SYMMETRY PRESERVING MOTION COORDINATION

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12.07.2022

European Control Conference

Workshop: *Rigidity Theory applied to Dynamic Systems from Parallel Robots to Multi-Agent Formations*

Technion - Israel Institute of Technology

DEPARTMENT OF AEROSPACE ENGINEERING
TECHNION Israel Institute of Technology

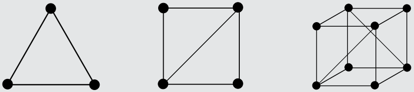
CONNECTLAB Cooperative Networks and Controls

Philadelphia Flight Control Laboratory

Formation Control Objective

Given a team of robots endowed with the ability to sense/ communicate with neighboring robots, design a control for each robot using only local information that

- moves the team into a desired spatial configuration - formation acquisition
- moves the team into a desired spatial configuration while simultaneously moving the formation through space as a rigid body - formation maneuvering



AGENT CONFIGURATIONS

- we consider a team of n agents in a metric space $\mathbb{R}^d, d \in \{2,3\}$,

$$p_i(t) \in \mathbb{R}^d$$

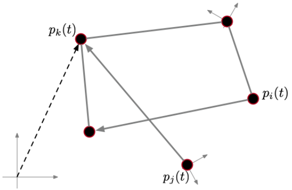
- the configuration of the agents at time t is the vector

$$p(t) = \begin{bmatrix} p_1(t) \\ \vdots \\ p_n(t) \end{bmatrix} \in \mathbb{R}^{nd}$$

- agent dynamics modeled as integrators

$$\dot{p}_i(t) = u_i(t),\ i = 1, \ldots, n$$

- agents interact according to a sensing graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$



- a framework is the pair $(\mathcal{G}, p)$

FORMATION CONSTRAINTS

- The desired formation is characterized by a set of M constraints, encoded in the function $F : \mathbb{R}^{nd} \to \mathbb{R}^M$, and a configuration $\mathbf{p}^\star$ satisfying the constraints.
- The set of all feasible formations is

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \,|\, F(p) = F(\mathbf{p}^\star)\}$$

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Formation Control Objective

For an ensemble of n agents with dynamics

$$\dot{p}_i = u_i,$$

with $p_i(t) \in \mathbb{R}^d$, an information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and formation constraint function $F : \mathbb{R}^{nd} \rightarrow \mathbb{R}^M$, design a distributed control law for each agent $i \in \{1, \ldots, n\}$ such that the set

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \,|\, F(p) = F(\mathbf{p}^\star)\},$$

is asymptotically stable.

Theorem - Distance Constrained Formation Control

Consider the potential function

$$V(p) = \frac{1}{4} \sum_{i \sim j} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2 \right)^2$$

and assume the desired distances $d_{ij}^\star$ correspond to a feasible formation. Then the gradient dynamical system

$$\dot{p} = -\nabla_p V(p) = -R^T(p)R(p)p + R^T(p)(d^\star)^2$$

asymptotically converges to the critical points of the potential function, i.e., $\frac{\partial V(p)}{\partial p} = 0$.

- $R(p)$ is the *rigidity matrix* for the framework $(\mathcal{G}, p)$
- rigidity theory used here to understand more about the equilibrium sets

PROOF SKETCH

(following De Queiroz '18)

Define some notations...

- relative positions: $\tilde{p}_{ij} = p_i - p_j$
- distance error: $e_{ij} = \|\tilde{p}_{ij}\| - d^\star_{ij}$
- intermediate variable: $z_{ij} = \|\tilde{p}_{ij}\|^2 - (d^\star_{ij})^2 = e_{ij}(e_{ij} + 2d^\star_{ij})$

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introduce Lyapunov candidate:

$$V(e) = \frac{1}{4}\sum_{i\sim j} z_{ij}^2 = z^T z$$

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time-derivative of Lyapunov function along trajectories

$$\dot{V} \quad \boxed{\dot{V} = z^T R(p) u}$$

IDEA: Design control u to ensure Lyapunov function is decreasing!

- Formation acquisition: $u = -R(p)^T z$
  ensures stable formation dynamics
  "classic" distance-constrained formation controller

FORMATION MANEUVERING

Formation maneuvering aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body.

Formation maneuvering aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body.

...recall our earlier Lyapunov function

$$\dot{W} = z^T R(p) u$$

choose $u = u_a + u_m$

- $u_a = -R(p)^T z$: used to attain desired formation
- $u_m = \mathbb{1} \otimes v_0 + \begin{bmatrix} \vdots \\ \omega_0 \times \tilde{q}_i \\ \vdots \end{bmatrix}$: rigid body translation (v_0) and rotation about a point ($\omega_0 \times \tilde{q}_i$)

Main Idea: rigid body rotations and translations are in the Kernel of the rigidity matrix!

if we relax our requirement to achieve formation shape, does it enrich the class of distance-preserving motions we can achieve?

Flexes of Frameworks

A framework $\mathcal{F} = (\mathcal{G}, p)$ is flexible if there exists a continuous motion of its joints (the points p_i) such that all pairs of joints connected by an edge remain at a constant distance, but between at least one pair of joints not joined by an edge, the distance changes.

- infinitesimal flexes can be found by examining the kernel of the rigidity matrix
- if the only infinitesimal flexes are the translations and rotations, the framework is rigid

Explore flexes of a framework that preserve some notion of symmetry

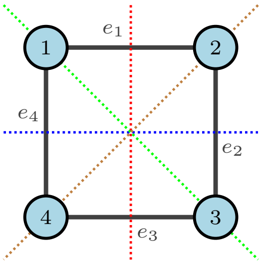
symmetric frameworks and rigidity well explored in the mathematics community

- B. Schulze and W. Whiteley, *Rigidity of Symmetric Frameworks* 2017
- B. Schulze, *The Orbit Rigidity Matrix of a Symmetric Framework* 2011
- R. Connelly, *Rigidity and Symmetry* 2014

Graph Automorphism

An automorphism of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of of its vertex set such that

$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$



Automorphisms encode graph symmetries

- identity: $\mathrm{Id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$
- $90°$ rotation: $\psi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
- $180°$ rotation: $\psi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
- $270°$ rotation: $\psi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

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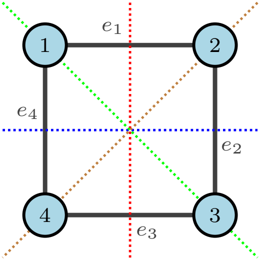


Automorphisms encode graph symmetries

- reflection: $\psi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
- reflection: $\psi_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
- reflection: $\psi_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$
- reflection: $\psi_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

Definition

Let X be a set, and let Γ be a collection of invertible functions $X \to X$. Then Γ is called a group if for any $\Gamma \ni f, g : X \to X$, both the composite function $f \circ g$ and the inverse function f^{-1} belong to Γ.

Automorphisms of a graph form a *group* - $\mathrm{Aut}(\mathcal{G})$

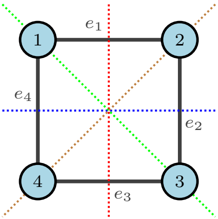
- $\mathrm{Aut}(\mathcal{G}) = \{\mathrm{Id}, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\}$
- subgroup: $\Gamma_1 = \{\mathrm{Id}, \psi_1, \psi_2, \psi_3\}$
- subgroup: $\Gamma_2 = \{\mathrm{Id}, \psi_2, \psi_4, \psi_5\}$
- subgroup: $\Gamma_3 = \{\mathrm{Id}, \psi_2\}$
- subgroup: $\Gamma_4 = \{\mathrm{Id}, \psi_6\}$
- subgroup: $\Gamma_5 = \{\mathrm{Id}, \psi_7\}$

Definition

A Γ-Symmetric graph is a graph for which there exists a group action $\theta : \Gamma \to \mathrm{Aut}(\mathcal{G})$. The action θ is free if $\theta(\gamma)(i) \neq i$ for all $i \in \mathcal{V}$ and non-trivial $\gamma \in \Gamma$.

Definition

For a Γ-symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and node $i \in \mathcal{V}$, the set $\Gamma^{(i)} = \{\theta(\gamma)(i) \,|\, \gamma \in \Gamma\}$ is called the vertex orbit. Similarly, for an edge $e = \{i, j\} \in \mathcal{E}$, the set $\Gamma^{(e)} = \{\{\theta(\gamma)(i), \theta(\gamma)(j)\} \,|\, \gamma \in \Gamma\}$ is termed the edge orbit.



Consider $\Gamma_3 = \{\mathrm{Id}, \psi_2\}$

- **Vertex Orbit:**
  $\Gamma^{(1)} = \Gamma^{(3)} = \{1, 3\},\ \Gamma^{(2)} = \Gamma^{(4)} = \{2, 4\}$
- **Edge Orbit:**
  $\Gamma^{(e_1)} = \Gamma^{(e_3)} = \{e_1, e_3\}$,
  $\Gamma^{(e_2)} = \Gamma^{(e_4)} = \{e_2, e_4\}$

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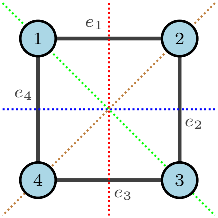
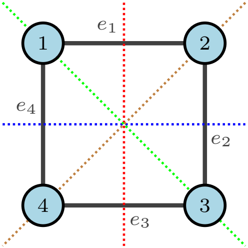


Consider $\Gamma_2 = \{\mathrm{Id}, \psi_2, \psi_4, \psi_5\}$

- **Vertex Orbit:**
  $\Gamma^{(i)} = \{1, 2, 3, 4\}$
- **Edge Orbit:**
  $\Gamma^{(e_i)} = \{e_1, e_2, e_3, e_4\}$

Definition

For a Γ-symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the multigraph $\mathcal{G}/\Gamma$ with vertex set $\mathcal{V}/\Gamma = \{\Gamma^{(i)} \mid i \in \mathcal{V}\}$ and edge set $\mathcal{E}/\Gamma = \{\Gamma^{(e)} \mid e \in \mathcal{E}\}$ is called the quotient graph.



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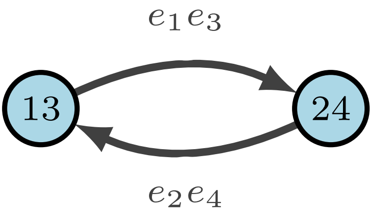

- nodes are the vertex orbits
- edges are the edge orbits

Definition

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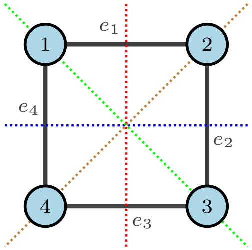

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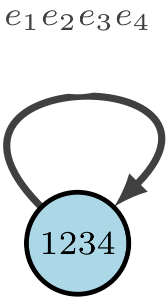
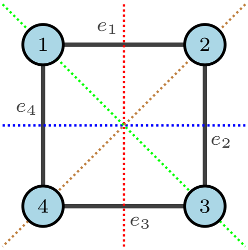


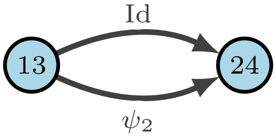
Definition

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a Γ-symmetric graph, where the group action $\theta : \Gamma \to \mathrm{Aut}(\mathcal{G})$ is free. Each edge orbit $\Gamma^{(e)}$ connecting $\Gamma^{(i)}$ and $\Gamma^{(j)}$ in $\mathcal{G}/\Gamma$ can be written as $\{\{\theta(\gamma)(i),\, \theta(\gamma) \circ \theta(\alpha)(j)\} \,|\, \gamma \in \Gamma\}$ for a unique $\alpha \in \Gamma$. For each $\Gamma^{(e)}$, orient $\Gamma^{(e)}$ from $\Gamma^{(i)}$ to $\Gamma^{(j)}$ in $\mathcal{G}/\Gamma$ and assign with the gain α. The resulting oriented quotient graph $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$, together with the gain labeling $w : E_0 \to \Gamma$, is the quotient Γ-gain graph $(\mathcal{G}_0, w)$ of $\mathcal{G}$.

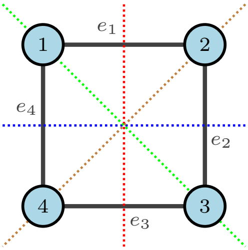
QUOTIENT Γ-GAIN GRAPHS



Consider $\Gamma_3 = \{\mathrm{Id}, \psi_2\}$



- vertex orbit $\{1, 3\}$ is adjacent to $\{2, 4\} = \{\mathrm{Id}(2), \mathrm{Id}(4)\}$ under identity element
- vertex orbit $\{1, 3\}$ is adjacent to $\{2, 4\} = \{\psi_2(2), \psi_2(4)\}$ under image of ψ_2
- note no self-loops - no vertex orbit is adjacent to itself under Id or ψ_2



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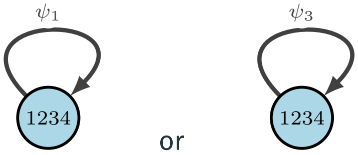


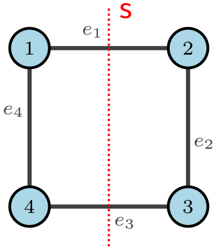
- vertex orbit $\{1, 2, 3, 4\}$ is adjacent to itself under ψ_1 or ψ_3
- no self loops with Id or ψ_2 - vertex orbit not adjacent to itself under these maps

Definition

Given a finite simple graph $\mathcal{G}$ and a map $p : \mathcal{V} \to \mathbb{R}^d$, a symmetry operation of the framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is an isometry x of $\mathbb{R}^d$ such that for some $\alpha_x \in \mathrm{Aut}(\mathcal{G})$ we have

$$x(p_i) = p_{\alpha_x(i)} \text{ for all } i \in \mathcal{V}.$$

The set of all symmetry operations of a framework $(\mathcal{G}, p)$ forms a group under composition, called the point group of $(\mathcal{G}, p)$.



- consider $\psi_4 \in \mathrm{Aut}(\mathcal{G})$ (reflection)
- isometry $x : (a, b) \mapsto (-a, b)$

satisfies $x(p_i) = p_{\alpha_x(i)}$ for all $i \in \mathcal{V}$.

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The set of all symmetry operations of a framework $(\mathcal{G}, p)$ forms a group under composition, called the point group of $(\mathcal{G}, p)$.

Let $\mathcal{R}_{(\mathcal{G},\Gamma)}$ denote set of all d-dimensional realizations of $\mathcal{G}$ whose point group is either equal to Γ or contains Γ as a subgroup.

- $\mathcal{R}_{(\mathcal{G},\Gamma)}$ consists of all realizations $(\mathcal{G}, p)$ for which there exists an action $\theta : \Gamma \to \mathrm{Aut}(\mathcal{G})$ such that

$$x(p(v)) = p(\theta(x)(v)) \text{ for all } v \in \mathcal{V} \text{ and all } x \in \Gamma.$$

Definition

For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a group action $\theta : \Gamma \to \mathrm{Aut}(\mathcal{G})$, and a homomorphism $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$, a framework $(\mathcal{G}, p)$ is Γ-symmetric if

$$\tau(\gamma)(p_i) = p_{\theta(\gamma)(i)} \text{ for all } \gamma \in \Gamma \text{ and } i \in \mathcal{V}.$$

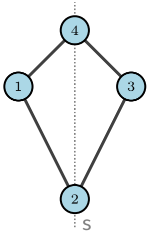
The symmetry group of a Γ-symmetric framework is the group

$$\tau(\Gamma) = \{\tau(\gamma) \,|\, \gamma \in \Gamma\}$$

of isometries of $\mathbb{R}^d$.

infinitesimal motions can also be studied in this framework

- $\tau(\gamma)(u_i) = u_{\theta(\gamma)(i)}$
- understanding symmetry structure means we only need to find infintesimal motion for one representative vertex in each vertex orbit



$(\mathcal{G}, p)$

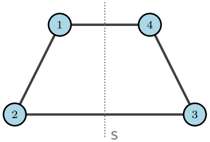
- $p_1 = (a, b)^T$
- $p_2 = (0, c)^T$
- $p_3 = (-a, b)^T$
- $p_4 = (0, d)^T$

Rigidity matrix

$$R(p) = \begin{bmatrix} (a \;\; b-c) & (-a \;\; c-b) & (0 \;\; 0) & (0 \;\; 0) \\ (a \;\; b-d) & (0 \;\; 0) & (0 \;\; 0) & (-a \;\; d-b) \\ (0 \;\; 0) & (a \;\; c-b) & (-a \;\; b-c) & (0 \;\; 0) \\ (0 \;\; 0) & (0 \;\; 0) & (-a \;\; b-d) & (a \;\; d-b) \end{bmatrix}$$

- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by
$(-1 \; 0 \; 0 \; \frac{a}{c-b} \; 1 \; 0 \; 0 \; \frac{a}{d-b})^T$
flex is symmetric! with respect to s
($x : (a, b) \mapsto (-a, b)$)



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- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-c, d)^T$
- $p_4 = (-a, b)^T$

Rigidity matrix

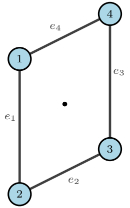
$$R(p) = \begin{bmatrix} (a-c \;\; b-d) & (c-a \;\; d-b) & (0 \;\; 0) & (0 \;\; 0) \\ (2a \;\; 0) & (0 \;\; 0) & (0 \;\; 0) & (-2a \;\; 0) \\ (0 \;\; 0) & (2c \;\; 0) & (-2c \;\; 0) & (0 \;\; 0) \\ (0 \;\; 0) & (0 \;\; 0) & (a-c \;\; d-b) & (c-a \;\; b-d) \end{bmatrix}$$

- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by

$$\left(-1 \;\; -1 \;\; -1 \;\; \frac{2(c-a)+b-d}{d-b} \;\; -1 \;\; -\frac{2(c-a)+b-d}{d-b} \;\; 1 \;\; 1\right)^T$$

flex is not symmetric with respect to s



$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-a, -b)^T$
- $p_4 = (-c, -d)^T$

Rigidity matrix

$$R(p) = \begin{bmatrix} (a-c\ b-d) & (c-a\ d-b) & (0\ 0) & (0\ 0) \\ (a+c\ b+d) & (0\ 0) & (0\ 0) & (-a-c\ \ -b-d) \\ (0\ 0) & (0\ 0) & (c-a\ d-b) & (a-c\ b-d) \\ (0\ 0) & (a+c\ b+d) & (-a-c\ \ -b-d) & (0\ 0) \end{bmatrix}$$

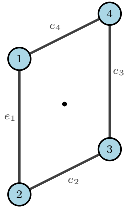
- 4-dimensional kernel - flexible framework
- 3 trivial motions

1-dimensional flex spanned by

$(-1\ 0\ \frac{cd-ab}{ad-bc}\ \frac{a^2-c^2}{ad-bc}\ 1\ 0\ -\frac{cd-ab}{ad-bc}\ -\frac{a^2-c^2}{ad-bc})^T$

flex is symmetric with respect to 180° rotation ($\mathcal{C}_2$)

ORBIT RIGIDITY MATRIX



$(\mathcal{G}, p)$

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-a, -b)^T$
- $p_4 = (-c, -d)^T$

Rigidity matrix

$$R(p) = \begin{bmatrix} (a-c\ b-d) & (c-a\ d-b) & (0\ 0) & (0\ 0) \\ (a+c\ b+d) & (0\ 0) & (0\ 0) & (-a-c\ \ -b-d) \\ (0\ 0) & (0\ 0) & (c-a\ d-b) & (a-c\ b-d) \\ (0\ 0) & (a+c\ b+d) & (-a-c\ \ -b-d) & (0\ 0) \end{bmatrix}$$

- 180° rotation of points corresponds to $\psi_2 \in \mathrm{Aut}(\mathcal{G})$
- recall: vertex orbits : $\{1, 3\}$, $\{2, 4\}$, edge orbits: $\{e_1, e_3\}$, $\{e_2, e_4\}$

symmetries make certain rows and columns of the rigidity matrix redundant

ORBIT RIGIDITY MATRIX

symmetries make certain rows and columns of the rigidity matrix redundant

$$R(p) = \begin{array}{c} \\ e_1 \\ e_4 \\ \mathcal{C}_2(e_1) \\ \mathcal{C}_2(e_4) \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 = \mathcal{C}_2(1) & 4 = \mathcal{C}_2(2) \end{array} \\ \begin{pmatrix} (a-c \ b-d) & (c-a \ d-b) & (0 \ 0) & (0 \ 0) \\ (a+c \ b+d) & (0 \ 0) & (0 \ 0) & (-a-c \ -b-d) \\ (0 \ 0) & (0 \ 0) & (c-a \ d-b) & (a-c \ b-d) \\ (a+c \ b+d) & (-a-c \ -b-d) & (0 \ 0) & \end{pmatrix} \end{array}$$

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Orbit Rigidity Matrix

$$\begin{array}{c} e_1 \\ e_4 \end{array} \overset{\begin{array}{cc} 1 & 2 \end{array}}{\begin{pmatrix} (p_1-p_2)^T & (p_2-p_1)^T \\ (p_1-C_2(p_2))^T & (p_2-C_2^{-1}(p_1))^T \end{pmatrix}} = \overset{\begin{array}{cc} 1 & 2 \end{array}}{\begin{pmatrix} (a-c,\; b-d) & (c-a,\; d-b) \\ (a+c,\; b+d)) & (c+a,\; d+b) \end{pmatrix}}$$

- 2 rows - one for each representative of edge orbits under action of $\mathcal{C}_2$
- 4 columns - nodes p_1, p_2 each have two dof; nodes $p_3 = \mathcal{C}_2(p_1)$ and $p_4 = \mathcal{C}_2(p_2)$ are uniquely determined by the symmetries

Definition [Shulze 2011]

For a Γ-symmetric framework $(\mathcal{G}, p)$ with quotient gain Γ-gain graph $(\mathcal{G}_0, w)$, the orbit rigidity matrix, $\mathcal{O}(\mathcal{G}_0, w, p)$, is the $|\mathcal{E}_0| \times d|\mathcal{V}_0|$ matrix defined as follows. Choose a representative vertex $\tilde{i}$ for each vertex $\Gamma^{(i)}$ in $\mathcal{V}_0$. The row corresponding to the edge $\tilde{e} = (\tilde{i}, \tilde{j})$ with gain $w(\tilde{e})$ in $\mathcal{E}_0$ is given by

$$(0 \ \cdots \ 0 \ \underbrace{p(\tilde{i}) - \tau(w(\tilde{e}))p(\tilde{j})}_{\tilde{i}} \ 0 \ \cdots \ 0 \ \underbrace{p(\tilde{j}) - \tau(w(\tilde{e}))^{-1}p(\tilde{j})}_{\tilde{i}} \ 0 \ \cdots \ 0).$$

If $\tilde{e} = (\tilde{i}, \tilde{i})$ is a loop at $\tilde{i}$, then the row corresponding to $\tilde{e}$ is given by

$$(0 \ \cdots \ 0 \ \underbrace{2p(\tilde{i}) - \tau(w(\tilde{e}))p(\tilde{i}) - \tau(w(\tilde{e}))^{-1}p(\tilde{i})}_{\tilde{i}} \ 0 \ \cdots \ 0 \, 0 \, 0 \ \cdots \ 0).$$

Theorem [Shulze 2011]

The kernel of the orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, w, p)$ is the space of (w, Γ)-symmetric infinitesimal motions of $(\mathcal{G}, p)$ restricted to the set of vertex orbits $\Gamma^{(i)}$ of $\mathcal{G}$.

- Orbit rigidity matrix can be used to identify symmetric infinitesimal flexes
- full-rank $\mathcal{O}(\mathcal{G}_0, w, p)$ implies none exist
- size of $\mathcal{O}(\mathcal{G}_0, w, p)$ does not depend on p, but only the graph and symmetry constraints

BACK TO MOTION COORDINATION

Symmetry preserving motion coordination aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body and preserving symmetry of configuration.

Symmetry preserving motion coordination aims to satisfy the formation control objective while simultaneously moving the formation through space as a rigid body and preserving symmetry of configuration.

...recall our earlier Lyapunov function

$$\dot{W} = z^T R(p) u$$

choose $u = u_a + u_m + u_s$

- $u_a = -R(p)^T z$: used to attain desired formation
- $u_m = \mathbb{1} \otimes v_0 + \begin{bmatrix} \vdots \\ \omega_0 \times \tilde{q}_i \\ \vdots \end{bmatrix}$: rigid body translation (v_0) and rotation about a point ($\omega_0 \times \tilde{q}_i$)
- u_s obtained from kernel of Orbit rigidity matrix

SYMMETRY PRESERVING MOTION COORDINATION

CONCLUDING REMARKS

- theory of symmetric frameworks and orbit rigidity matrix promising for complex motion coordination applications
- analytical challenges associated with identifying symmetries and automorphisms
- extensions for bearing rigidity

Acknowledgments

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