Rigidity Theory in SE(2) for Unscaled Relative Position Estimation using only Bearing Measurements

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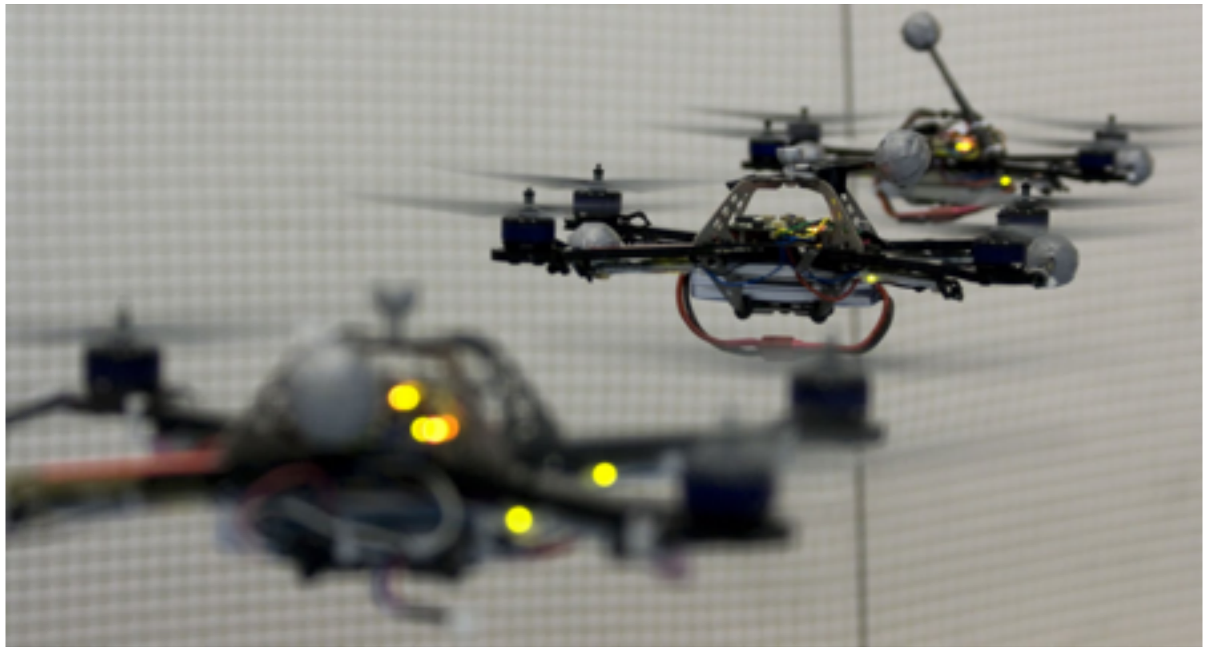
Challenges in Multi-Robot Systems



Solutions to coordination problems in multi-robot systems are *highly* dependent on the sensing and communication mediums available!

Sensing

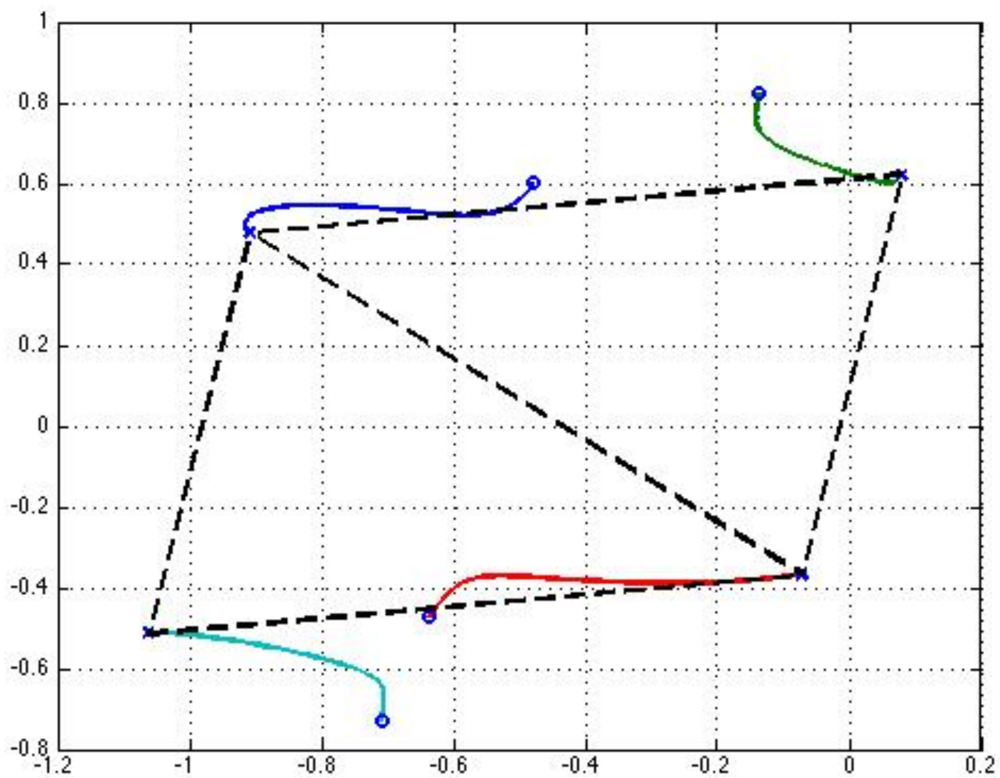
- GPS
- Relative Position Sensing
- Range Sensing
- Bearing Sensing

Communication

- Internet
- Radio
- Sonar
- MANet

selection criteria depends on mission requirements, cost, environment…

Formation Control: Distance-Based Approaches



robots modeled as integrators

$$\dot{p}_i = u_i$$

agents can sense range to neighbors determined by a (fixed) sensing graph

$$\|p_i - p_j\|^2$$

desired formation is specified by a vector of distances

$$d_{ij}^2$$

$$\dot{p}_i = \sum_{j \sim i} \left(\|p_i - p_j\|^2 - d_{ij}^2 \right) (p_j - p_i)$$

desired formation is (locally) asymptotically stable if the sensing graph is ***infinitesimally rigid***

[Krick2007, Anderson2008, Dimarogonas2008, Dörfler2010]

Rigidity Theory

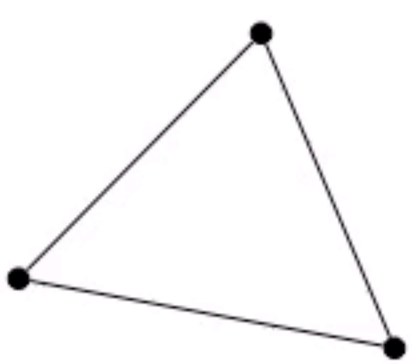
Rigidity is a combinatorial theory for characterizing the "stiffness" or "flexibility" of structures formed by rigid bodies connected by flexible linkages or hinges.

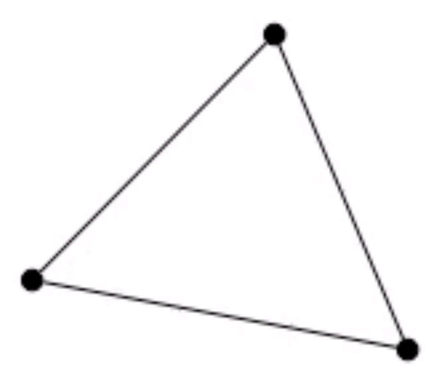
Distance Rigidity

- maintain distance pairs
- rigid body rotations and translations

Parallel Rigidity

- maintain angles (shape)
- rigid body translations and dilations



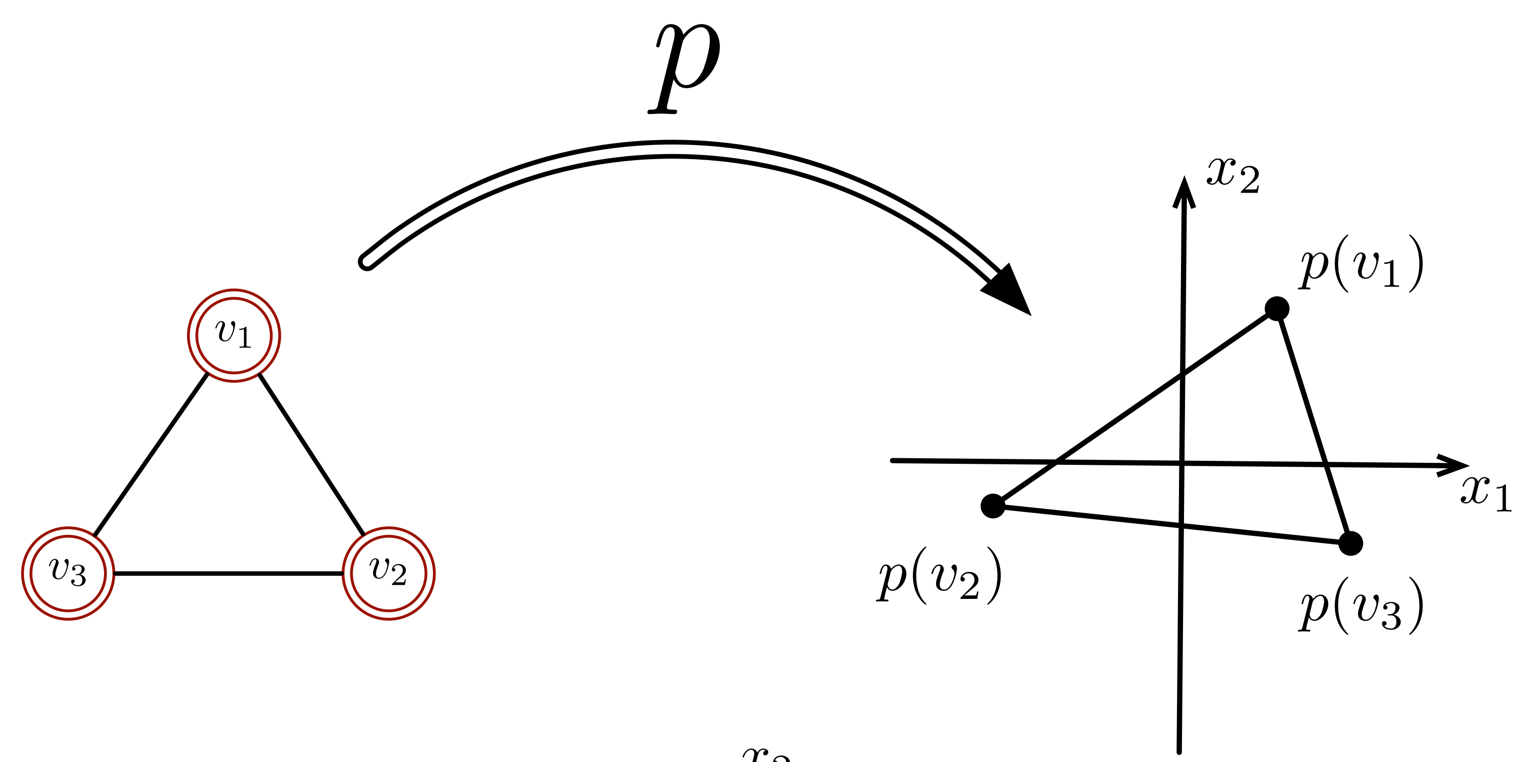


Rigidity Theory

bar-and-joint frameworks

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

$$p \; : \; \mathcal{V} \rightarrow \mathbb{R}^2$$

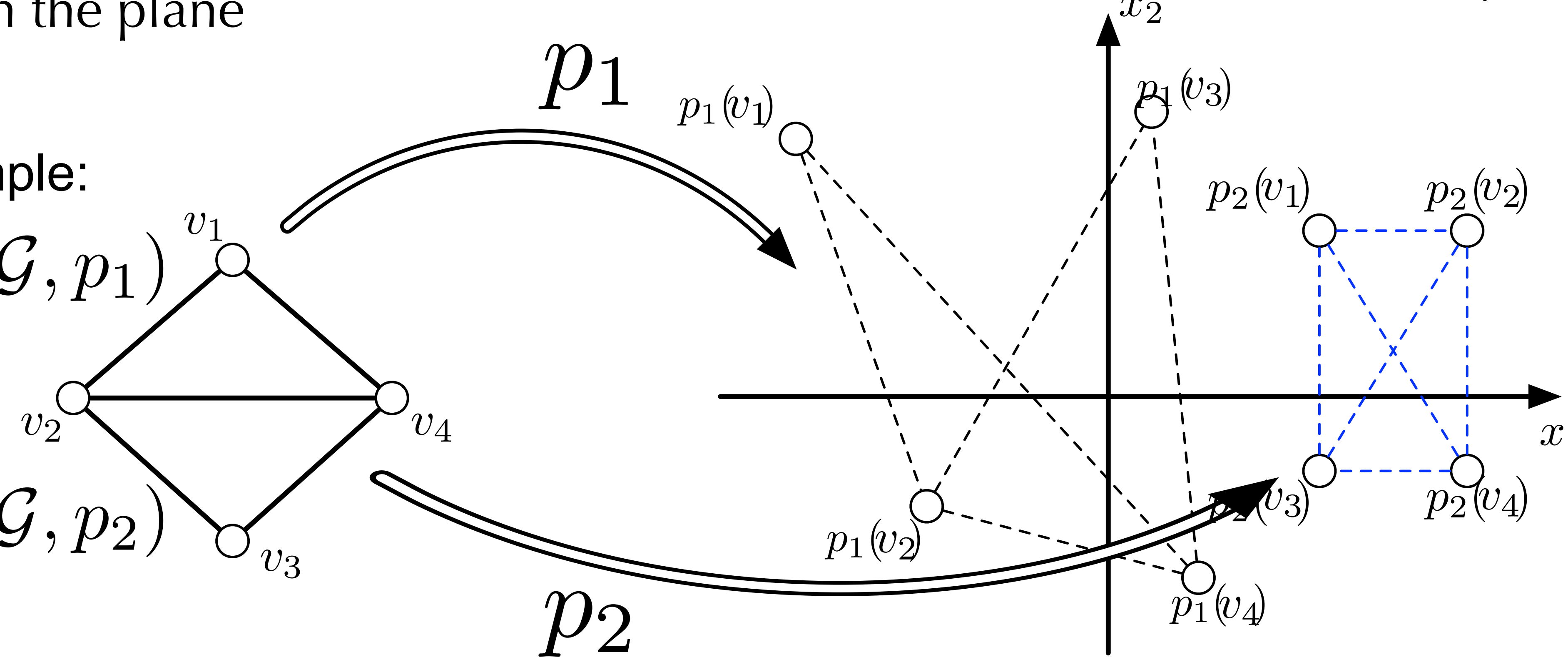


maps every vertex to a point in the plane

example:

$$\mathcal{F}_1 = (\mathcal{G}, p_1)$$

$$\mathcal{F}_2 = (\mathcal{G}, p_2)$$



Rigidity Theory

Rigidity is a combinatorial theory for characterizing the "stiffness" or "flexibility of structures formed by rigid bodies connected by flexible linkages or hinges.

Distance Rigidity

infinitesimal motions

$$(p(u) - p(v))^T (\xi(u) - \xi(v)) = 0$$

Rigidity Matrix

$$R(p)\xi = 0$$

Parallel Rigidity

infinitesimal motions

$$((p(u) - p(v))^{\perp})^T (\xi(u) - \xi(v)) = 0$$

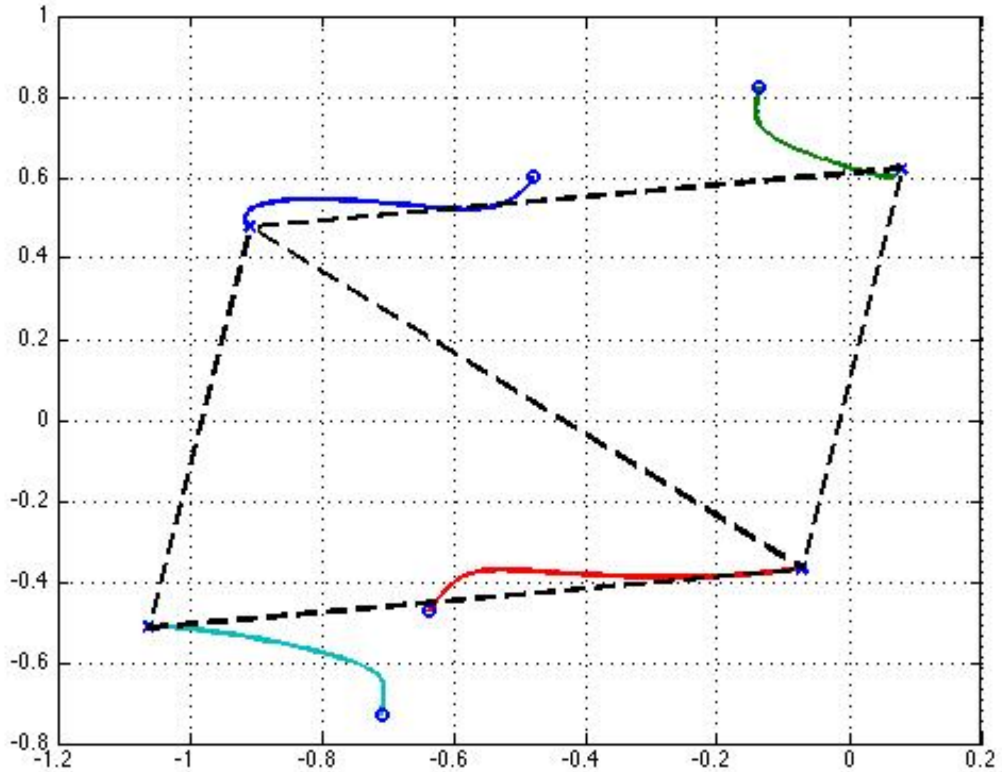
Parallel Rigidity Matrix

$$R_{||}(p)\xi = 0$$

Theorem

A framework is infinitesimally rigid if and only if the rank of the rigidity matrix is $2|\mathcal{V}| - 3$

Formation Control: Distance-Based Approaches

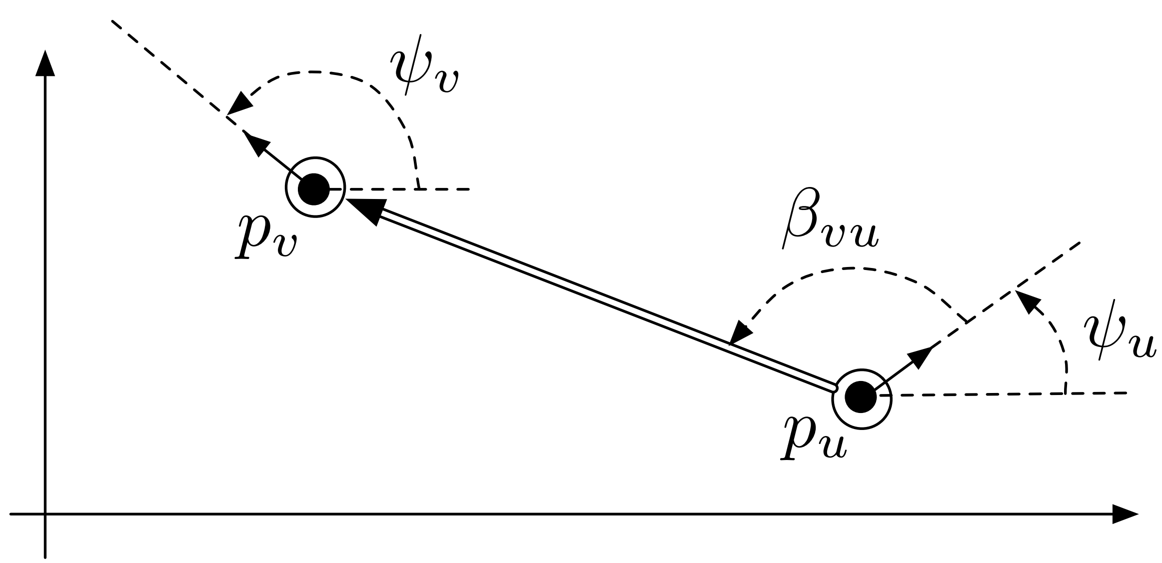


$$\dot{p}_i = \sum_{j \sim i} \left(\|p_i - p_j\|^2 - d_{ij}^2 \right) (p_j - p_i)$$

Important Assumptions

- point masses
- bidirectional sensing
- range measurements*
- *common reference frame is implicit*

A more "practical" approach…



- agents represented by points in SE(2) (position and orientation)
- bearing measurements with respect to *body-frame*
- unidirectional sensing

Rigidity Theory in SE(2)

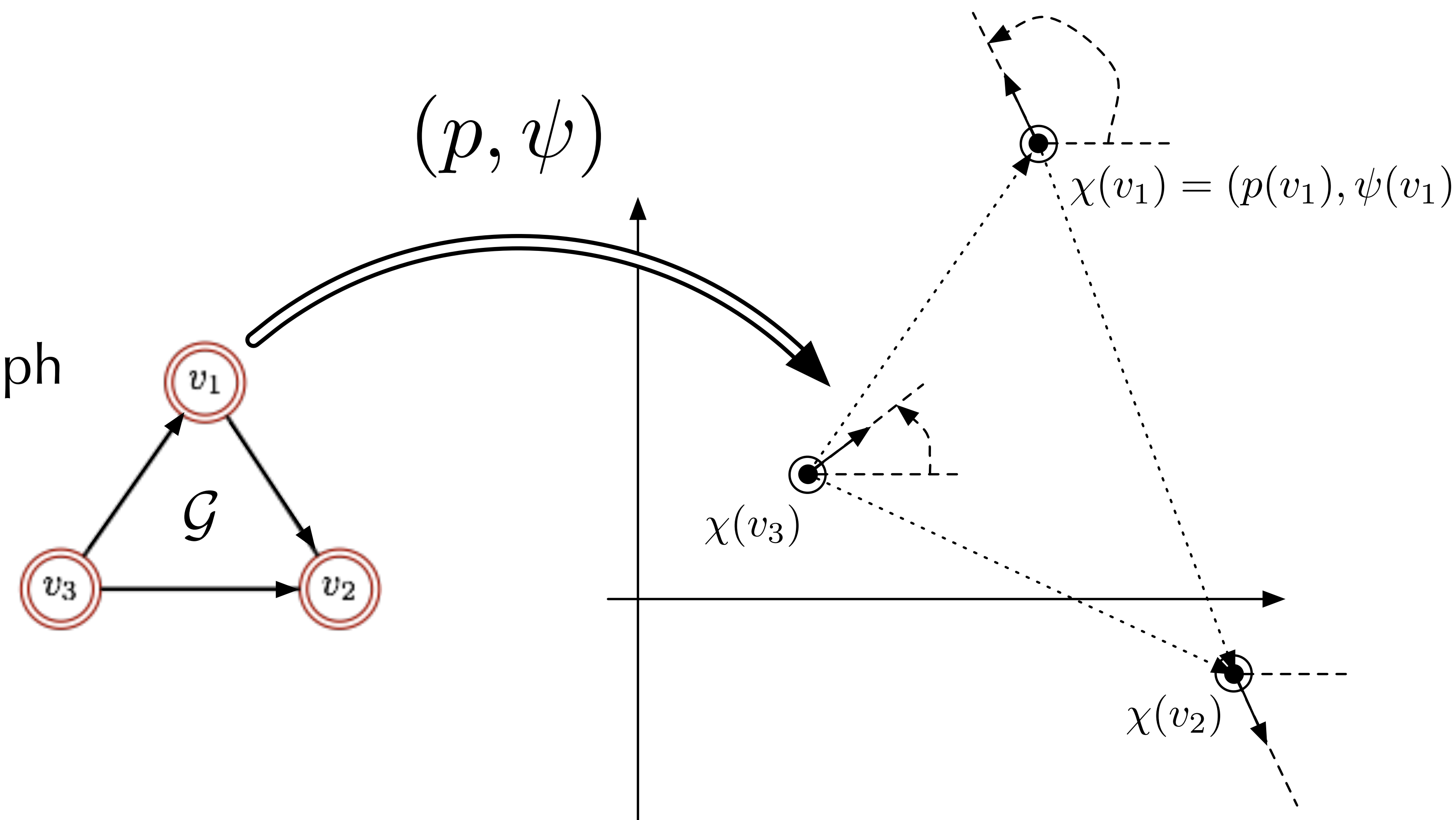
bar-and-joint frameworks in SE(2)

$(\mathcal{G}, p, \psi)$

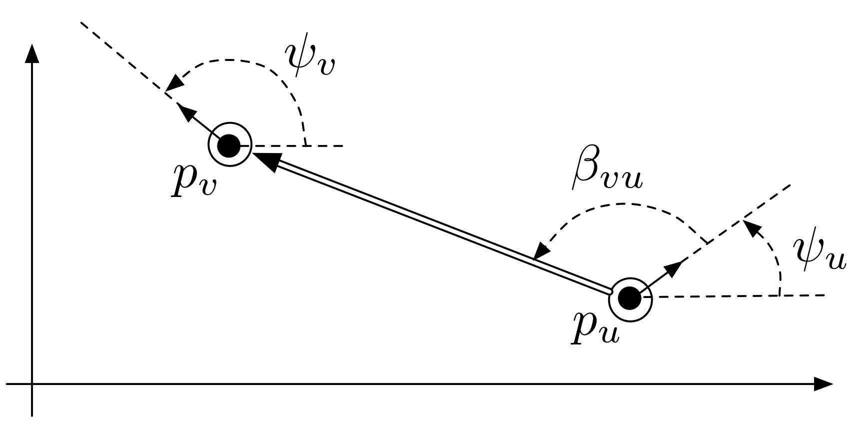
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a *directed* graph

$p : \mathcal{V} \to \mathbb{R}^2$

$\psi : \mathcal{V} \to \mathcal{S}^1$



a directed edge indicates availability of relative bearing measurement



stacked vector of entire framework

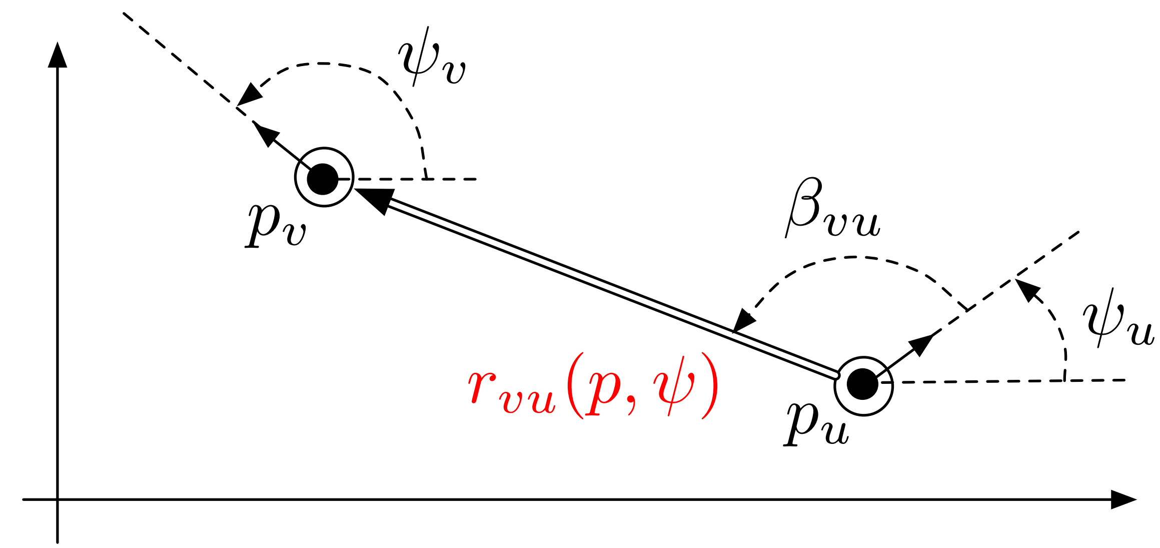
$$\chi_p = p(\mathcal{V}) \in \mathbb{R}^{2|\mathcal{V}|}$$

$$\chi_\psi = \psi(\mathcal{V}) \in \mathcal{S}^{1^{|\mathcal{V}|}}$$

Rigidity Theory in SE(2)

bar-and-joint frameworks in SE(2)

$$(\mathcal{G}, p, \psi)$$



directed bearing rigidity function

$$b_{\mathcal{G}} : SE(2)^{|\mathcal{V}|} \rightarrow {\mathcal{S}^1}^{|\mathcal{E}|}$$

$$b_{\mathcal{G}}(\chi(\mathcal{V})) = \left[\begin{array}{ccc} \beta_{e_1} & \cdots & \beta_{e_{|\mathcal{E}|}} \end{array} \right]^T$$

bearing can be expressed as a unit vector

$$r_{vu}(p,\psi) = \left[\begin{array}{c} r_{vu}^x \\ r_{vu}^y \end{array} \right] = \left[\begin{array}{c} \cos(\beta_{vu}) \\ \sin(\beta_{vu}) \end{array} \right]$$

$$= \underbrace{\left[\begin{array}{cc} \cos(\psi(v)) & \sin(\psi(v)) \\ -\sin(\psi(v)) & \cos(\psi(v)) \end{array} \right]}_{T(\psi(v))} \frac{(p(u) - p(v))}{\|p(v) - p(u)\|}$$

Rigidity Theory in SE(2)

Definition (Rigidity in SE(2))

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph and $K_{|\mathcal{V}|}$ be the complete directed graph on $|\mathcal{V}|$ nodes. The $SE(2)$ framework $(\mathcal{G}, p, \psi)$ is *rigid* in $SE(2)$ if there exists a neighborhood S of $\chi(\mathcal{V}) \in SE(2)^{|\mathcal{V}|}$ such that

$$b_{K_{|\mathcal{V}|}}^{-1}(b_{K_{|\mathcal{V}|}}(\chi(\mathcal{V}))) \cap S = b_{\mathcal{G}}^{-1}(b_{\mathcal{G}}(\chi(\mathcal{V}))) \cap S,$$

where $b_{K_{|\mathcal{V}|}}^{-1}(b_{K_{|\mathcal{V}|}}(\chi(\mathcal{V}))) \subset SE(2)$ denotes the pre-image of the point $b_{K_{|\mathcal{V}|}}(\chi(\mathcal{V}))$ under the directed bearing rigidity map.
The $SE(2)$ framework $(\mathcal{G}, p, \psi)$ is *roto-flexible* in $SE(2)$ if there exists an analytic path $\eta : [0,\ 1] \to SE(2)^{|\mathcal{V}|}$ such that $\eta(0) = \chi(\mathcal{V})$ and

$$\eta(t) \in b_{\mathcal{G}}^{-1}(b_{\mathcal{G}}(\chi(\mathcal{V}))) - b_{K_{|\mathcal{V}|}}^{-1}(b_{K_{|\mathcal{V}|}}(\chi(\mathcal{V})))$$

for all $t \in (0,\ 1]$.

Rigidity Theory in SE(2)

Definition (Equivalent and Congruent SE(2) Frameworks)

Frameworks $(\mathcal{G}, p, \psi)$ and $(\mathcal{G}, q, \phi)$ are *bearing equivalent* if

$$T(\psi(u))^T \overline{p}_{uv} = T(\phi(u))^T \overline{q}_{uv},$$

for all $(u, v) \in \mathcal{E}$ and are *bearing congruent* if

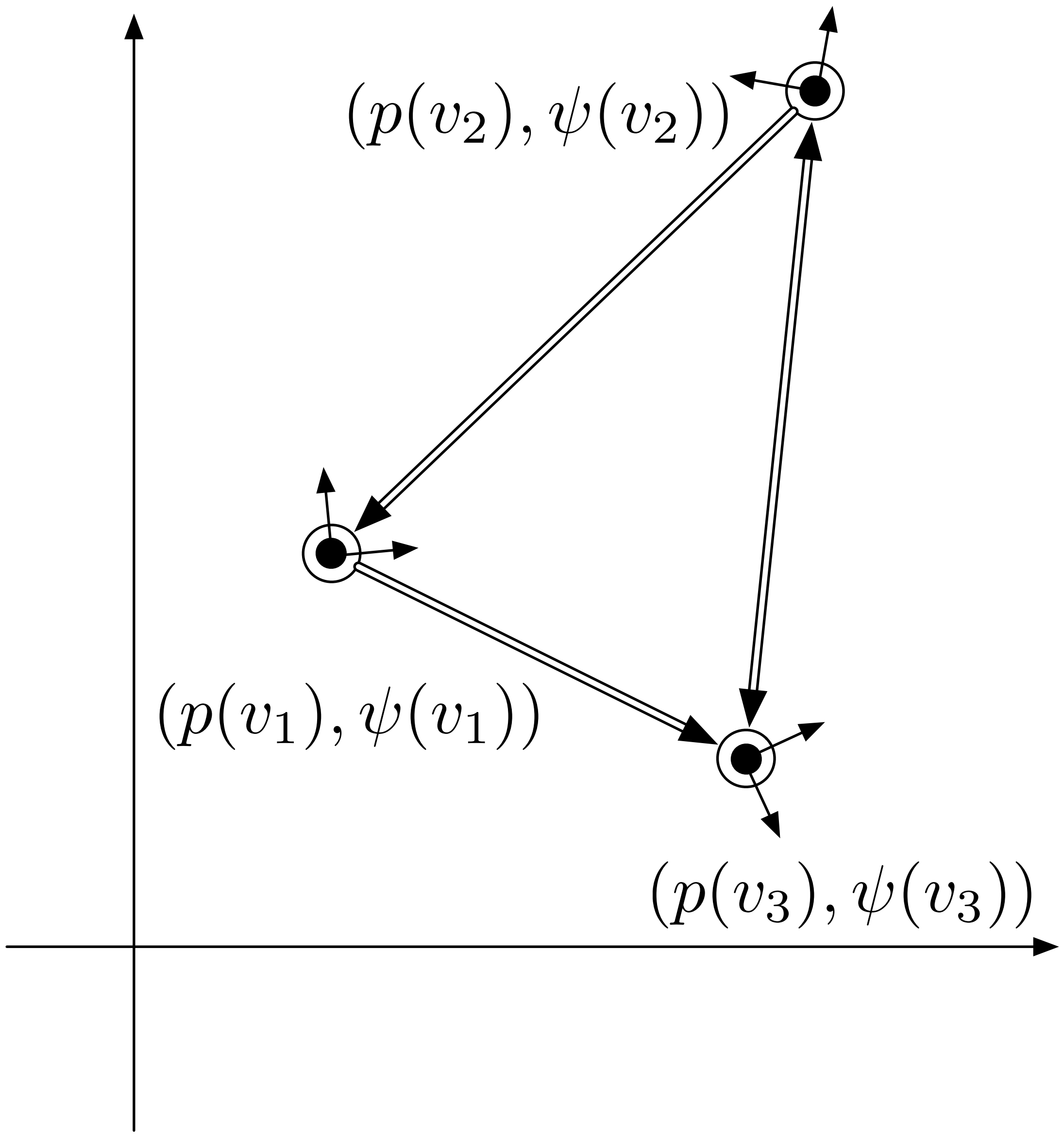
$$\begin{aligned} T(\psi(u))^T \overline{p}_{uv} &= T(\phi(u))^T \overline{q}_{uv} \text{ and} \\ T(\psi(v))^T \overline{p}_{vu} &= T(\phi(v))^T \overline{q}_{vu}, \end{aligned}$$

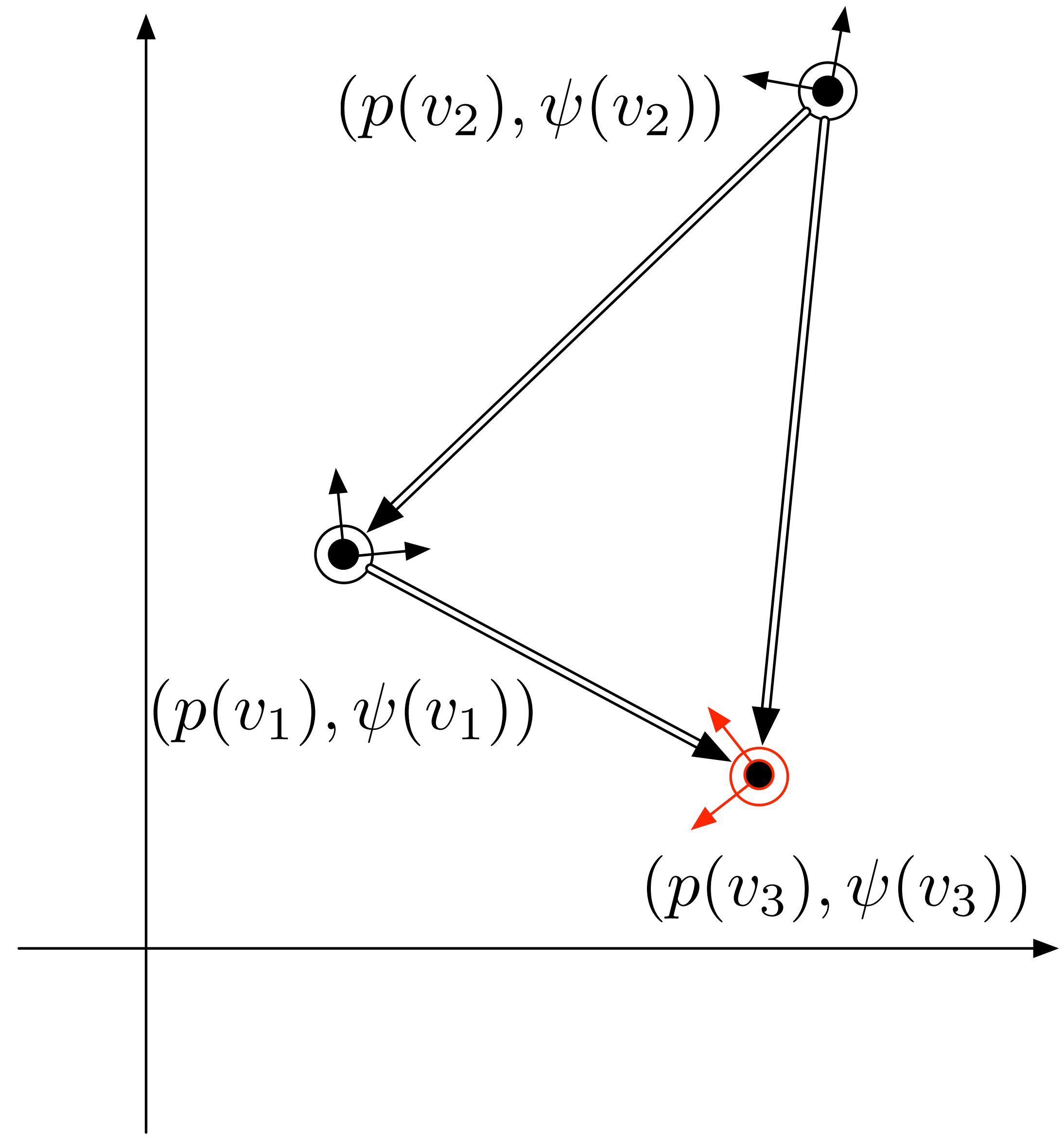
for all $u, v \in \mathcal{V}$.

Definition (Global Rigidity of SE(2) Frameworks)

A framework $(\mathcal{G}, p, \psi)$ is *globally rigid* in $SE(2)$ if every framework which is bearing equivalent to $(\mathcal{G}, p, \psi)$ is also bearing congruent to $(\mathcal{G}, p, \psi)$.

Rigidity Theory in SE(2)





both frameworks are *parallel rigid* (i.e., internal angles are fixed)

agent 3 maintains no bearing angles and is free to "spin" —> framework is *not* globally rigid in SE(2)!

Rigidity Theory in SE(2)

a "linearized" version of bearing rigidity

$$b_{\mathcal{G}}(\chi(\mathcal{V}) + \delta\chi) = b_{\mathcal{G}}(\chi(\mathcal{V})) + (\nabla_{\chi} b_{\mathcal{G}}(\chi(\mathcal{V})))\, \delta\chi + h.o.t.$$

Directed Bearing Rigidity Matrix

$$\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V})) := \nabla_{\chi} b_{\mathcal{G}}(\chi(\mathcal{V})) \in \mathbb{R}^{|\mathcal{E}| \times 3|\mathcal{V}|}$$

Theorem

An $SE(2)$ framework is infinitesimally rigid if and only if

$$\mathbf{rk}[\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V}))] = 3|\mathcal{V}| - 4$$

Rigidity Theory in SE(2)

a "linearized" version of bearing rigidity

$$b_{\mathcal{G}}(\chi(\mathcal{V}) + \delta\chi) = b_{\mathcal{G}}(\chi(\mathcal{V})) + (\nabla_{\chi} b_{\mathcal{G}}(\chi(\mathcal{V})))\,\delta\chi + h.o.t.$$

Directed Bearing Rigidity Matrix

$$\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V})) := \nabla_{\chi} b_{\mathcal{G}}(\chi(\mathcal{V})) \in \mathbb{R}^{|\mathcal{E}| \times 3|\mathcal{V}|}$$

$$\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V})) = \left[\begin{array}{cc} D_{\mathcal{G}}^{-1}(\chi_p) R_{\|}(\chi_p) & \overline{E}(\mathcal{G})^T \end{array}\right]$$

$$D_{\mathcal{G}}(\chi_p) = \mathbf{diag}\{\ldots, \|p(u) - p(v)\|^2, \ldots\}$$

$$[\overline{E}(\mathcal{G})]_{ik} = \begin{cases} 1, & \text{if } e_k = (v_i, v_j) \in \mathcal{E} \\ 0, & \text{o.w.} \end{cases}$$

Infinitesimal Motions in SE(2)

recall…

Distance Rigidity

- maintain distance pairs
- rigid body rotations and translations

$$R(p)\xi = 0$$

Parallel Rigidity

- maintain angles (shape)
- rigid body translations and dilations

$$R_{||}(p)\xi = 0$$

What are the infinitesimal motions in SE(2)?

Theorem

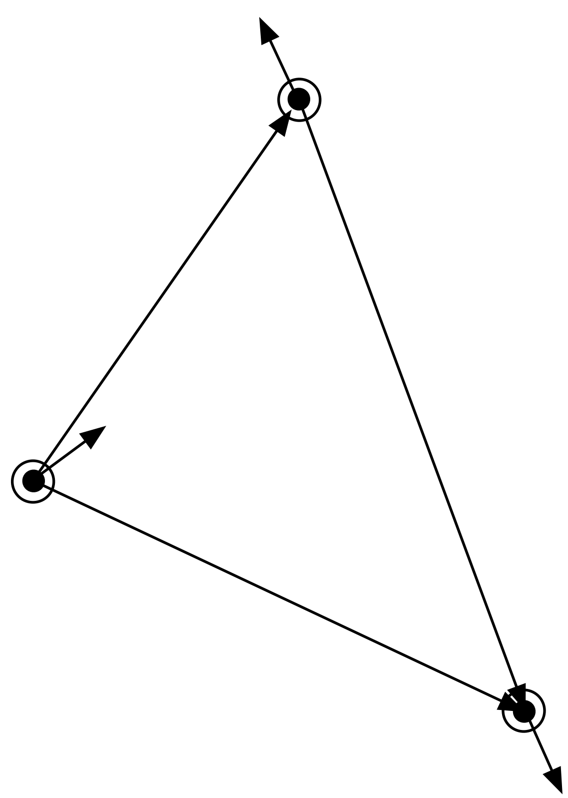
Every infinitesimal motion $\delta\chi \in \mathcal{N}\left[\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V}))\right]$ satisfies

$$R_{||}(\chi_p)\delta\chi_p = -D_{\mathcal{G}}(\chi_p)\overline{E}^T(\mathcal{G})\delta\chi_\psi$$

Infinitesimal Motions in SE(2)

$$R_{\|}(\chi_p)\delta\chi_p = -D_{\mathcal{G}}(\chi_p)\overline{E}^T(\mathcal{G})\delta\chi_\psi$$

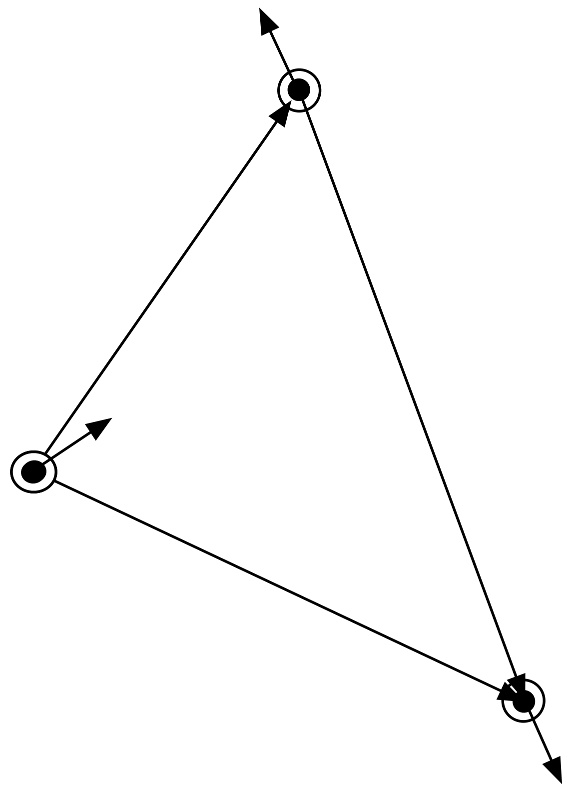
if all agents maintain attitude, infinitesimal motions are the ***translations*** and ***dilations*** of the framework



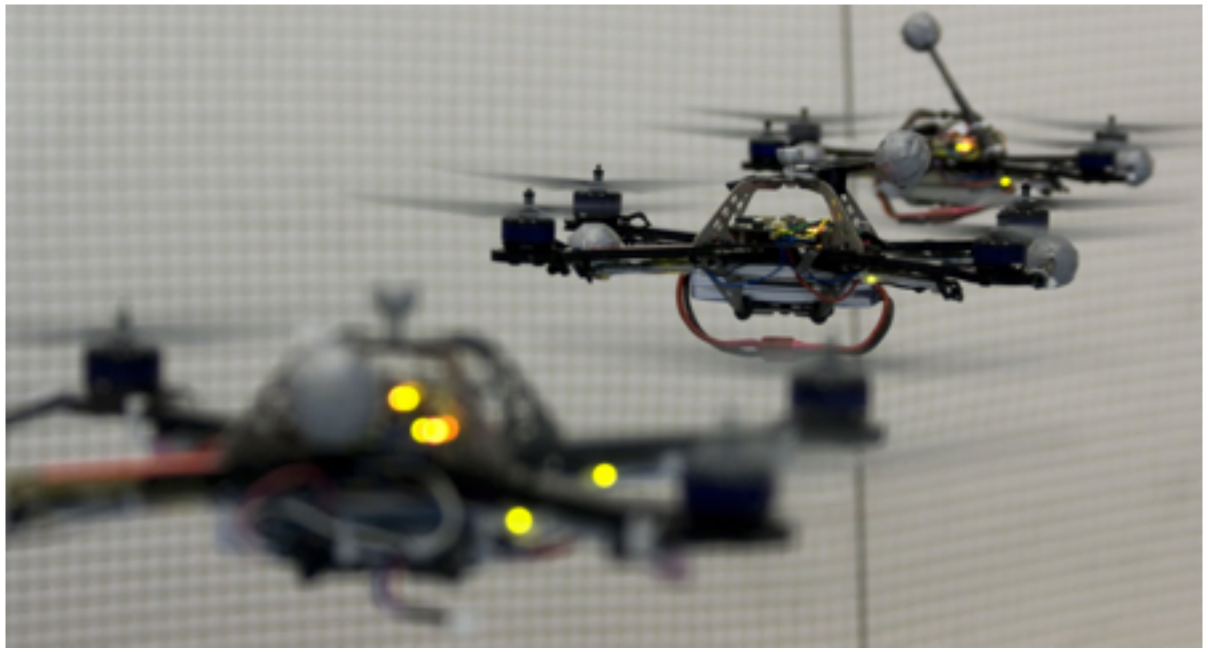
Infinitesimal Motions in SE(2)

$$R_{\|}(\chi_p)\delta\chi_p = -D_{\mathcal{G}}(\chi_p)\overline{E}^T(\mathcal{G})\delta\chi_\psi$$

if angular velocities are non-zero,
the infinitesimal motions are the
coordinated rotations of the framework



Estimation of Relative Positions



high level coordination objectives (formation keeping, localization, sensor fusion) require robots to know the transformation between local body frames - **relative positions** and **relative orientation**

<u>A distributed gradient descent estimator</u>

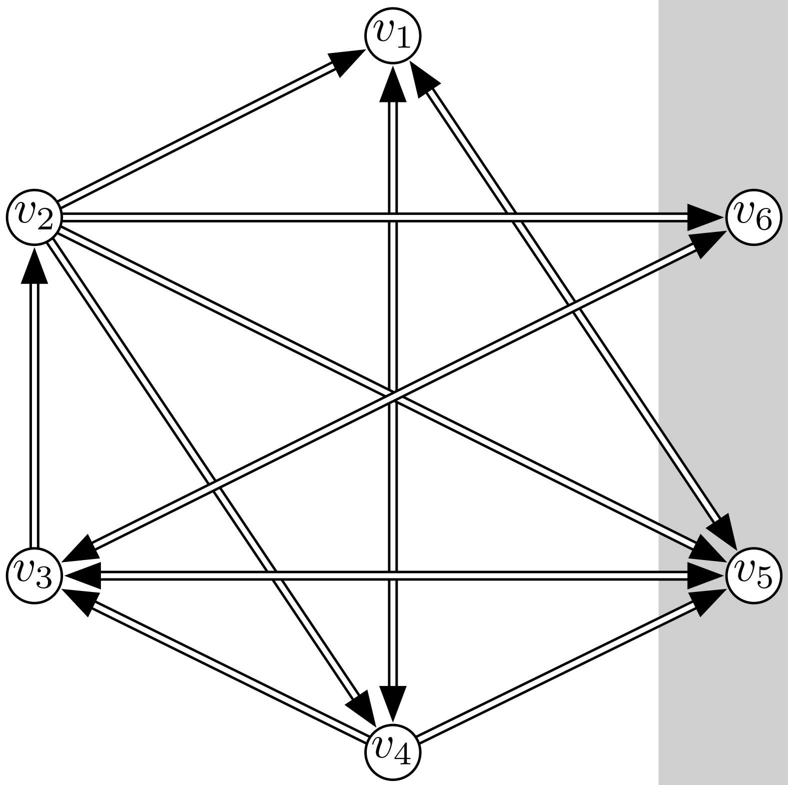
Bearing Error:

$$e(\hat{\xi}, \hat{\vartheta}, p, \psi) = b_{\mathcal{G}}(\chi(\mathcal{V})) - \hat{b}_{\mathcal{G}}(\hat{\xi}, \hat{\vartheta})$$

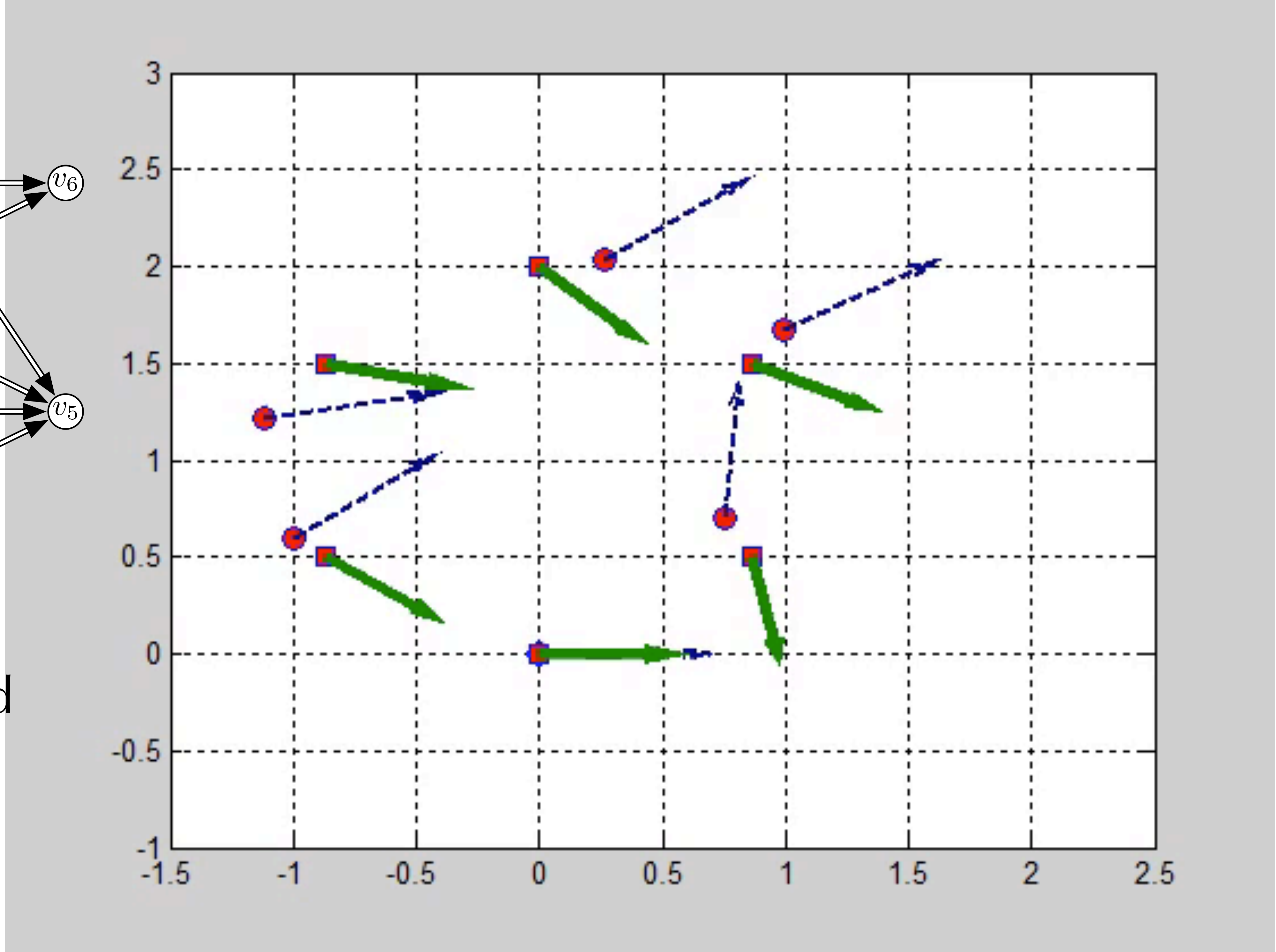
Cost Function:

$$J(e) = \frac{1}{2}\left(k_e\|e(\hat{\xi}, \hat{\vartheta}, p, \psi)\|^2 + k_1\|\hat{\xi}_{\iota\iota}\|^2 + k_2(\|\hat{\xi}_{\iota\kappa}\|^2 - 1)^2 + k_3(1 - \cos\hat{\vartheta}(\iota))\right)$$

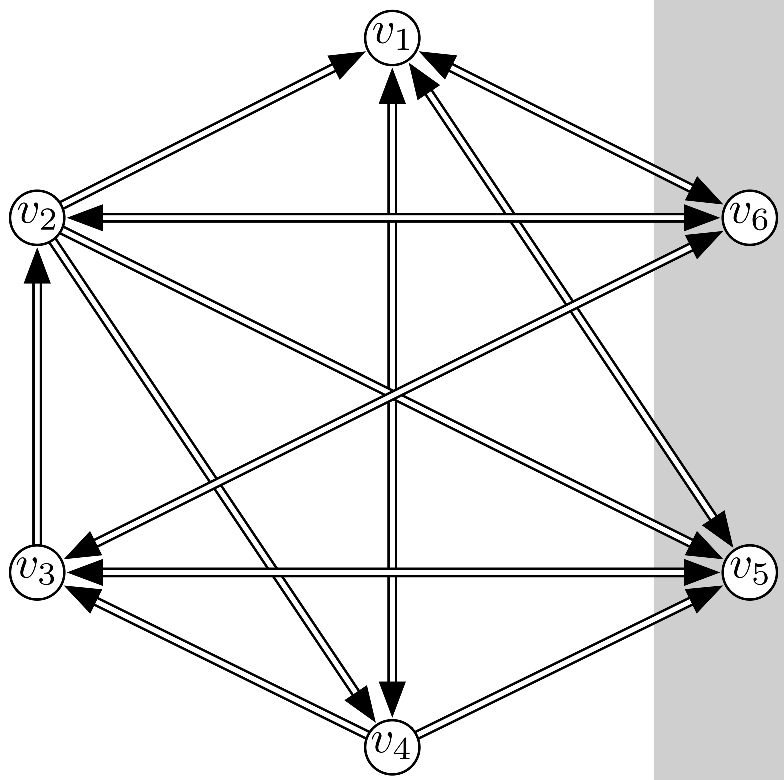
Estimation of Relative Positions



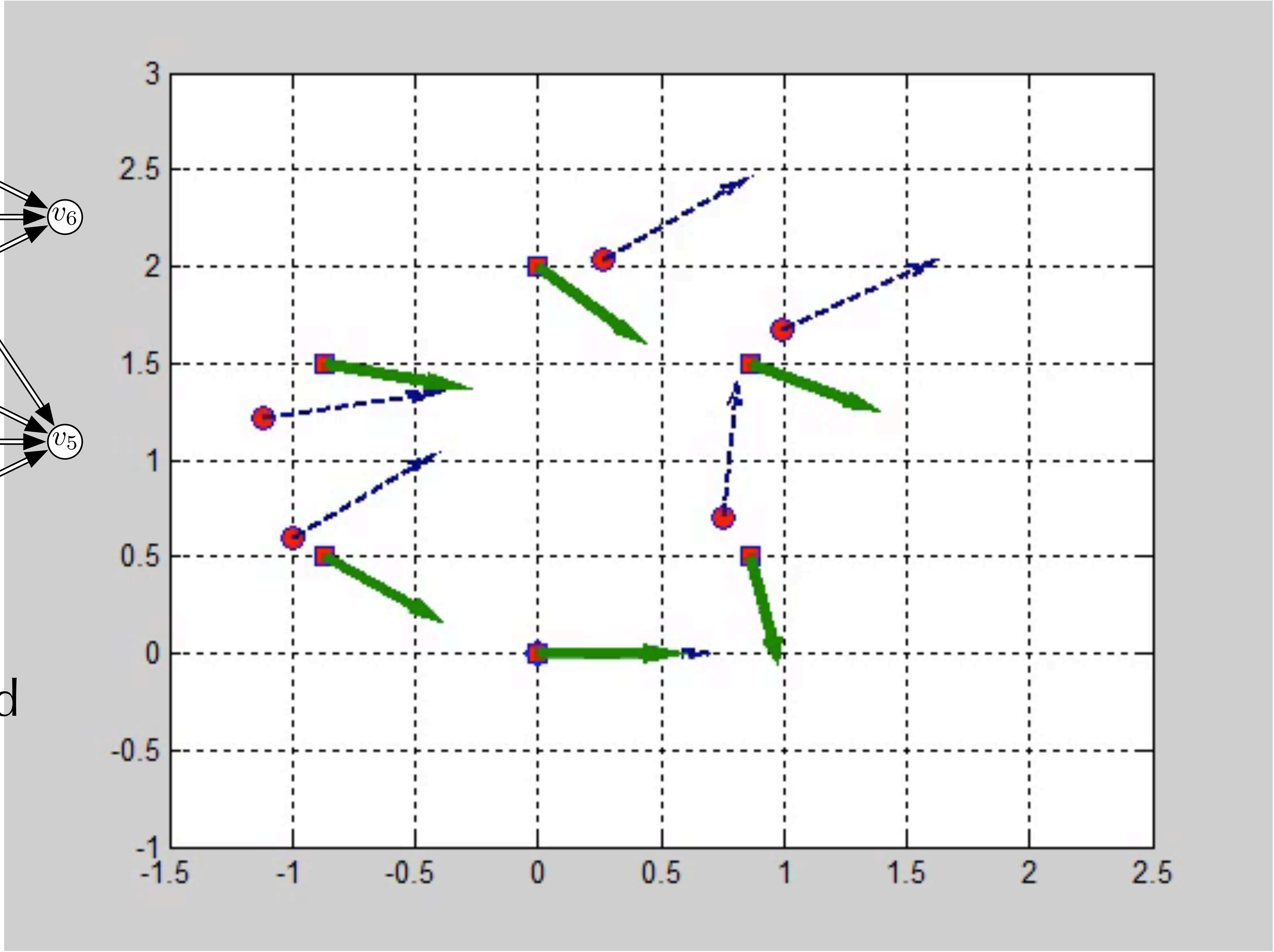
not SE(2)
infinitesimally rigid



Estimation of Relative Positions



SE(2)
infinitesimally rigid



Conclusions and Outlook

- coordination methods for multi-agent systems depend on sensing and communication mediums
- systems with *bearing* only sensing is a practical solution for many multi-agent systems

- extension of rigidity to concepts to frameworks in SE(2)
- SE(2) rigidity used to distributedly estimate relative positions from only bearing measurements

Conclusions and Outlook

- deeper results for bearing rigidity
- extensions to SE(3)
- estimation filter combined with higher-level tasks (formation keeping)
- control and estimation with field-of-view constraints

Acknowledgements

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Questions?

Rigidity Theory in SE(2)

a "linearized" version of bearing rigidity

$$b_{\mathcal{G}}(\chi(\mathcal{V}) + \delta\chi) = b_{\mathcal{G}}(\chi(\mathcal{V})) + (\nabla_{\chi} b_{\mathcal{G}}(\chi(\mathcal{V})))\,\delta\chi + h.o.t.$$

Directed Bearing Rigidity Matrix

$$\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V})) := \nabla_{\chi} b_{\mathcal{G}}(\chi(\mathcal{V})) \in \mathbb{R}^{|\mathcal{E}| \times 3|\mathcal{V}|}$$

Definition **(Infinitesimal Rigidity in SE(2))**

An $SE(2)$ framework $(\mathcal{G}, p, \psi)$ is *infinitesimally rigid* if $\mathcal{N}\left[\mathcal{B}_{\mathcal{G}}(\chi(\mathcal{V}))\right] = \mathcal{N}\left[\mathcal{B}_{K_{|\mathcal{V}|}}(\chi(\mathcal{V}))\right]$. Otherwise, it is *infinitesimally roto-flexible* in $SE(2)$.