Formation Control via Rotation Symmetry Constraints

IAAC Graduate Students in Systems and Control

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CONNECT LAB
Cooperative Networks and Controls

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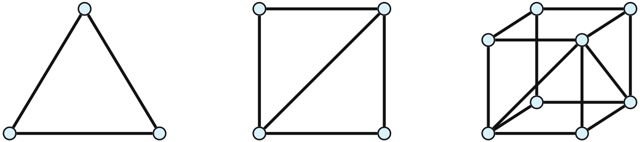
Introduction

Formation Control Objective

Given a team of agents that can sense or communicate with neighboring agents, design a distributed control law using only local information to drive the team to a desired spatial configuration



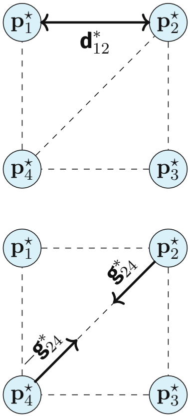
Introduction

Formation control arises naturally in a wide range of multi-agent applications

- UAV Swarms: mapping and surveillance
- Satellite constellations: uniform spacing for coverage
- Multi-robot systems: distributed sensing and transportation

Motivation

- Standard approaches impose explicit geometric constraints between agents (e.g. distances or bearings)



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- S. Zhao and D. Zelazo, “Bearing rigidity theory and its applications for control and estimation of network systems: Life beyond distance rigidity,” IEEE Control Syst. Mag., vol. 39, no. 2, pp. 66–83, Apr. 2019.

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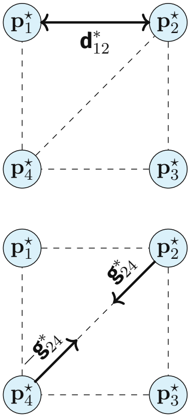
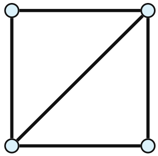
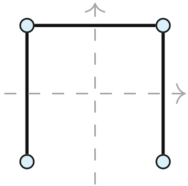
For a team of n agents, $2n-3$ constraints are required for the implementation in $\mathbb{R}^2$



Motivation

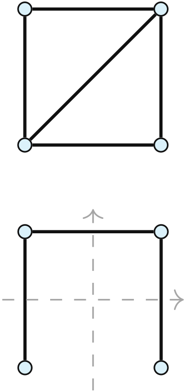
- Standard approaches impose explicit geometric constraints between agents (e.g. distances or bearings)
- Many formations exhibit inter-agent symmetries





Motivation

- Standard approaches impose explicit geometric constraints between agents (e.g. distances or bearings)
- Many formations exhibit inter-agent symmetries
- Exploiting these symmetries can significantly reduce the number of constraints required to characterize the formation, potentially down to $n - 1$



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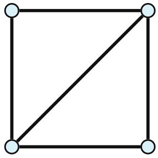
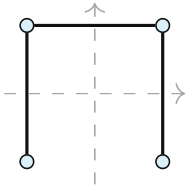
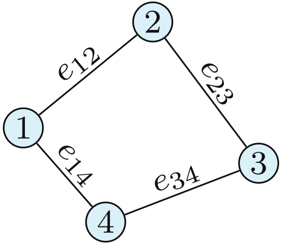




Can we achieve this using symmetry constraints alone?

Multi-agent Systems

- A team of n agents interact according to an information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$



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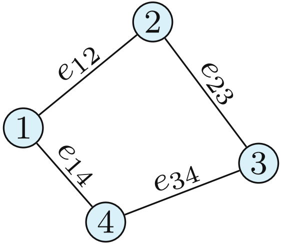
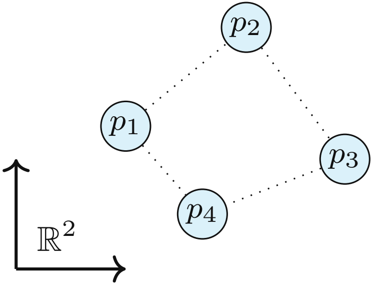


- The graph can be embedded in Euclidean space $\mathbb{R}^d$ as a framework $(\mathcal{G}, p)$. The agent positions are given by $p(t) = \left[p_1^T, \ldots, p_n^T\right]^T \in \mathbb{R}^{nd}$

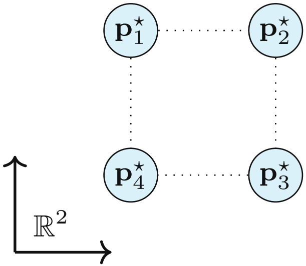


Formation Constraints

- The desired formation is encoded by M constraints $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, satisfied at $\mathbf{p}^\star$
- The set of all feasible formations is

$$\mathcal{F}(p) = \{p \in \mathbb{R}^{nd} \mid F(p) = F(\mathbf{p}^\star)\}$$

- A feasible formation is defined as a framework $(\mathcal{G}, \mathbf{p}^\star)$



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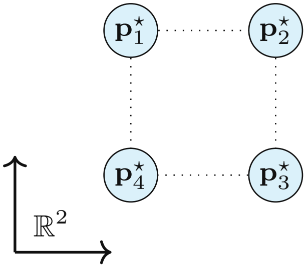
How do we drive the agents toward $\mathcal{F}(p)$?

- A feasible formation is defined as a framework $(\mathcal{G}, \mathbf{p}^\star)$



Formation Constraints

Formation Control Objective

For an ensemble of n agents with dynamics

$$\dot{p}_i = u_i,$$

with $p_i(t) \in \mathbb{R}^d$, information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and formation constraint function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, design a distributed control law for each agent $i \in \{1, \ldots, n\}$ such that the set

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is asymptotically stable.

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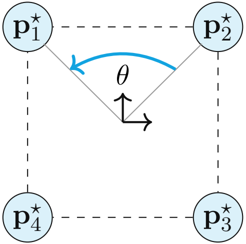
F will encode inter-agent rotational symmetries of the target configuration

Example

Square formation with its centroid located at the origin



- The angle between any two adjacent agents is $\theta = 90°$
- The constraints encoded in F capture symmetric relationships between agent positions, e.g., $\mathbf{p}_1^\star = R(\theta)\,\mathbf{p}_2^\star$

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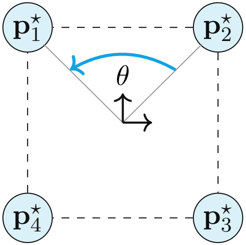

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This structure reflects an underlying *graph symmetry*

Symmetry in Graphs

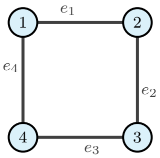
Automorphisms encode graph symmetries

Graph Automorphism

An automorphism of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation $\psi : \mathcal{V} \to \mathcal{V}$ of its vertex set such that

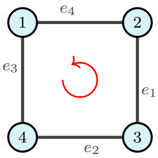
$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$

Example: Cycle graph C_4



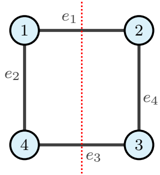
Identity:

$$\mathrm{Id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$



90° rotation:

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

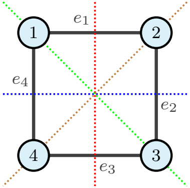


reflection:

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Symmetry in Graphs

- Additional permutations can be found for the given graph considering all possible reflections and rotations



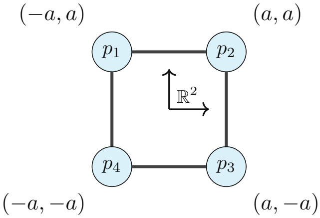
- The set of all automorphisms of $\mathcal{G}$ form a *group* - $\mathrm{Aut}(\mathcal{G})$
  - $\mathrm{Aut}(\mathcal{G}) = \{\mathrm{Id}, \psi_1, \psi_2, ...\}$
- For any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$, we say that $\mathcal{G}$ is Γ-symmetric, which define specific symmetries in $\mathcal{G}$

Definition

A framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called $\tau(\Gamma)$-symmetric if

$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all} \quad \gamma \in \Gamma \quad \text{and all} \quad i \in \mathcal{V}$$

Graph symmetries can be realized in Euclidean space by assigning to each element of Γ an orthogonal matrix τ representing a point group isometry.



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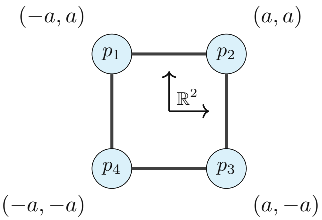
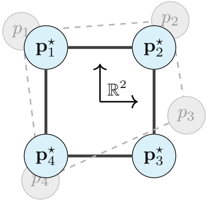


- Consider $\Gamma = \{\mathrm{Id}, \psi_1, \dots\}$ (All rotations of C_4)

  - $\tau(\psi_1)p_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} -a \\ a \end{bmatrix} = p_1$
  - $\tau(\psi_1^2)p_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ -a \end{bmatrix} = \begin{bmatrix} -a \\ a \end{bmatrix} = p_1$

Symmetry in Frameworks

For any cycle graph C_n:



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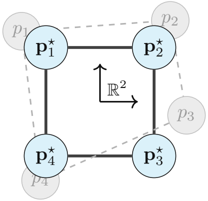

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- In cycle notation:
  $\Gamma = \{\mathrm{Id}, \psi_1, \psi_1^2, \ldots, \psi_1^{n-1}\} \subseteq \mathrm{Aut}(C_n)$

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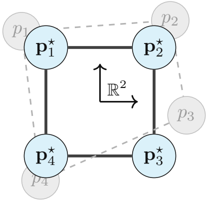

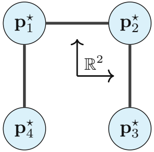

$$\mathbf{p}_4^\star = \tau(\psi_1^3)\,\mathbf{p}_3^\star$$

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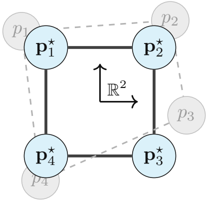

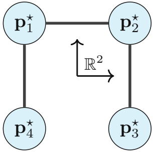

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An *interaction graph* $\mathcal{G}_I = (\mathcal{V}, \mathcal{E}_I)$, chosen as a spanning tree of C_n is sufficient to define the target configuration

A Gradient Approach

Formation Control Objective

Design $u_i(t)$ such that for every edge $ij \in \mathcal{E}_I$,

$$\lim_{t \to \infty} \|p_i(t) - \tau(\gamma_{ji})\, p_j(t)\| = 0$$

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Design $u_i(t)$ such that for every edge $ij \in \mathcal{E}_I$,

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Define a symmetry-forcing potential function:

$$F(p(t)) = \frac{1}{2} \sum_{ij \in \mathcal{E}_I} \|p_i(t) - \tau(\gamma_{ji}) p_j(t)\|^2.$$

Gradient descent $u(t) = -\nabla F(p(t))$ defines the control law:

$$\dot{p}_i(t) = \sum_{ij \in \mathcal{E}_I} (\tau(\gamma_{ji}) p_j(t) - p_i(t))$$

Symmetry Based Control Law

Compact state-space form:

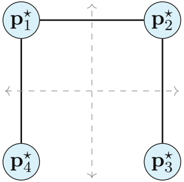
$$\dot{p}(t) = -Qp(t)$$

$Q \in \mathbb{R}^{2n \times 2n}$ is a *symmetry-constrained* matrix-weighted Laplacian of $\mathcal{G}_I$, with block entries

$$[Q]_{ij} = \begin{cases} d(i)I_2, & i = j,\ i \in \mathcal{V} \\ -\tau(\gamma_{ji}), & ij \in \mathcal{E}_I \\ 0, & \text{o.w.} \end{cases}$$

- $d(i)$ denotes the degree of node i

Example



$$Q = \begin{bmatrix} I_2 & -R(\frac{\pi}{2}) & 0 & 0 \\ -R(\frac{\pi}{2})^T & 2I_2 & -R(\frac{\pi}{2}) & 0 \\ 0 & -R(\frac{\pi}{2})^T & 2I_2 & -R(\frac{\pi}{2}) \\ 0 & 0 & -R(\frac{\pi}{2})^T & I_2 \end{bmatrix}$$

$$Q\mathbf{p}^\star = 0$$

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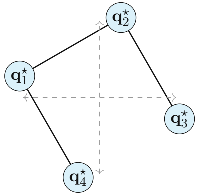

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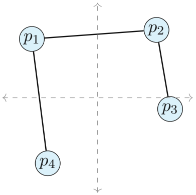

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$$Q\mathbf{p} \neq 0$$

Symmetry Based Control Law

Example

$$Q = \begin{bmatrix} I_2 & -R(\frac{\pi}{2}) & 0 & 0 \\ -R(\frac{\pi}{2})^T & 2I_2 & -R(\frac{\pi}{2}) & 0 \\ 0 & -R(\frac{\pi}{2})^T & 2I_2 & -R(\frac{\pi}{2}) \\ 0 & 0 & -R(\frac{\pi}{2})^T & I_2 \end{bmatrix}$$

Properties of Q

- Positive semi-definite
- Its nullspace corresponds to the trivial motions of the formation — translations and uniform scalings

Symmetry Based Control Law

let

$$\mathcal{F} = \{p \in \mathbb{R}^{2n} | \tau(\gamma_{ji}) p_i = p_{\gamma_{ji}(i)}, \quad \forall ij \in \mathcal{E}, \ i \in \mathcal{V}\},$$

be the set of $\mathcal{C}_n$-symmetric configurations

Theorem

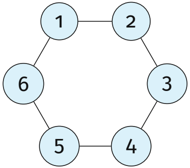
For any initial condition $p(0) \in \mathbb{R}^{2n}$, the control law $\dot{p}(t) = -Qp(t)$ renders $\mathcal{F}$ exponentially stable. Moreover, the agents converge to

$$\lim_{t\to\infty} p_i(t) = \frac{1}{n} S_i \sum_{k=1}^{n} S_k^T p_k(0).$$

where S_i is the product of rotation matrices along the spanning tree path from node 1 to node i, and $p(\infty)$ is the orthogonal projection of $p(0)$ onto $\mathcal{F}$.

Example

Setup: $n = 6$ agents, tasked with attaining a $\mathcal{C}_6$-symmetric formation (hexagon)



Underlying Graph C_6

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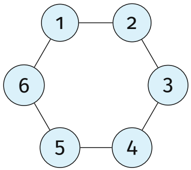

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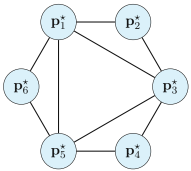

A distance based approach
requires 9 edges

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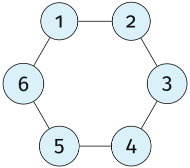

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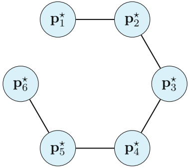
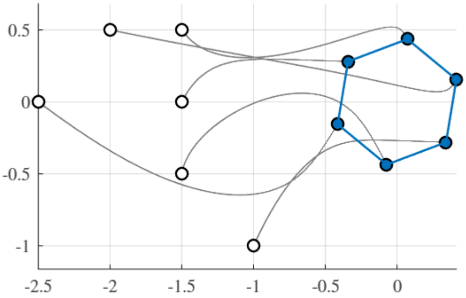
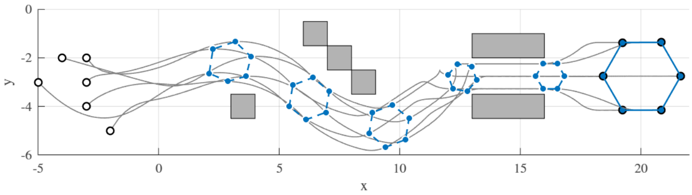


A symmetry based approach requires 5 edges



Formation Maneuvering

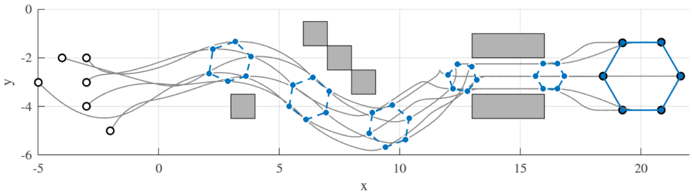


- Formation maneuvering aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body
- Secondary objective:

$$\lim_{t\to\infty} ||\dot{p}_i(t) - v_i(t)|| = 0$$

where $v_i \in \mathbb{R}^d$ is the desired rigid body velocity for each agent

Formation Maneuvering



- Formation maneuvering aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body
- Secondary objective:

$$\lim_{t \to \infty} ||\dot{p}_i(t) - v_i(t)|| = 0$$

where $v_i \in \mathbb{R}^d$ is the desired rigid body velocity for each agent

Idea: We augment the control law with a time-varying virtual trajectory.

Formation Maneuvering

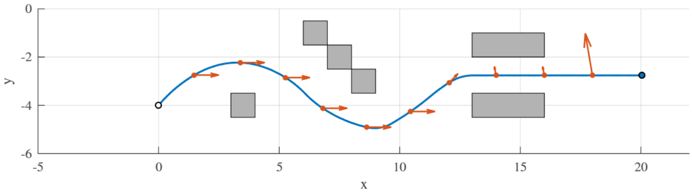
Assumption

All agents have access to a virtual trajectory predefined by a

i) translation $r(t) \in \mathbb{R}^2$ with $\dot{r}(t) = v(t)$;

ii) rotation $\mathcal{R}(t) \in SO(2)$ with $\dot{\mathcal{R}}(t) = \Omega(t)\mathcal{R}(t)$ where $\Omega(t) = \begin{bmatrix} 0 & -\omega(t) \\ \omega(t) & 0 \end{bmatrix}$, and $\omega(t)$ is the desired angular velocity of the formation;

iii) scale factor $s(t) \in \mathbb{R}^+$, with $\dot{s}(t) = \alpha(t)s(t)$, $\alpha(t) \in \mathbb{R}$.



Formation Maneuvering

Let $c_i(t) = p_i(t) - r(t)$ denote the position of agent i relative to the reference trajectory, and define the set of all shifted $\mathcal{C}_n$-symmetric configurations as

$$\mathcal{F}_c = \{p \in \mathbb{R}^{2n} \,|\, \tau(\gamma_{ji})c_i = c_{\gamma_{ji}(i)}, \; \forall ji \in \mathcal{E}, \; i \in \mathcal{V}\}$$

We introduce the state $\zeta(t) \in \mathbb{R}^{2n}$ to be the states $p(t) \in \mathbb{R}^{2n}$ expressed in a frame moving along the virtual trajectory,

$$\zeta(t) = \frac{1}{s(t)}\left(I_n \otimes \mathcal{R}(t)^T\right)c(t) \in \mathbb{R}^{2n}.$$

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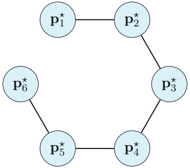
Theorem

For any initial condition $p(0) \in \mathbb{R}^{2n}$, the augmented Control Law:

$$u(t) = -Qc(t) + \mathbb{1}_n \otimes v(t) + (I_n \otimes \Omega(t) + \alpha(t))c(t)$$

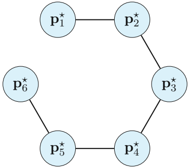
renders $\mathcal{F}_c$ exponentially stable.

Example



Setup: $n = 6$ agents tasked with attaining a $\mathcal{C}_6$-symmetric formation (hexagon) while maneuvering along a predefined virtual trajectory

Example



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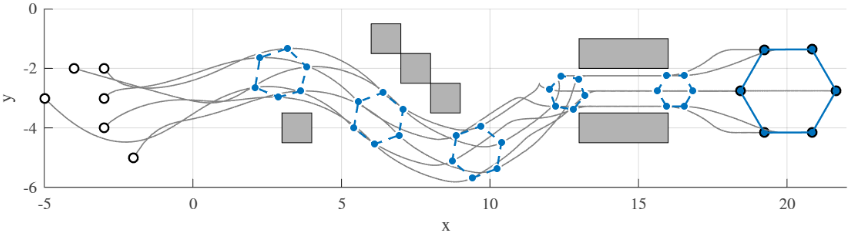
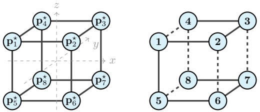
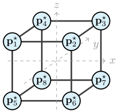
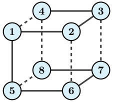


Extension to $\mathbb{R}^3$

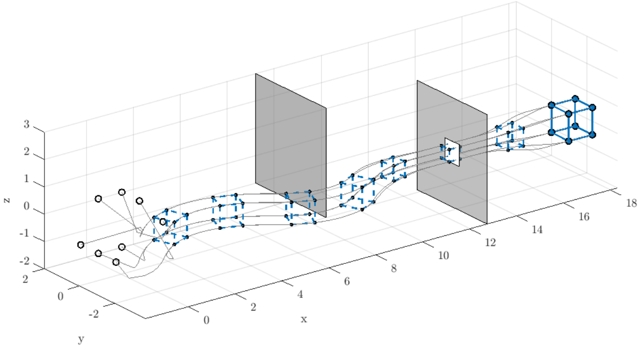


Setup: $n = 8$ agents tasked with attaining a cube formation while maneuvering along a predefined virtual trajectory





Setup: $n = 8$ agents tasked with attaining a cube formation while maneuvering along a predefined virtual trajectory



Concluding Remarks

Summary

- Cyclic formations can be achieved using only rotation symmetry constraints
- The proposed method requires only $n-1$ edges (minimal connectivity), fewer than ”classical” approaches

Future Work

- Extension to broader point-group symmetries in $\mathbb{R}^3$
- Explore leader-follower architectures enabling fully distributed agreement on time-varying virtual trajectories
- Incorporate additional geometric constraints alongside symmetry to enable a broader class of target formations

Concluding Remarks

Thank you

- Z. Martinez and D. Zelazo, “Formation Control via Rotation Symmetry Constraints,” in American Control Conference, New Orleans, LA, USA, May 2026.
- Z. Martinez and D. Zelazo, “Symmetry-Based Formation Control on Cycle Graphs Using Dihedral Point Groups,” in IFAC World Congress, Busan, South Korea, Aug. 2026.