A PASSIVITY ANALYSIS FOR NONLINEAR CONSENSUS ON BALANCED DIGRAPHS

EUROPEAN CONTROL CONFERENCE 2025

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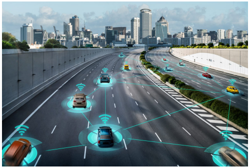
June 25, 2025

THE STEPHEN B. KLEIN FACULTY OF AEROSPACE ENGINEERING

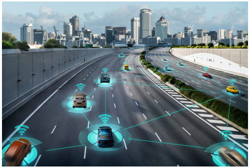
CONNECTLAB
Cooperative Networks and Controls

Applications:

- Formation flying
- Power grids
- Automated transportation networks...

MULTI-AGENT NETWORKS



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Fundamental problem: Output consensus, Stabilization

Multi-agent networks: A group of SISO agents Σ_i interact over a graph $\mathcal{G}$

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i) \end{cases}, i \in [1, n]$$

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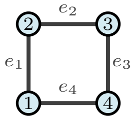

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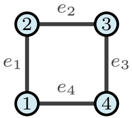

Graph $\mathcal{G}$

Edges: $\{e_1, e_2, e_3\}$

Distributed controllers

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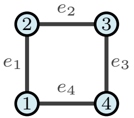
$$\Pi_k : \begin{cases} \dot{\eta}_k = \phi_k(\eta_k, \zeta_k) \\ \mu_k = \psi_k(\eta_k, \zeta_k) \end{cases}, k \in [1, m]$$



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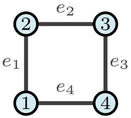

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Distributed controllers

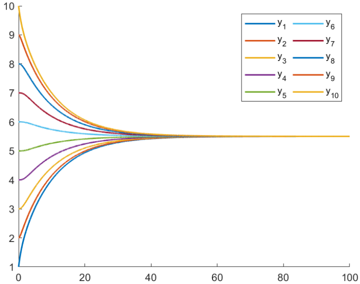
Output consensus problem:

Design distributed Π_k's, such that

$$\lim_{t \to \infty} (y_i(t) - y_j(t)) = 0, \ \forall i, j$$

$$\Leftrightarrow \lim_{t \to \infty} y(t) \in S$$

where $S = \text{span}(\mathbb{1})$ denotes the agreement space.



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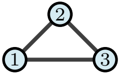

Undirected $\mathcal{G}$

$$E_{\mathcal{G}} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Incidence matrix

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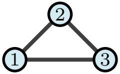

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GRAPHS AND LINEAR CONSENSUS PROTOCOLS

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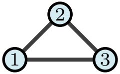

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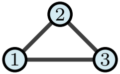

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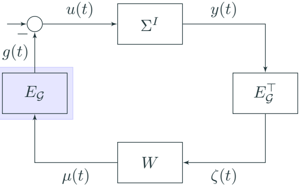

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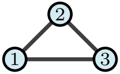

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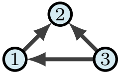

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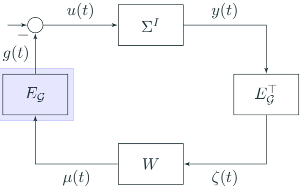

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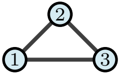

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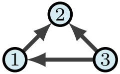

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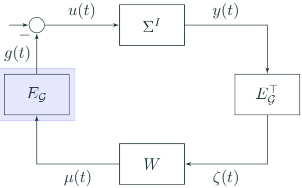

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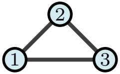

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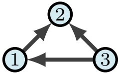

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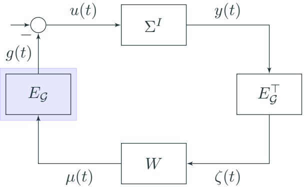

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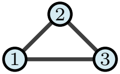

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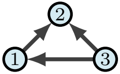

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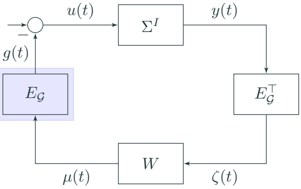

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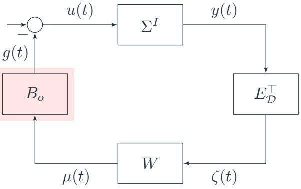
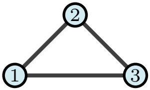


Directed Networks

GRAPH TOPOLOGIES MATTER!



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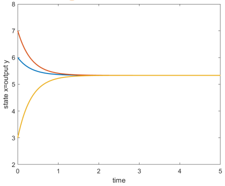
$$\lim_{t\to\infty} y(t) = \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top x(0)$$

Average consensus



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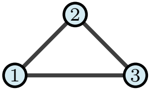

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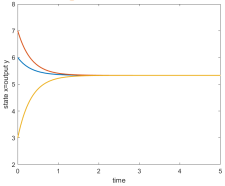
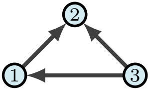
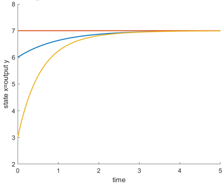


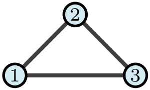


Directed $\mathcal{D}$

$$\lim_{t\to\infty} y(t) = (q_1^\top x(0))\mathbb{1}_n$$

Regular consensus





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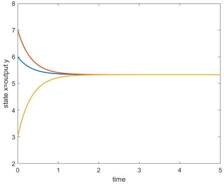

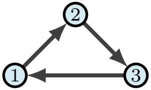

Balanced $\mathcal{D}_b$

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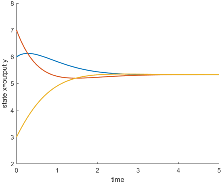

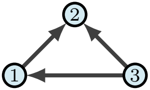

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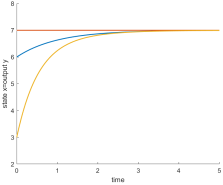

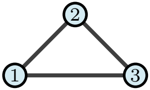

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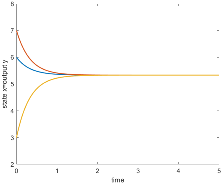

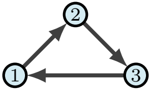

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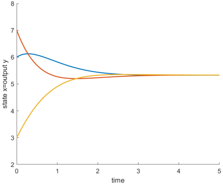

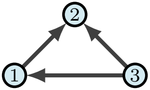

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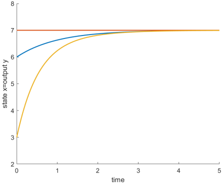
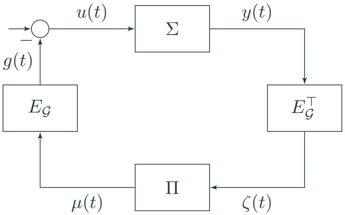


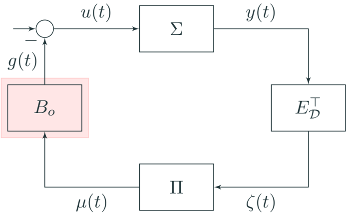
For $\mathcal{G}$ and $\mathcal{D}_b$:

- Achieve average consensus
- Trajectories may differ

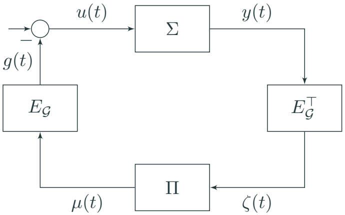
NETWORK SYSTEMS AND PASSIVITY



Undirected $(\Sigma, \Pi, \mathcal{G})_E$

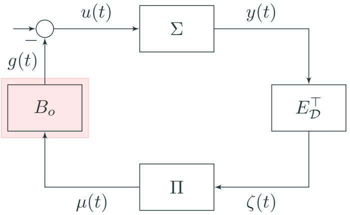


Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$



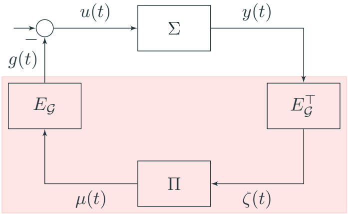
Undirected $(\Sigma, \Pi, \mathcal{G})_E$

Diffusively-coupled networks

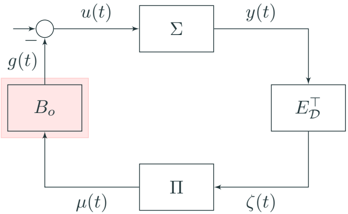


Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

NETWORK SYSTEMS AND PASSIVITY

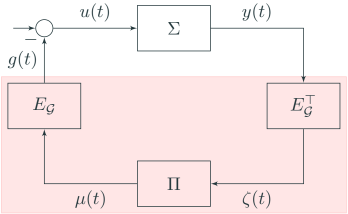


Undirected $(\Sigma, \Pi, \mathcal{G})_E$

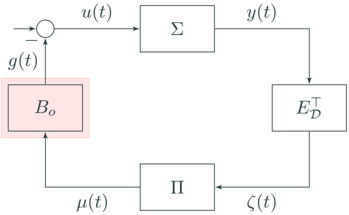


Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Symmetric operator** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$

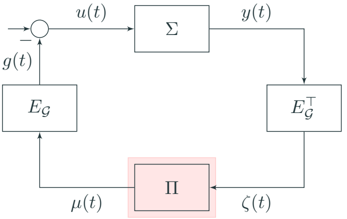


Undirected $(\Sigma, \Pi, \mathcal{G})_E$

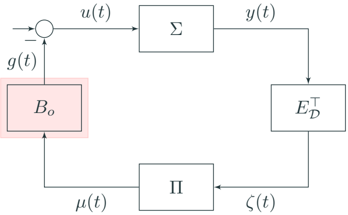


Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Symmetric operator** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$
- **Passivity Analysis** $\checkmark$



Undirected $(\Sigma, \Pi, \mathcal{G})_E$



Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Symmetric operator** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$
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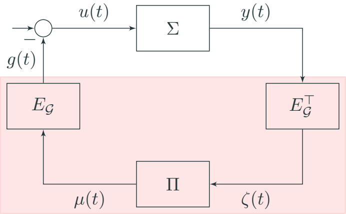
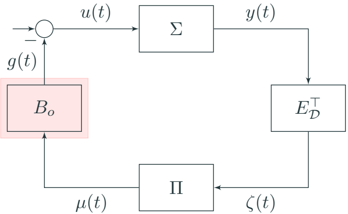
Passive Π $\quad \mu^{\top}(t)\zeta(t) \geq \dot{V}(\eta(t))$



Undirected $(\Sigma, \Pi, \mathcal{G})_E$



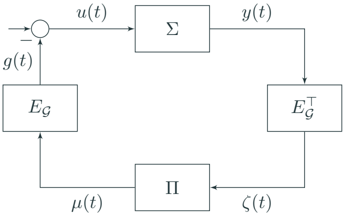
Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Symmetric operator** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$
- **Passivity Analysis** $\checkmark$

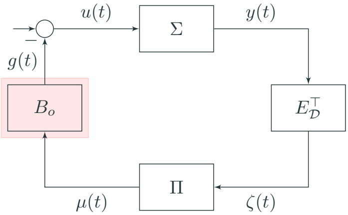
Passive Π $\quad \mu^{\top}(t)\zeta(t) \geq \dot{V}(\eta(t))$

$\Rightarrow$ Passive $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^{\top}$

$$g^{\top}(t)y(t) = \mu^{\top}(t)E_{\mathcal{G}}^{\top}y(t) = \mu^{\top}(t)\zeta(t)$$



Undirected $(\Sigma, \Pi, \mathcal{G})_E$



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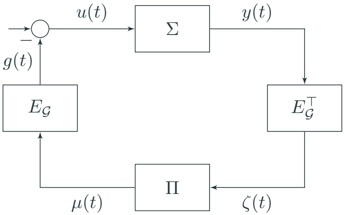
- A decoupled analysis
- Convergence, stability



Undirected $(\Sigma, \Pi, \mathcal{G})_E$

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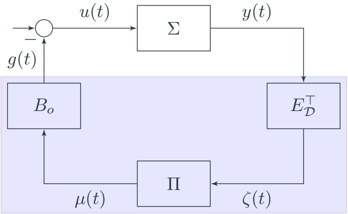
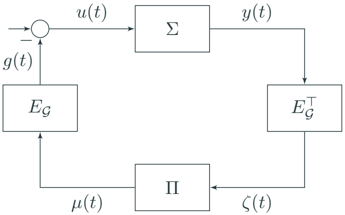


Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Asymmetric** operator $B_o \Pi E_{\mathcal{D}}^\top$

NETWORK SYSTEMS AND PASSIVITY



Undirected $(\Sigma, \Pi, \mathcal{G})_E$

- **Symmetric operator** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^\top$
- **Passivity Analysis** ✓

Passive Π $\quad \mu^\top(t)\zeta(t) \geq \dot{V}(\eta(t))$

$\Rightarrow$ Passive $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^\top$

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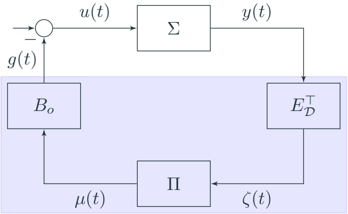


Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

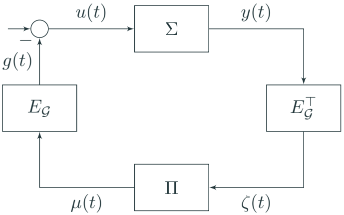
- **Asymmetric** operator $B_o \Pi E_{\mathcal{D}}^\top$

Proposition

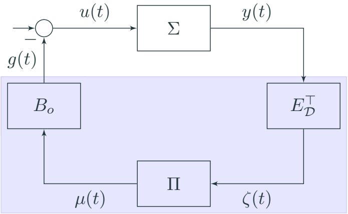
If the following hold:

- $\Pi = I : \zeta(t) = \mu(t)$
- $\mathcal{D}$ is Balanced,

then the operator $B_o \Pi E_{\mathcal{D}}^\top$ is passive.



Undirected $(\Sigma, \Pi, \mathcal{G})_E$



Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Symmetric operator** $E_\mathcal{G} \Pi E_\mathcal{G}^\top$
- **Passivity Analysis** ✓

Passive Π $\quad \mu^\top(t)\zeta(t) \geq \dot{V}(\eta(t))$

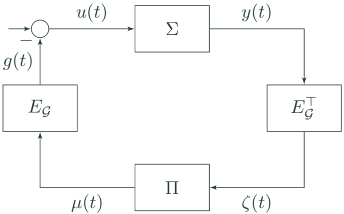
$\Rightarrow$ Passive $E_\mathcal{G} \Pi E_\mathcal{G}^\top$

$$g^\top(t)y(t) = \mu^\top(t)E_\mathcal{G}^\top y(t) = \mu^\top(t)\zeta(t)$$

  - A decoupled analysis
  - Convergence, stability

- **Asymmetric** operator $B_o \Pi E_\mathcal{D}^\top$

  Doesn't preserve passivity



Undirected $(\Sigma, \Pi, \mathcal{G})_E$

- **Symmetric operator** $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^\top$
- **Passivity Analysis** $\checkmark$

Passive Π $\quad \mu^\top(t)\zeta(t) \geq \dot{V}(\eta(t))$

$\Rightarrow$ Passive $E_{\mathcal{G}} \Pi E_{\mathcal{G}}^\top$

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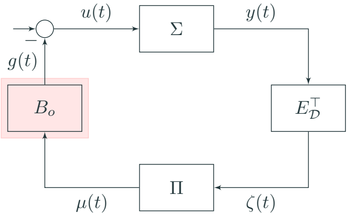
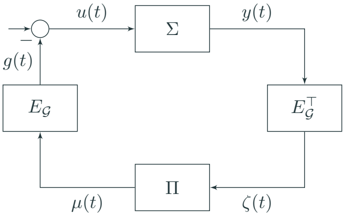
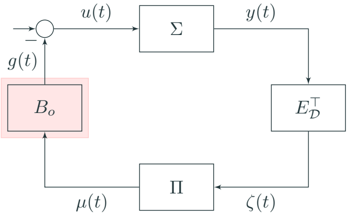


Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

- **Asymmetric** operator $B_o \Pi E_{\mathcal{D}}^\top$
  Doesn't preserve passivity
- **Passivity Analysis?**

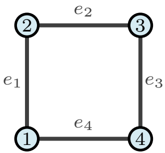


Undirected $(\Sigma, \Pi, \mathcal{G})_E$

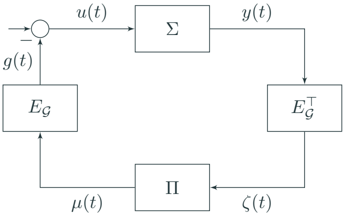


Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

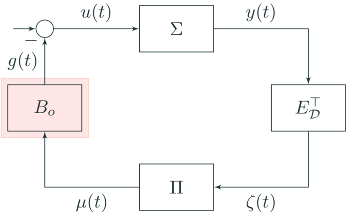
$$E_{\mathcal{D}} = B_o + B_i$$



$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



Undirected $(\Sigma, \Pi, \mathcal{G})_E$

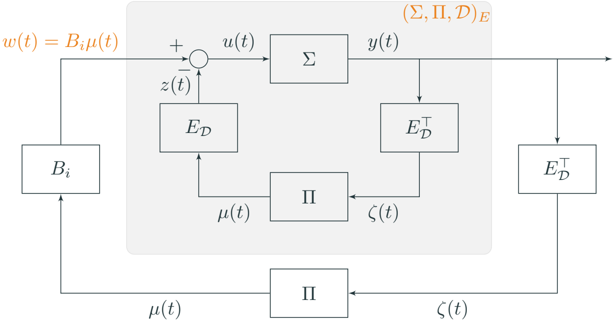


Directed $(\Sigma, \Pi, \mathcal{D})_{B_o}$

$$E_{\mathcal{D}} = B_o + B_i$$

Decompose the feedback path of $(\Sigma, \Pi, \mathcal{D})_{B_o}$

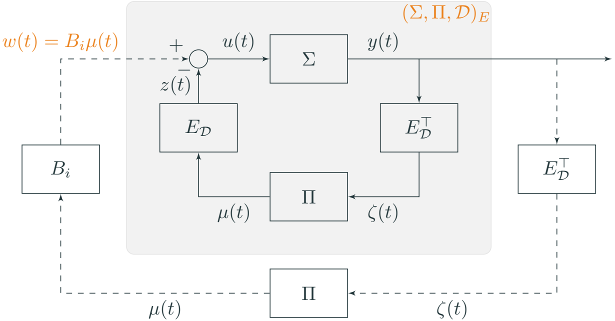
$E_{\mathcal{D}} = B_o + B_i$



$$(\Sigma, \Pi, \mathcal{D}, w) = (\Sigma, \Pi, \mathcal{D})_{B_o}$$

- ”External” input: $w(t) = B_i\mu(t)$
- Agent input: $u(t) = -B_o\mu(t)$
- Controller input: $\zeta(t) = E_{\mathcal{D}}^\top y(t)$

$E_{\mathcal{D}} = B_o + B_i$



$$(\Sigma, \Pi, \mathcal{D}, w) = (\Sigma, \Pi, \mathcal{D})_{B_o}$$

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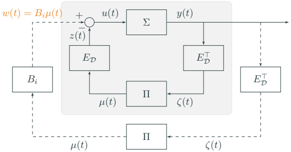
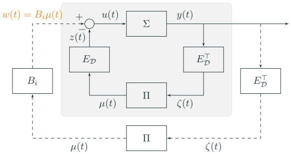
Passivity



$(\Sigma, \Pi, \mathcal{D}, w)$, Digraphs

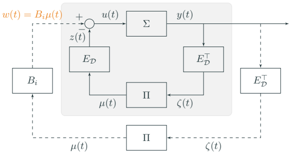
Passivity



$(\Sigma, \Pi, \mathcal{D}, w)$, Digraphs

Output Consensus $\lim_{t\to\infty} y(t) \in \mathrm{span}(\mathbb{1}) = S$

Passivity



$(\Sigma, \Pi, \mathcal{D}, w)$, Digraphs

Submanifold $\lim_{t\to\infty} \mathrm{Proj}_{S^{\perp}}(y(t)) = 0$

Output Consensus $\lim\limits_{t\to\infty} y(t) \in \mathrm{span}(\mathbb{1}) = S$

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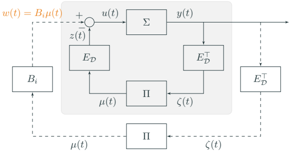

$(\Sigma, \Pi, \mathcal{D}, w)$, Digraphs

Submanifold & Passivity Relations[1]

$$\lim_{t\to\infty} \operatorname{Proj}_{S^\perp}(y(t)) = 0$$

$$u(t)^\top \operatorname{Proj}_{S^\perp}(y(t)) \geq l\|u(t)\|^2 + e\|\operatorname{Proj}_{S^\perp}(y(t))\|^2$$

$$z(t)^\top \operatorname{Proj}_{S^\perp}(y(t)) \geq l\|z(t)\|^2 + e\|\operatorname{Proj}_{S^\perp}(y(t))\|^2$$

Output Consensus

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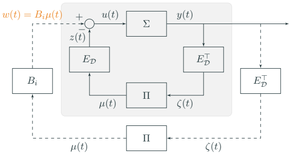
[1] J. M. Montenbruck, M. Arcak, and F. Allgöwer, "An input-output framework for submanifold stabilization," IEEE TAC, 2017.

PASSIVITY AND OUTPUT CONSENSUS

Passivity



$(\Sigma, \Pi, \mathcal{D}, w)$, Digraphs

Balanced graphs, output strict passivity

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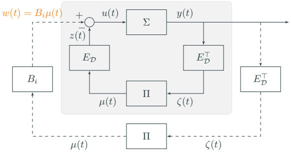
Passivity



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Output Consensus

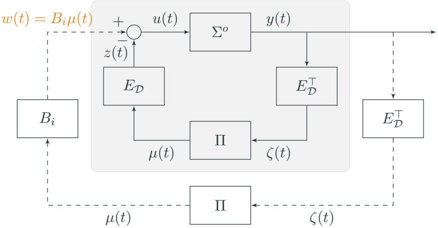
$$\lim_{t\to\infty} y(t) \in \mathrm{span}(\mathbb{1}) = S$$

Stabilization

$$\lim_{t\to\infty} y(t) = \mathbb{0} \in S$$

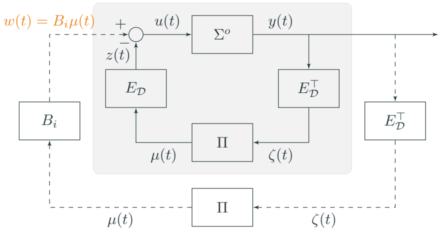
[1] J. M. Montenbruck, M. Arcak, and F. Allgöwer, "An input-output framework for submanifold stabilization," IEEE TAC, 2017.

Network systems over balanced digraphs & Passivity



$$(\Sigma^o, \Pi, \mathcal{D}_b, w)$$

Network systems over balanced digraphs & Passivity

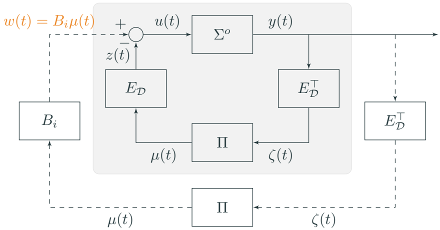


$$(\Sigma^o, \Pi, \mathcal{D}_b, w)$$

$$\Sigma_i^o : \begin{cases} \dot{x}_i(t) = f_i(x_i(t), u_i(t)), \\ y_i(t) = h_i(x_i(t)), \quad i \in [1, n] \end{cases}$$

(Outputs only depend on states)

Network systems over balanced digraphs & Passivity



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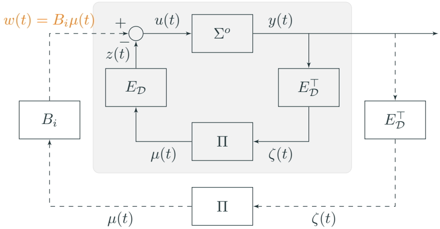
(Outputs only depend on states)

How can we connect output strict passivity and stabilization?

Network systems over balanced digraphs & Passivity



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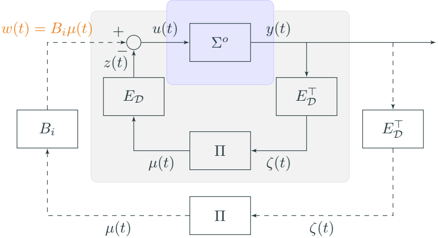
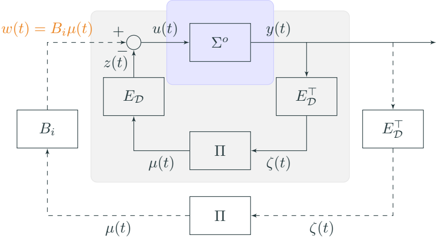
3-Step passivity-based analysis

STEP 1: FORWARD PATH



- $(\Sigma^o, \Pi, \mathcal{D}_b, w)$
  - $\max(D_o)$: maximal out-degree
  - $u = -B_o\mu$
- Σ_i^o's are:
  - Continuously differentiable
  - Output strictly passive (OP)

STEP 1: FORWARD PATH



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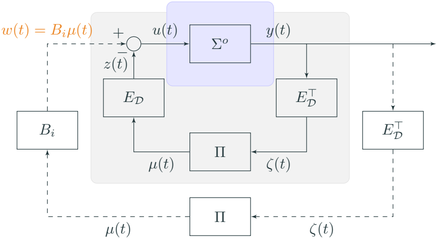
OP Σ_i^o: $$u_i y_i \geq \dot{Q}_i(x_i) + \varepsilon_i y_i^2, \quad \varepsilon_i > 0$$

STEP 1: FORWARD PATH



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  - $\max(D_o)$: maximal out-degree
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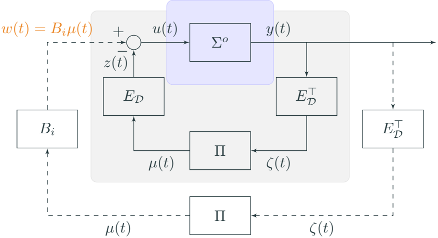
OP Σ_i^o: $\quad u_i y_i \geq \dot{Q}_i(x_i) + \varepsilon_i y_i^2, \;\; \varepsilon_i > 0$

OP Σ^o: $\quad u^\top y \geq \sum \dot{Q}_i(x_i) + \varepsilon \|y\|_2^2, \; \varepsilon = \min(\varepsilon_i)$

STEP 1: FORWARD PATH



- $(\Sigma^o, \Pi, \mathcal{D}_b, w)$
  - $\max(D_o)$: maximal out-degree
  - $u = -B_o\mu$
- Σ_i^o's are:
  - Continuously differentiable
  - Output strictly passive (OP)
- Goal:

$$u^\top \operatorname{Proj}_{S^\perp}(y) \geq l\|u\|^2 + e\|\operatorname{Proj}_{S^\perp}(y)\|^2$$

OP Σ_i^o: $\quad u_i y_i \geq \dot{Q}_i(x_i) + \varepsilon_i y_i^2, \;\; \varepsilon_i > 0$

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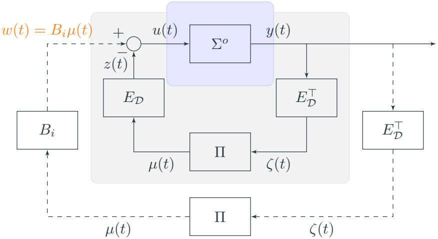

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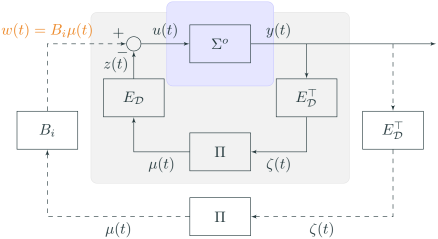
$$u^\top \operatorname{Proj}_{S^\perp}(y) = u^\top y - \frac{1}{n}u^\top \mathbb{1}\mathbb{1}^\top y \geq \sum \dot{Q}_i(x_i) - \frac{1}{n}u^\top \mathbb{1}\mathbb{1}^\top y + \varepsilon\|y\|_2^2$$



- $(\Sigma^o, \Pi, \mathcal{D}_b, w)$
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Proposition: Forward Path Passivity Relation

Assume that Σ_i^o's are OP-ε_i and with initial conditions that are asymptotically reachable from $\{0\}$. Let $\varepsilon = \min_i(\varepsilon_i)$. Then, it follows that

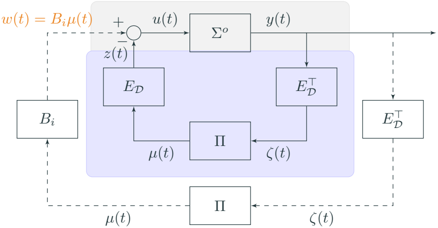
$$u^\top \operatorname{Proj}_{S^\perp}(y) \geq \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i(x_i) - \|u\|_2\|y\|_2 + \varepsilon\|\operatorname{Proj}_{S^\perp}(y)\|_2^2,$$

and the passivity relation satisfies,

$$\langle u^\tau, \operatorname{Proj}_{S^\perp}(y^\tau)\rangle \geq -\frac{\max(D_o)}{\varepsilon}\|\mu^\tau\|_{\mathscr{L}_2}^2 + \varepsilon\|\operatorname{Proj}_{S^\perp}(y^\tau)\|_{\mathscr{L}_2}^2.$$

$\cdot^\tau$: Truncate signals to $S = \operatorname{span}(\mathbb{1})$ after time $t > \tau$

STEP 2: INNER-FEEDBACK PATH



- $(\Sigma^o, \Pi, \mathcal{D}_b, w)$
- Π_k's are:
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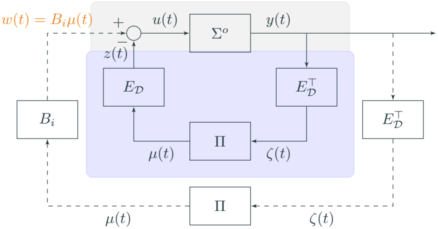

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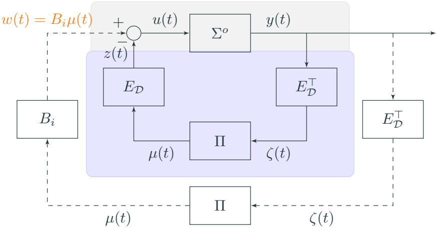
Property of the incidence matrices of $\mathcal{D}_b$:

$$E^\top y = \mathrm{Proj}_{S^\perp}(E^\top y) = E^\top \mathrm{Proj}_{S^\perp}(y).$$



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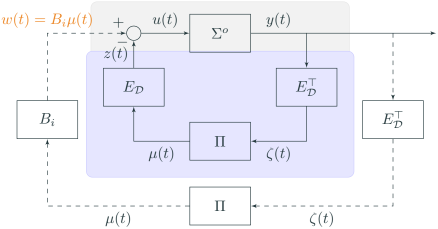
OP Π_k: $\qquad \mu_k\zeta_k \geq \dot{W}_k + \alpha_k\mu_k^2,\ \alpha = \min(\alpha_k)$



- $(\Sigma^o, \Pi, \mathcal{D}_b, w)$
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  $z^\top \operatorname{Proj}_{S^\perp}(y) \geq l\|z\|^2 + e\|\operatorname{Proj}_{S^\perp}(y)\|^2$

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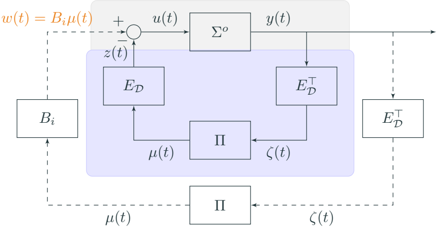

- $(\Sigma^o, \Pi, \mathcal{D}_b, w)$

Proposition: Inner-Feedback Path Passivity Relation

Assume that the controllers Π_k's are OP-α_i. Then, it follows that,

$$\mu^\top \zeta = \mu^\top E^\top y = z^\top \operatorname{Proj}_{S^\perp}(y) \geq \sum_{k=1}^{|\mathbb{E}|} \dot{W}_k(\eta_k) + \alpha \|\mu\|_2^2,$$

and the passivity relation satisfies,

$$\langle z^\tau, \operatorname{Proj}_{S^\perp}(y^\tau) \rangle \geq -\lim_{t \to -\infty} \textstyle\sum_{k=1}^{|\mathbb{E}|} W_k(\eta_k(t)) + \alpha \|\mu^\tau\|_{\mathscr{L}_2}^2.$$

STEP 3: A PASSIVITY-BASED ANALYSIS AND STABILIZATION

Directed networks with balanced diagraphs $(\Sigma^o, \Pi, \mathcal{D}_b, w)$

Agents $\Sigma_i^o : \dot{x}_i(t) = f_i(x_i(t), u_i(t)),\ y_i(t) = h_i(x_i(t)),\ i \in [1, n]$

Theorem

Suppose the following conditions hold:

1. f_i and h_is are continuously differentiable
2. OP $\Sigma_i^o \Rightarrow \langle u^\tau, \mathrm{Proj}_{S^\perp}(y^\tau)\rangle \geq -\frac{\max(D_o)}{\varepsilon}\|\mu^\tau\|_{\mathscr{L}_2}^2 + \varepsilon\|\mathrm{Proj}_{S^\perp}(y^\tau)\|_{\mathscr{L}_2}^2$
3. OP $\Pi_k \Rightarrow \langle z^\tau, \mathrm{Proj}_{S^\perp}(y^\tau)\rangle \geq \lim_{t\to-\infty} \sum_{k=1}^{|\mathbb{E}|} W_k(\eta_k(t)) + \alpha\|\mu^\tau\|_{\mathscr{L}_2}^2$

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Then, the network $(\Sigma^o, \Pi, \mathcal{D}_b, w)$ is stabilized.

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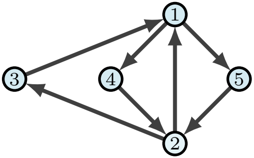
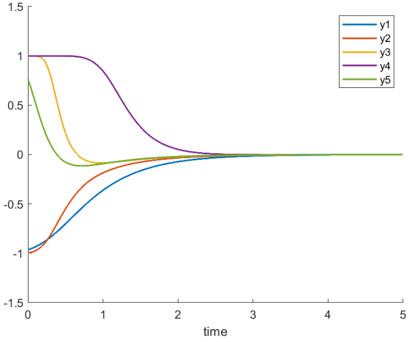
Recent results (Submit to CDC2025):

- Extend to **general** digraphs;
- A theorem on **output consensus** for systems consisting of passive agents.

- Systems
  $\Sigma_i^o : \dot{x}(t) = -a_i x_i(t) + u_i(t),\ y_i(t) = \tanh(x_i(t)),\ a_i > 0$ OP-a_i
- Maximal out-degree: $\max(D_o) = 2$
- Minimal passivity index of agents: $\varepsilon = \frac{3}{2}$, $\alpha \geq \frac{4}{3}$



A balanced digraph



Outputs of agents

CASE STUDY: NEURAL NETWORK

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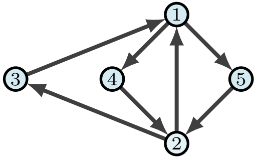

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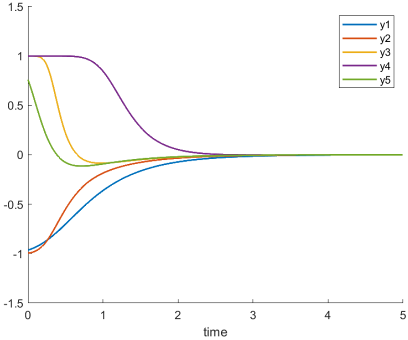

Outputs of agents

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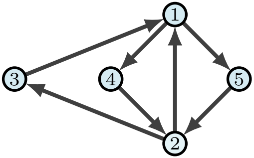
- **A sufficient condition**



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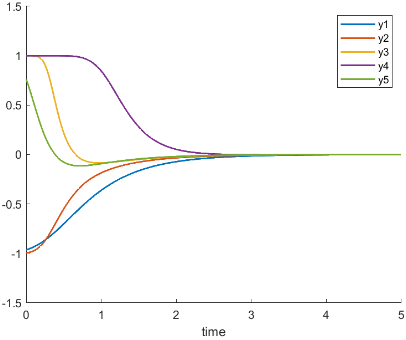

Outputs of agents

Summary:

- The passivity of operator $B_o \Pi E_{\mathcal{D}}^{\top}$: passive only when $\mathcal{D}$ is balanced.
- Loop decomposition: A general approach that enables a passivity analysis for the network systems with directed coupling.
- Stabilization of network systems over balanced digraphs: passivity-based conditions

Future work:

- complex dynamics, other passivity properties.
- A sufficient and necessary condition.

Thank-You!

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CONNECT LAB
Cooperative Networks and Controls