathbb{R}$

 $\dot{x}(t) = u(t)$ $y(t) = x(t)$

 $S(x(t)) = \frac{1}{2}(x(t) - y)^2$ $\frac{1}{2} (x(t) - y)^2$ storage function

 $\overline{}$

 \sum Equilibrium input-c Equilibrium input-output map is not a function! Equilibrium input-output map is *not* a *function*!

 \mathbf{u}

y

Monotone Relations

 M onotone Polations complete non-decreasing curves Maximal Monotone Relations - complete non-decreasing curves in \mathbb{R}^2

a relation is *maximal monotone* if it cannot be embedded *into a larger monotone relation* \overline{O} into a larger monotone relation

$$
(i) \quad \text{for arbitrary } (u, y) \in k_y \text{ and } (u', y') \in k_y
$$
\n
$$
k_y \text{ is maximal monotone} \Leftrightarrow \quad (ii) \quad \text{for arbitrary } (u, y) \le (u', y') \text{ or } (u, y) \ge (u', y')
$$
\n
$$
\text{s.t. neither } (u, y) \le (u', y') \text{ nor } (u, y) \ge (u', y')
$$

Maximal EIP

Definition

The dynamical SISO system

$$
\dot{x}(t) = f(x(t), u(t), w)
$$

$$
y(t) = h(x(t), u(t), w)
$$

is *maximal equilibrium independent passive* if there exists a maximal monotone relation $k_y \text{ }\subset \mathbb{R}^2$ such that for all $(u, y) \in k_y$ there exists a positive semi-definite storage function $S(x(t))$ satisfying

$$
\frac{d}{dt}S(x(t)) \le (y(t) - y)(u(t) - u).
$$

Furthermore, it is *output-strictly maximal equilibrium independent passive* if additionally there is a constant $\rho > 0$ such that

$$
\frac{d}{dt}S(x(t)) \le (y(t) - y)(u(t) - u) - \rho ||y(t) - y||^2.
$$

Maximal EIP

... ... *G G* Σ_1 $\left(\Sigma_{2}\right)$ Π_1 $\sqrt{\Pi_2}$ $\begin{bmatrix} \prod_m \end{bmatrix}$ \sum_n $E(G)$ $E(G)^{T}$ *u*(*t*) *y*(*t*) $\mu(t)$ $|\tilde{ }$ ($\overline{ \Pi_2}$) $|\tilde{ }$ $\zeta(t)$ output strictly maximal EIP

maximal EIP

Necessary Conditions (revisited)

Lemma

If the networked system has a steady-state solution, u, y , then the solution must satisfy $u \in \mathcal{R}(E(\mathcal{G})), y \in \mathcal{N}(E^T(\mathcal{G})) = \text{span} \{1\}$

and

$$
\mathbf{y} \in k_y(\mathbf{u})
$$

A "network feasibility problem"

Duality in Cooperative Control duality in Cooperative Cooperative Control of the Cooperative Cooperative Control of the Cooperative Cooperativ
The Cooperative Cooperative Cooperative Cooperative Cooperative Cooperative Cooperative Cooperative Cooperativ **Monotone Relations and Convex Functions**

Theorem [Rockafellar, *Convex Analysis*]

The sub-differential for the closed proper and relations of the sub-differential for the closed proper convex functions on R are the maximal **Theorem:** Theorem: The subditions from $\mathbb R$ to $\mathbb R$. *R.T. Rockafellar: Convex Analysis* $\begin{array}{ccc} \text{I} & \text{I} & \text{I} & \text{I} & \text{I} \\ \text{I} & \text{I} & \text{I} & \text{I} & \text{I} \end{array}$ on R are the maximal monotone relations from R to R.

integral function of equilibrium i/o map

u

 $k_{y}(\mathrm{u})=h(k_{x}(\mathrm{u}),\boldsymbol{\rm u}))$

Duality in Cooperative Control

 $\partial K_i(\mathbf{u}_i) = k_{\mathbf{v},i}(\mathbf{u}_i).$ integral function of Equilibrium I/O Maps are *Convex!*

ky,i

Maximal passivity Kⁱ is proper, closed *convex function*.

Duality in Cooperative Control \mathbf{f}_{max} for \mathbf{f}_{max} are the equilibrium input-output relations, i.e., \mathbf{f}_{max} ly in cooperative cor

$$
K_i(\mathbf{u}_i) \ (= \mathbf{K}(\mathbf{u})) \qquad \min_{\mathbf{y}_i} \sum_{i=1}^{|\mathbf{V}|} K_i^*(\mathbf{y}_i) \ (= \mathbf{K}^*(\mathbf{y}))
$$

$$
E\mathbf{\mu} = 0.
$$
 s.t. $E^{\top}\mathbf{y} = 0.$

Optimal Potential Problem

$$
K_i^*(\mathrm{y}_i)=\sup_{\mathrm{u}_i}\{\mathrm{y}_i\mathrm{u}_i-K_i(\mathrm{u}_i)\}
$$

solution to (OFP1)*, (ii)* y *is an optimal solution to* (OPP1)*, and (iii) in the steady-state* (OFP1) *and* (OPP1) *have*

ⁱ = *K*?

Duality in Cooperative Control \mathbf{f}_{max} for \mathbf{f}_{max} are the equilibrium input-output relations, i.e., \mathbf{f}_{max} ly in cooperative cor

Theorem The form is of the form of an optimal flow problem (4). The cost on the divergences (in/out-flows) u 2 R*_|V*_|V[|] prem is in the form (5). The conjugates of the integral functions of the equilibrium input-to-output-to-output-to-output-to-output-to-output-to-output-to-output-to-output-to-output-to-output-to-output-to-output-to-ou

Assume all the node dynamics are maximal EIP. If the networ Assume all the node dynamics are maximal EIP. If the networked enforces a balancie of the potential over the complete network. The problem can be written in the standard form (5), by choosing *Cten k* solution for the point $\mathbf{u}, \mathbf{y}, \text{other}$ system has a steady-state solution u*,* y, then

- Optimal Potential Problem: Dual to the optimal flow problem, we define the following *optimal potential* 1) **u** is an optimal solution of OFP1, *ⁱ*=1 *^Ki*(u*i*) and ^K?(y) := ^P*|*V*[|]*
	- 2) **y** is an optimal solution to OPP1, The main result of this paper is that the output agreement steady-states in a network of maximal equilibrium ϵ

3)
$$
\sum_{i=1}^{|\mathcal{V}|} K_i(u_i) + \sum_{i=1}^{|\mathcal{V}|} K_i^{\star}(y_i) = 0
$$

^k = 0.

ⁱ = *K*?

solution to (OFP1)*, (ii)* y *is an optimal solution to* (OPP1)*, and (iii) in the steady-state* (OFP1) *and* (OPP1) *have*

ⁱ=1 *K*?

ⁱ (y*i*).

Duality in Cooperative Control \mathbf{f}_{max} for \mathbf{f}_{max} are the equilibrium input-output relations, i.e., \mathbf{f}_{max} **Dually in Cooperative Contracts**

 $\partial K_i(\mathbf{u}_i) = k_{y,i}(\mathbf{u}_i)$ maximal (strongly) monotone relat $K_i(u_i) = k_{vi}$ (u_i) maximal (strongly) monotone relations $\mathbf{F}_{i}(\mathbf{u}_{i})$ over the potential over the problem complete network. $\partial K_i(\mathbf{u}_i) = k_{y,i}(\mathbf{u}_i)$ maximal (strongly) monotone relations

- \Rightarrow OFP1 is feasible and strictly convex $-D1$ is foosible and strictly servey. -P| is teasible and strictly conv *ⁱ*=1 *^Ki*(u*i*) and ^K?(y) := ^P*|*V*[|]* \Rightarrow OFP1 is feasible and strictly convex \Rightarrow OFP1 is feasible and strictly convex
	- *|*V*|* T suong addiny, only one solution to Ori $n!$ \Rightarrow by strong duality, only one solution to OPP1 exists
	- c tly y*i* \overline{C} (*one outp* \Rightarrow exactly one output agreement solution exists

ⁱ = *K*?

Duality in Cooperative Control ^y˜ 2 @*K(***u***)*for the optimal multiplier **^y**˜. Thus, since @*K(***u***)* = **^k**y*(***u***)*, Ω r*K*? *ⁱ (*y*i)* = *^k*¹

How do the controls generate the correct inputs? the multiplier satisfies **^y**˜ 2 **^k**y*(***u***)*.

 \int ... *G G* $\begin{pmatrix} 1 & y & z \\ z & z & z \end{pmatrix}$ Π_1 $\left(\Pi_{2}\right)$ ${\rm \overline{I}}{}_{m}$ $E(\mathcal{G})$ *T µ*(*t*) ⇣(*t*) $\prod_{i} \qquad \dot{m}_i \qquad (1) \qquad \qquad \qquad$ $\begin{array}{ccc} \mathbf{1}_{\mathcal{K}} & \mathbf{1$ $\mu_k(\nu) = \left[\begin{array}{c} \psi_k(\nu_k(\nu), \zeta_k(\nu)) \end{array} \right]$ if ^y˜ 62 N *(E*>*)* then *^s(***y**˜*)* is unbounded below. For **^y**˜ 2 N *(E*>*)* it $F(G)$ \vert $\mathcal{L}(9) \vert$ $\begin{matrix} \overbrace{\Pi_1} & \overbrace{\Pi_2} & \overbrace{\Pi_3} & \overbrace{\Pi_4} & \overbrace{\Pi_5} & \overbrace{\Pi_6} & \overbrace{\Pi_7} & \overbrace{\Pi_8} & \overbrace{\Pi_9} & \overbrace{\Pi_1} & \overbrace{\Pi_1} & \overbrace{\Pi_1} & \overbrace{\Pi_1} & \overbrace{\Pi_2} & \overbrace{\Pi_3} & \overbrace{\Pi_4} & \overbrace{\Pi_5} & \overbrace{\Pi_6} & \overbrace{\Pi_7} & \overbrace{\Pi_8} & \overbrace{\Pi_9} & \overbrace{\Pi_1} & \overbrace{\Pi_1} & \overbrace{\Pi_1}$ $\mu(t)$ \Box the optimality conditions for the dual pair of optimality conditions \mathbf{C} $\binom{n}{\prod_{m}}$ optimal to both problems (α P1) and (α P1) and (α P1), it must be a saddle- α *4.2. The control level* It remains to investigate when the controller dynamics (8) can be realize an output agreement steady state. In particular, in the $\mathcal{L}(\boldsymbol{\alpha})$ $\mathcal{L}(\mathcal{G})$ that corresponds to the desired control input. Suppose a so-control input. Suppose a so-contro lution **u** to (16) is known, then the controller must be such that $f\left(\left\vert \cdot\right\vert \right)$ $\overline{}$ Find $\eta \in \mathcal{R}(E^{\top})$ s.t. $\mathbf{u} = -E \mathbf{\psi}(\mathbf{\eta})$. $\Pi_k: \quad \dot{\eta}_k(t) \quad = \quad \zeta_k(t)$ $\mu_k(t) = \left| \begin{array}{ll} \psi_k(\eta_k(t),\zeta_k(t)) \end{array} \right.$

Duality in Cooperative Control Duality in Cooperative Control

How do the controls generate the correct inputs? w do the compons generale

 \int $\begin{pmatrix} 1 & y & z \\ z & z & z \end{pmatrix}$ | \ $S_{\rm{S}}$ output y(t) \sim - $\begin{array}{ccc} \Pi_k: & \dot{\eta}_k(t) & = & \zeta_k(t) \end{array}$ $\mu_k(t) = \int \psi_k(\eta_k(t), \zeta_k(t))$ assume strongly monotone

Let $P_k(\eta)$ be such that $\nabla P_k = \psi_k$.

Duality in Cooperative Control *Optimal Potential Problem (OPP2) Optimal Flow Problem (OFP2)* \mathbf{S} solution and the steady-solution has additional inverse optimality properties. The see this, consider this, consider the see this, consider the see this, consider the see this, consider the see this, consider the the following pair of the set of the pair of the p Optimal Potential Pro oblem^e Optimal Flow Probl min h*,*v \sum *|*E*|* $k=1$ $P_k(\eta_k) + \sum$ *|*V*| i*=1 $\mathrm{u}_i \mathrm{v}_i$ s.t. $\mathbf{n} = E^{\top} \mathbf{v}$. $(\mathcal{P}^{\bullet}\mathcal{K})$ $\begin{array}{ccc} \mathbf{0} & \mathbf$ **tity in cooperative control** Optimal Flow Problem: The dual problem to (OPP2) is the following *optimal flow problem* min μ \sum *|*E*|* $k=1$ $P_k^{\star}(\mathfrak{\mu}_k)$ s.t. $\mathbf{u} + E\mathbf{u} = 0$,

Theorem is a structure, (OPP2) is a structure, (OPP2) is a structure, (OPP2) is a structure of \mathcal{L} *^k* is the convex conjugates of *Pk*, and u 2 *R*(*E*) is a given constant vector. The problem is in compliance

Assume all the node dynamics are maximal EIP. and the controller dynamics that $\nabla P_k(\eta_k) = \psi_k(\eta_k)$. Then the networked system has an output agreement $r_{\rm GGI}$ *|*E*| P*? are such that all output maps ψ_k are strongly monotone and define P_k such steady-state solution. Furthermore, let η be the state-state of the controller in output agreement, then and the controller dynamics (8) α *are strongly monotone.*

 \boldsymbol{n} α ¹¹ *^k* (µ*k*) *Then* η is an optimal solution of OPP2,

 $\mu = \psi(n)$ is an o₁ *state of the controller in output agreement, then (i)* h *is an optimal solution to* (OPP2)*, (ii)* µ = (h) *is an optimal* 2) $\mu = \psi(\eta)$ is an optimal solution to OFP2,

3)
$$
\sum_{k=1}^{|\mathcal{E}|} P_k^{\star}(\mu_k) + \sum_{k=1}^{|\mathcal{E}|} P_k(\eta_k) = \mu^T \eta
$$

The Closed L The Closed Loop

From SISO to MIMO - Cyclic Monotonicity

Definition

Consider $R \subset \mathbb{R}^n \times \mathbb{R}^n$. The relation R is *Cyclicly Monotone* if for any $N \geq 1$ and any pairs $(u_i, y_i) \in R$, $i = 1, \ldots, N$, $\sum y_i^T (u_i - u_{i-1}) \geq 0.$ *N* $i=1$

Theorem [Rockafellar, 1966]

A relation $R \subset \mathbb{R}^n \times \mathbb{R}^n$ is cyclicly monotone if and only if it is contained in the subgradient of a convex function $\psi : \mathbb{R}^n \to \mathbb{R}$.

An Example - Vehicle Platooning Platooning Vehicles: Prediction and Control An Example - Vehicle Example venis N_{e} \ln \ln \ln

P PLATODIC TREE *Platooning Vehicles* Microscopic Traffic Model *Microscopic manner*

[Helbing *et al.,* 2001]

 $p_i = v_i$ $v_{\rm{max}}=v_{\rm$ $\dot{\mathbf{n}} = \mathbf{n}$. $v_i = \kappa_i[V_i^{\circ} + V_i(\nabla \mathbf{p}) - v_i]$ $\dot{p}_i = v_i$ $\dot{v}_i = \kappa_i[V_i^0 + V_i(\nabla \mathbf{p}) - v_i]$ *p*˙*ⁱ* = *vⁱ* $\overline{}$ $\overline{\$ *i* $\frac{1}{2}$ $\frac{1}{2}$

Drivers adjustment: velocity adjustment Velocity adjustment: Velocity adjustment (control) Velocity adjustment:

 $i \in \mathcal{N}(i)$ if $\bigcup_{i \in \mathcal{N}(i)}$ country $P(i)$ $i \in \mathcal{N}$ i $V_i(\nabla \mathbf{p}) = V$ X $i \in N(t)$ $\tanh(p_i - p_i)$ $V_i(\nabla \mathbf{p}) = V_i^1$ \sum $i \in \mathcal{N}(i)$ $\tanh(p_j - p_i)$

An Example - Vehicle Platooning Platooning Vehicles: Prediction and Control Extensions to Clustering Analysis Example venis $\sqrt{1 + 1 + 1}$ Illustrative Example: Platooning Vehicles

P Platonic Light Platonic Ligh Microscopic Hallic IVII

 $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ $[Helbing et al., 2001]$

 $\dot{p}_i = v_i$ $\sum_{i=1}^{n} \frac{1}{i} \sum_{j=1}^{n} \frac{1}{j} \sum_{j=1}^{n} \frac{1}{j$ *p*˙*ⁱ* = *vⁱ v*˙*ⁱ* = *i*[*V* ⁰ $\dot{v}_i = \kappa_i[V_i^0 + V_i(\nabla \mathbf{p}) - v_i]$ *p*˙*ⁱ* = *vⁱ* $\mathbf{v}_i = \kappa_i[\mathbf{v}_i + \mathbf{v}_i(\mathbf{v}_i) - \mathbf{v}_i]$

drivers and the contract of th *ⁱ* + *Vi*(rp) *vi*] Velocity adjustment: Velocity adjustment (control)

$$
V_i(\nabla \mathbf{p}) = V_i^1 \sum_{i \in \mathcal{N}(i)} \tanh(p_j - p_i)
$$

i∈N(i) Each vehicle/driver has a "preferred" velocity *V* ⁰ *ⁱ* ("disturbance"). Each vehicle/driver has a "preferred" velocity *V* ⁰ *ⁱ* ("disturbance"). output coupling function is monotone, but not strongly monotone!

הפקולתה להנדסת אוירונוטיקה וחלל Faculty of Aerospace Engineering

Solution to the network of Aerospace Engineering

Solution to the network of Aerospace Engineering

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 $\frac{1}{2}$

An Example - Vehicle Platooning Γ canonical model Γ II LAGIIIUIC - VCIII

Network Optimization approach can still be used! $\frac{1}{2}$ $\frac{1}{2}$

Optimal Flow Problem (OFP1) *C*entuation of the network optimation problems on the plant level in the matter of the plant level in the p

$$
\min_{\mathrm{u}_i,\mu_k}\,\sum_{i=1}^{|\mathbf{V}|}K_i(\mathrm{u}_i)
$$

s.t.
$$
\mathbf{u} = E\mu
$$

\n
$$
\|\mu\|_{\infty} \le 1
$$

Optimal Potential Problem *(OPP1)* $(OFP1)$ ($OPP1$)

$$
\min_{\mathbf{u}_i, \mu_k} \sum_{i=1}^{|\mathbf{V}|} K_i(\mathbf{u}_i)
$$
\n
$$
\min_{\mathbf{y}_i, \zeta_k} \sum_{i=1}^{|\mathbf{V}|} K_i^{\star}(\mathbf{y}_i) + \sum_{k=1}^{|\mathbf{E}|} |\zeta_k|
$$
\n
$$
\text{s.t. } \mathbf{u} = E\mu
$$
\n
$$
\text{s.t. } \zeta = E^{\top} \mathbf{y}
$$

An Example - Vehicle Platooning we can spice them on Extensions to Clustering Analysis Extensions to Clustering Analysis

An Example - Vehicle Platooning icl_0 $D!$:
1970 - Paris
1970 - Paris \bigcup $\lim_{n\to\infty}$

clustering phenomena can be explained $\frac{1}{2}$ stadying the solations of the static
network optimization problems by studying the solutions of the static network optimization problems

> min \mathbf{u}_i , μ_k $\sqrt{ }$ *|*V*| i*=1 $\sum K_i(u_i)$ u*i,µ^k* $\frac{1}{\sqrt{1}}$

$$
s.t. u = E\mu \qquad \qquad s.t. \zeta =
$$

 $\|\mu\|_{\infty} \leq 1$

 $k < 1$

$$
\sum_{i=1}^{|V|} K_i(u_i)
$$
\n
$$
\min_{y_i, \zeta_k} \sum_{i=1}^{|V|} K_i^*(y_i) + \sum_{k=1}^{|E|} |\zeta_k|
$$
\n
$$
= E\mu
$$
\n
$$
\text{s.t. } \zeta = E^{\top} \mathbf{y}
$$

y*i,*⇣*^k*

Towards a Synthesis Procedure \vec{r} **f** \vec{r} τ (*houtput a* Synthesis Procedi *u(x*) By FILIICSIS THULCU *µ(t)* Controller output µ Flow **^u** + *^E*µ = ⁰ *^P*?

 \sim wapu relations. However, now the cost function ? flow variables µ*k*. Design controllers to achieved a *desired* output agreement state.

Generalized Optimal Potential Problem **We also define the generalized optimal potential problem** *(OPP1)* \int anc also the dynamic variable *v(t)*, which corresponds to the potential problem as

$$
\min_{\mathbf{y}, \zeta} \sum_{i=1}^{|\mathbf{V}|} K_i^{\star}(\mathbf{y}_i) + \sum_{k=1}^{|\mathbf{E}|} \Gamma_k(\zeta_k)
$$

s.t. $\zeta = E^{\top} \mathbf{y}$.

- 1. Find a convex function Γ such that the desired output agreement state, y^* satis y^* satisfies $\zeta^* = E^T y^*$ and minimizes the GOPP.
- $\overline{2}$ Find M $subdiff_0$ of FIP gratoma whose stoody state input out The governs whose seedy searchip are deput maps are the are related to the asymptotic behavior of ι . 2. Find Maximal EIP systems whose steady-state input-output maps are the subdifferentials of the functions Γ_i .

Theorem 5.2 (*Generalized Network Convergence Theorem*)**.** *Con-*

Summary

Passivity based cooperative control

- maximal EIP systems
- connection to dual network optimization problems
- maximal EIP implies agreement solution is *inverse optimal*
- s tor cooperative control problems! • duality relation exists for cooperative control problems!

Outlook

Passivity based cooperative control

- further extensions to MIMO systems
- controller synthesis
- "Duality" as a systems property

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