Fiedler-Based Characterization and Identification of Leaders in Semi-Autonomous Networks

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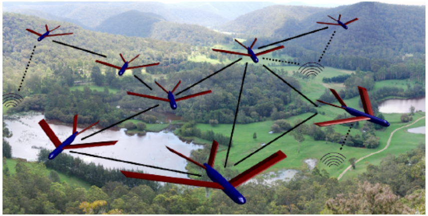
Introduction

- **Multi-agent systems (MAS)** consist of autonomous agents interacting to achieve a common goal.
- MAS are characterized by **decentralized control**, scalability, and robustness to individual agent failures.
- The security of MAS is vulnerable to cyber-physical attacks, especially through network topology identification.
- This work explores **identifying leader agents** in networked dynamic systems under a semi-autonomous consensus protocol.



Main Contributions

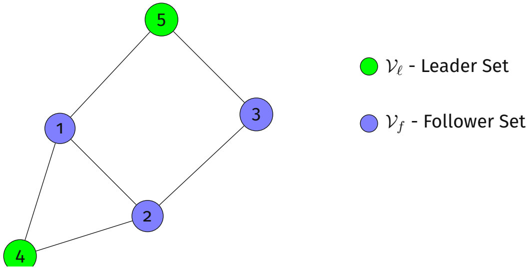
- Establish analytical conditions under which the components of the Fiedler vector corresponding to leaders and followers are separable.
- Development of a data-driven algorithm for identifying leader nodes in semi-autonomous consensus networks.

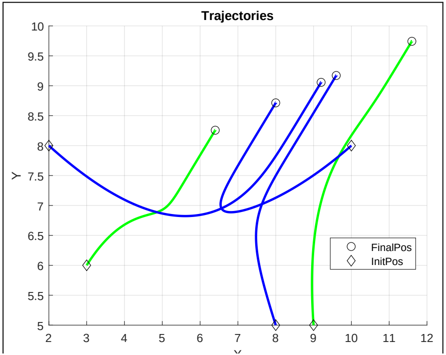
Problem Statement

Each agent has the closed-loop dynamics

$$\dot{x}_i = \begin{cases} \sum_{j\sim i}(x_j - x_i) + (u_i^{\text{ex}} - x_i), & i \in \mathcal{V}_\ell, \\ \sum_{j\sim i}(x_j - x_i), & i \in \mathcal{V}_f. \end{cases}$$

Network Topology





Problem Statement

$$\dot{x}_i = \begin{cases} \sum_{j\sim i}(x_j - x_i) + (u_i^{\text{ex}} - x_i), & i \in \mathcal{V}_\ell, \\ \sum_{j\sim i}(x_j - x_i), & i \in \mathcal{V}_f. \end{cases}$$

In a matrix form, we write

$$\dot{x} = -L_B(\mathcal{G})x + \begin{bmatrix} I_{|\mathcal{V}_\ell|} \\ 0_{|\mathcal{V}_f|\times|\mathcal{V}_\ell|} \end{bmatrix} \begin{bmatrix} u_1^{ex} \\ \vdots \\ u_{|\mathcal{V}_\ell|}^{ex} \end{bmatrix}.$$

- $L_B(\mathcal{G})$ is called the grounded Laplacian
- the eigen-pair (λ_F, v_F) of $L_B(\mathcal{G})$ corresponding to the smallest eigenvalue of are termed the Fiedler eigenvalue and eigenvector

Problem Statement

Assumption 1

The follower set is non-empty ($\mathcal{V}_f \neq \emptyset$).

Assumption 2

The external inputs u_i^{ex} are *constant signals*.

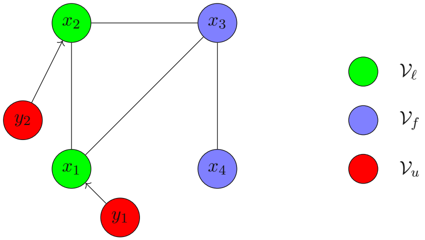
Problem Statement

We transform the semi-autonomous system into an autonomous-like structure:

- Introduce a state variable y representing external control inputs.
- Assume the external input remains constant, giving the dynamics:

$$\dot{y}_i = 0, \quad y_i(0) = u_i^{ex}.$$

Our graph $\bar{\mathcal{G}}$ is directed and consists of three groups: $\mathcal{V}_\ell, \mathcal{V}_f, \mathcal{V}_u$.



Problem Statement

The augmented dynamics is given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\bar{L} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} L_B & \bar{L}_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where $\bar{L} = L(\bar{\mathcal{G}})$ is the directed graph Laplacian of $\bar{\mathcal{G}}$. The submatrices are given by:

$$L_B = L(\mathcal{G}) + \begin{bmatrix} I_{|\mathcal{V}_\ell|} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{L}_{12} = \begin{bmatrix} I_{|\mathcal{V}_\ell|} \\ 0 \end{bmatrix}.$$

Problem Statement

The relation between the eigenvalues and the eigenvectors of $\bar{L}$ and L_B is as follows

$$\lambda_i(\bar{L}) = \begin{cases} 0, & i = 1, \ldots, m \\ \lambda_{i-m}(L_B), & i = m+1, \cdots, n+m \end{cases}.$$

and

$$v_i(\bar{L}) = \begin{bmatrix} v_i(L_B)^T & \mathbf{0}_m^T \end{bmatrix}^T, \quad i = m+1, \cdots, m+n.$$

Result

The Fiedler eigenvalue of L_B is also the smallest non-zero eigenvalue for $\bar{L}$.

Problem Statement

Lemma

If $\mathcal{G}$ is connected, then the following properties hold for L_B:

- The **Fiedler Eigenvalue** $\lambda_{\mathbf{F}}$ (smallest eigenvalue) of L_B is positive and simple and satisfies
$$0 < \lambda_F \leq 1$$
- The upper bound of the Fiedler eigenvalue is attained iff all nodes in $\mathcal{G}$ are leaders.
- The **Fiedler Eigenvector** v_F is unique (up to scaling) and is the only positive eigenvector.

Problem Statement

Objective

In this work we are interested in two problems:

i) Characterize a class of graphs such that it is possible to identify the leader nodes from the trajectories.

ii) Using the trajectory data, construct an algorithm to identify the leader nodes.

In the sequel we present our solution to the above problem.

Part 1: Leader-Identifiable Graphs

Semi-Normalized Adjacency Matrix

Define the **semi-normalized adjacency matrix** as

$$\hat{A} = \hat{D}_{\lambda_F}^{-1} A \in \mathbb{R}^{n\times n},$$

where $\hat{D}_{\lambda_F} \in \mathbb{R}^{n\times n}$ is given by

$$[\hat{D}_{\lambda_F}]_{ii} = \begin{cases} d(i) + 1 - \lambda_F & i \in \mathcal{V}_\ell \\ d(i) - \lambda_F & i \in \mathcal{V}_F \end{cases}.$$

Proposition 1

The semi-normalized adjacency matrix $\hat{A}(\mathcal{G})$ has real, simple, greatest eigenvalue equal to 1. Furthermore, the corresponding eigenvector is v_F, the Fiedler vector of $\bar{L}_{11}$.

Graph Sequence

To study how network structure affects the spectral properties of the grounded Laplacian, we introduce the notion of a *graph sequence*.

We will examine a sequence of expanding graphs $\mathcal{G}^{\sigma(i)}$ with some structure constraints:

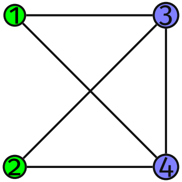
- The leaders set remains constant.
- The leader degree is constant.
- Leader nodes are not connected to each other.

Let $\mathcal{G}_f^{\sigma(i)}$ denote the graph obtained by removing all leader nodes and their incident edges from $\mathcal{G}^{\sigma(i)}$. The additional property in the sequence is as follows:

- The minimum degree in $\mathcal{G}_f^{\sigma(i)}$ is strictly increasing (denoted as $\underline{d}_F$).

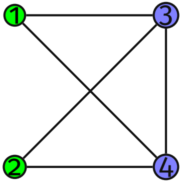
Graph Sequence

$\underline{d}_F$=1

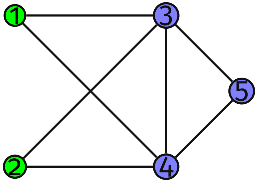


Graph Sequence

$\underline{d}_F$**=1**

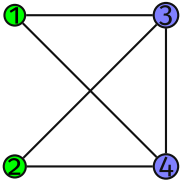


$\underline{d}_F$**=2**

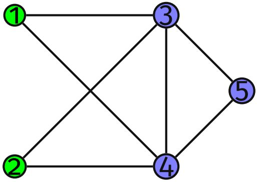


Graph Sequence

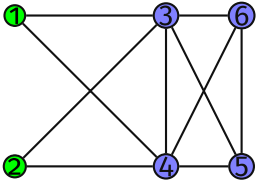
d_F=1



d_F=2

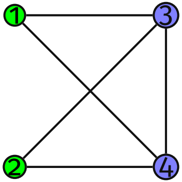


d_F=3

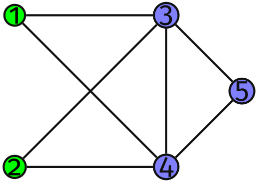


Graph Sequence

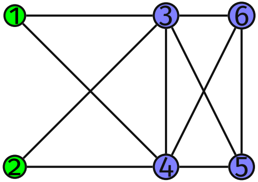
d_F=1



d_F=2

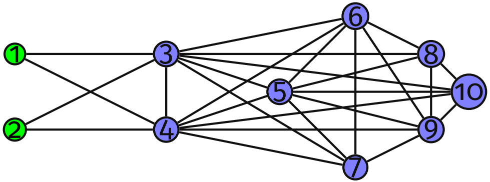


d_F=3



d_F=5

. . .



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Fiedler Vector Convergence

Lemma

Let $\{\mathcal{G}^{\sigma(i)}\}$ be a graph sequence satisfying the graph sequence definition. Then, $\lim_{i\to\infty}[\hat{A}^{\sigma(i)}(\mathcal{G})\bar{v}_F^{\sigma(i)}]_j = \lim_{i\to\infty}\bar{v}_F^{\sigma(i)}$, where

$$[\bar{v}_F^{\sigma(i)}]_j = \begin{cases} 1, & j \in \mathcal{V}_f^{\sigma(i)}, \\ \dfrac{\mathbf{d}_j}{\mathbf{d}_j + 1 - \lambda_F^{\sigma(i)}}, & j \in \mathcal{V}_\ell, \end{cases}$$

where $\lambda_F^{\sigma(i)}$ is the Fiedler value of the grounded Laplacian associated with the graph $\mathcal{G}^{\sigma(i)}$ and leader set $\mathcal{V}_\ell$.

Leader-Identifiable Graphs

Theorem

Let $\mathcal{G}$ be graph where the nodes separated into two groups, leaders $\mathcal{V}_\ell^{\mathcal{G}}$ and followers $\mathcal{V}_f^{\mathcal{G}}$. If the following conditions are met:

i) $\mathcal{G}$ is connected;

ii) $k \notin \mathcal{N}(j)$ for all $k, j \in \mathcal{V}_\ell$ (leader nodes are not connected to each other);

iii) with $\phi(j) := \frac{\deg(j)}{(\deg(j)+1-\lambda_F)}$, one has

$$1 - \max_{j \in \mathcal{V}_\ell} \phi(j) > \max_{j \in \mathcal{V}_\ell} \min_{k \in \mathcal{V}_\ell} |\phi(j) - \phi(k)|$$

iv) $\underline{d}_F$ is sufficient large,

then the Fiedler vector v_F of $\mathcal{G}$ satisfies

$$\min [v_F]_i - \max [v_F]_i \;>\; \max \min \big| [v_F]_i - [v_F]_j \big|.$$

Relative Tempo

- **The relative tempo** is the ratio of velocities of agents,

$$[\bar{\tau}(t)]_i = \frac{\dot{\bar{x}}_i(t)}{\dot{x}_{\text{ref}}(t)}$$

where $\dot{x}_{\text{ref}}$ is the velocity of a specific agent chosen as a common divisor for all others.

- The solution for $\dot{\bar{x}} = -\bar{L}\bar{x}$ is given by:

$$\bar{x}(t) = \mathrm{e}^{-\lambda_1(\mathcal{G})t}(\bar{p}_1\bar{x}_0)p_1 + \mathrm{e}^{-\lambda_2(\mathcal{G})t}(\bar{p}_2\bar{x}_0)p_2 + \cdots + \mathrm{e}^{-\lambda_{n+m}(\mathcal{G})t}(\bar{p}_{n+m}\bar{x}_0)p_{n+m}$$

where $\bar{p}_i$ is the ith row of P^{-1}, and p_i is the ith column of P and $\bar{L} = P\Lambda P^{-1}$.

Relative Tempo

$$[\bar{\tau}(t)]_i = \frac{\dot{\bar{x}}_i(t)}{\dot{x}_{\text{ref}}(t)}$$

- Substituting the solution derivative into the relative tempo expression yields

$$\tau_i = \frac{-\lambda_F(p_{m+1}^T\bar{x}_0)[v_F]_i e^{-\lambda_F t} + \sum_{k=m+2}^{m+n} -\lambda_k(p_k^T\bar{x}_0)[v_k]_i e^{-\lambda_k t}}{-\lambda_F(p_{m+1}^T\bar{x}_0)[v_F]_{ref} e^{-\lambda_F t} + \sum_{k=m+2}^{m+n} -\lambda_k(p_k^T\bar{x}_0)[v_k]_{ref} e^{-\lambda_k t}}, \quad i \in [1, \dots, n].$$

- For sufficient time T:

$$\boxed{[\bar{\tau}(t)]_i \simeq [v_F]_i, \quad t > T}$$

H. Shao and M. Mesbahi, *Degree of relative influence for consensus-type networks*, Portland, OR, USA, 2014, pp. 2676-2681.

Leaders Identification Algorithm

Assuming existence theorem conditions, we have

$$\min_{i\in\mathcal{V}_f}[v_F]_i - \max_{i\in\mathcal{V}_\ell}[v_F]_i > \max_{i,j\in\mathcal{V}_\ell, i>j} |[v_{F_s}]_i - [v_{F_s}]_j|.$$

Then we use the following algorithm to identify the leaders

Algorithm

- Step 1: Measure the agents velocities to an external constant input until steady state.
- Step 2: Calculate the relative tempo and compute the Fiedler vector.
- Step 3: Sort the Fiedler vector $v_{F_s} = \mathrm{sort}(v_F)$ where $[v_{F_s}]_i \le [v_{F_s}]_{i+1}$.
- Step 4: Calculate the number of leaders n_l with

$$n_l = |\mathcal{V}_\ell| = \arg \max_{j\in\{1,2,3,\cdots,n-1\}} \{[v_{F_s}]_{j+1} - [v_{F_s}]_j\}.$$

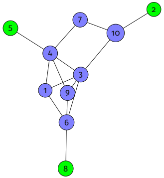
Example

In this example, we demonstrate a 2D scenario. We consider a system with $n = 10$ agents, where $\{2, 5, 8\} \in \mathcal{V}_L$. Recall the protocol dynamics:

$$\dot{x}_i = \begin{cases} \sum_{j\sim i}(x_j - x_i) + (u_i^{\text{ex}} - x_i), & i \in \mathcal{V}_\ell, \\ \sum_{j\sim i}(x_j - x_i), & i \in \mathcal{V}_f. \end{cases}$$

The external input provided to the leaders is

$$u = \begin{bmatrix} 40 & 35 & 48 & 44 & 16 & 45 \end{bmatrix}^T$$



Example Cont.

The grounded Laplacian and the Fiedler vector is given by:

$$L_B = \begin{bmatrix} 3 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 5 & -1 & 0 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 5 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 4 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 3 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 3 \end{bmatrix}, \quad v_F = \begin{bmatrix} 0.37 \\ \mathbf{0.18} \\ 0.37 \\ 0.35 \\ \mathbf{0.19} \\ 0.34 \\ 0.37 \\ \mathbf{0.19} \\ 0.37 \\ 0.32 \end{bmatrix}$$

Example Cont.

Next, we verify the conditions outlined in the Theorem:

- Leaders are not connected to each other.
- Degree distribution condition.
- $\underline{d}_F$ is sufficient large.

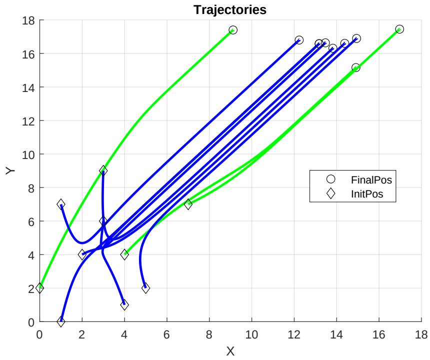
Since all conditions are satisfied, the leaders can be identified using the suggested algorithm.

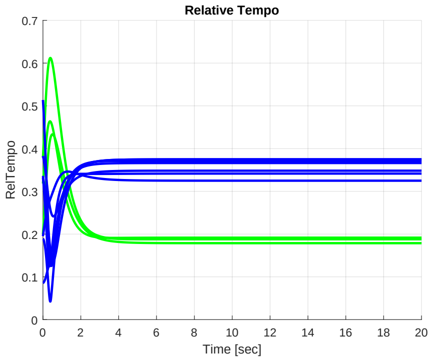
Example Cont.

I. Measure the velocities and calculate the relative tempo:

$$\tau = \begin{bmatrix} 0.37 & 0.18 & 0.37 & 0.35 & 0.19 & 0.34 & 0.37 & 0.19 & 0.37 & 0.32 \end{bmatrix}^T$$

We note that this is equal to the Fiedler vector v_F.





Example Cont.

II. Identify Leaders

- Sort the Fiedler vector $v_{F_s} = \text{sort}(v_F)$ where $[v_{F_s}]_i \leq [v_{F_s}]_{i+1}$:

$$v_{F_s} = \begin{bmatrix} 0.18 & 0.19 & 0.19 & 0.32 & 0.34 & 0.35 & 0.37 & 0.37 & 0.37 & 0.37 \end{bmatrix}^T$$

$$\text{Index} = \begin{bmatrix} 2 & 5 & 8 & 10 & 6 & 4 & 7 & 9 & 3 & 1 \end{bmatrix}^T$$

- Calculate the number of leaders n_l with

$$n_l = |\mathcal{V}_\ell = \arg \max_{j \in \{1,2,3,\cdots,n-1\}} \{[v_{F_s}]_{j+1} - [v_{F_s}]_j\} = 3.$$

- The leaders correspond to the smallest n_l components in v_{F_s}:

$$\mathcal{V}_\ell = \{2, 5, 8\}$$

Conclusion

- Certain graph structures are more likely to be associated with separation in the components of the Fiedler vector.
- Such graphs can facilitate leader identification through external observation in scenarios with constant external input.