

SYMMETRY-FORCED FORMATION CONTROL

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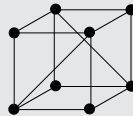
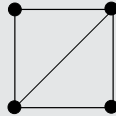
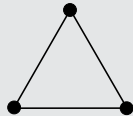


Technion Control & Systems Theory Seminar

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Formation Control Objective

Given a team of robots endowed with the ability to sense/ communicate with neighboring robots, design a control for each robot using only local information that moves the team into a desired spatial configuration.



- The **desired formation** is characterized by a set of M constraints, encoded in the function $F : \mathbb{R}^{nd} \rightarrow \mathbb{R}^M$, and a configuration \mathbf{p}^* satisfying the constraints.
- The set of all **feasible formations** is

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \mid F(p) = F(\mathbf{p}^*)\}$$

Formation Control Objective

For an ensemble of n agents with dynamics

$$\dot{p}_i = u_i,$$

with $p_i(t) \in \mathbb{R}^d$, an information exchange graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and formation constraint function $F : \mathbb{R}^{nd} \rightarrow \mathbb{R}^M$, design a distributed control law for each agent $i \in \{1, \dots, n\}$ such that the set

$$\mathcal{F}(p) = \{p \in \bar{\mathcal{D}} \mid F(p) = F(\mathbf{p}^*)\},$$

is asymptotically stable.

Theorem - Distance Constrained Formation Control

[Krick 2009]

Consider the potential function

$$F_f(p) = \frac{1}{4} \sum_{ij \in \mathcal{E}} (\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^2)^2$$

and assume the desired distances d_{ij}^* correspond to a feasible formation. Then the gradient dynamical system

$$u_i = -\nabla_{p_i} F_f(p) = \sum_{ij \in \mathcal{E}} (\|p_i - p_j\|^2 - \mathbf{d}_{ij}^2) (p_j - p_i)$$

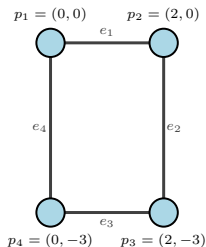
asymptotically converges to the critical points of the potential function, i.e., $\frac{\partial F_f(p)}{\partial p} = 0$.

$$F_f(p) = \frac{1}{4} \sum_{ij \in \mathcal{E}} (\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^*)^2$$

- formation potential can be written in terms of a **rigidity function**

$$F_f(p) = \frac{1}{2} \|r_{\mathcal{G}}(p) - r_{\mathcal{G}}(\mathbf{p})\|^2$$

- $r_{\mathcal{G}} : p \mapsto \left[\dots \quad \frac{1}{2} \|p_i - p_j\|^2 \quad \dots \right]^T$: distances between neighbors
- \mathbf{p} : a configuration satisfying distance constraints (i.e., $\|\mathbf{p}_i - \mathbf{p}_j\|^2 = \mathbf{d}_{ij}^2$)



$$r_{\mathcal{G}}(p) = \begin{bmatrix} \|p_1 - p_2\|^2 \\ \|p_2 - p_3\|^2 \\ \|p_3 - p_4\|^2 \\ \|p_4 - p_1\|^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 4 \\ 9 \end{bmatrix}$$

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- \mathbf{p} : a configuration satisfying distance constraints (i.e., $\|\mathbf{p}_i - \mathbf{p}_j\|^2 = \mathbf{d}_{ij}^2$)
- rigidity theory looks for **distance-preserving infinitesimal motions**

$$r_{\mathcal{G}}(p + \delta p) = r_{\mathcal{G}}(p) + \frac{\partial r_{\mathcal{G}}(p)}{\partial p} \delta p + \text{h.o.t}$$

- infinitesimal motions satisfy $\frac{\partial r_{\mathcal{G}}(p)}{\partial p} \delta p = 0$
- the **Rigidity matrix**: $R(p) = \frac{\partial r_{\mathcal{G}}(p)}{\partial p} \in \mathbb{R}^{|\mathcal{E}| \times 2|\mathcal{V}|}$
- "rigid body" rotations and translations are always distance preserving: **trivial motions**
- A framework (\mathcal{G}, p) is **infinitesimally rigid** if the only infinitesimal motions are trivial

our formation control

$$u_i = -\nabla_{p_i} F_f(p) = \sum_{ij \in \mathcal{E}} (\|p_i - p_j\|^2 - \mathbf{d}_{ij}^2) (p_j - p_i)$$

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can be expressed with rigidity matrix

$$u = -R^T(p)(R(p)p - \mathbf{d}^2)$$

our formation control

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a proof sketch

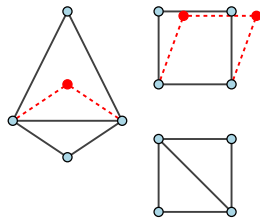
- define error dynamics for distance error: $e = R(p)p - \mathbf{d}^2$

$$\dot{e} = -R(p)R^T(p)e$$

- Lyapunov argument $V(e) = \frac{1}{2}\|e\|^2$
 - when $R(p)R^T(p) > 0$, we have (local) exponential convergence to desired formation
 - **good frameworks** are i) infinitesimally rigid, and ii) full row-rank (**isostatic frameworks**)

Rigidity theory helps us understand

- how many constraints are required to ensure **uniqueness** of formation shape (modulo translations, rotations, and flip ambiguities)
- how the constraints should be **distributed** in the network

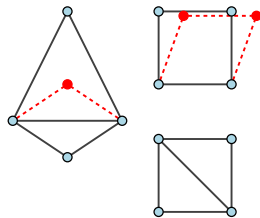


A widely accepted architectural requirement for distance constrained formation control is that **isostatic** frameworks are required. Equivalent to:

$$\text{rk } R(p) = 2|\mathcal{V}| - 3 \text{ and } |\mathcal{E}| = 2|\mathcal{V}| - 3 \quad (\text{in } \mathbb{R}^2)$$

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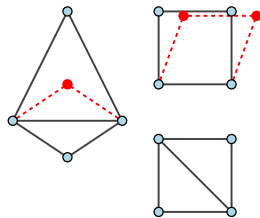
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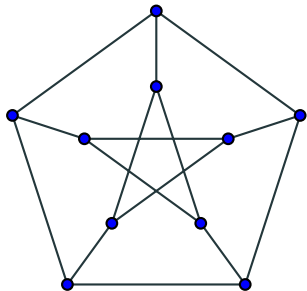
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Q: is this a necessary condition? (can we solve the problem with fewer edges?)

A: Impose additional **symmetry** constraints without requiring more information exchange (in fact, less!)

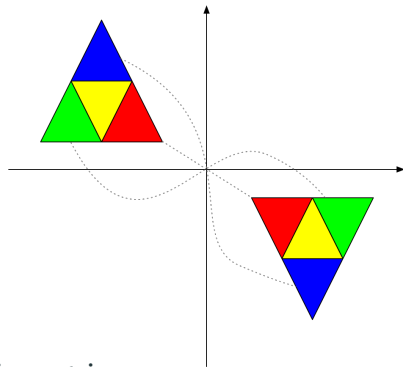
GRAPH SYMMETRIES AND POINT GROUPS

Graph Symmetries



- graph automorphisms

Point Groups



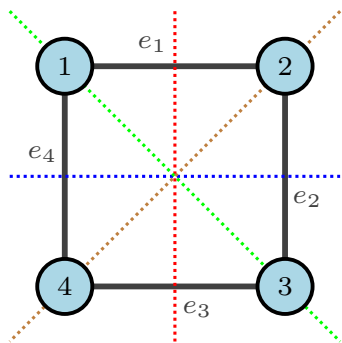
- isometries

Graph Automorphism

An **automorphism** of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of its vertex set such that

$$\{v_i, v_j\} \in \mathcal{E} \Leftrightarrow \{\psi(v_i), \psi(v_j)\} \in \mathcal{E}$$

Automorphisms encode graph **symmetries**



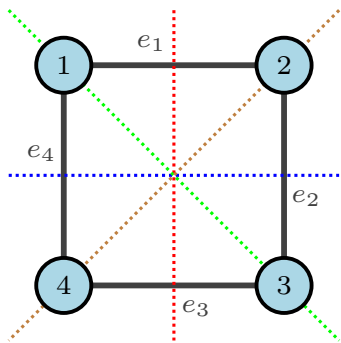
- identity: $\text{Id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$
- 90° rotation: $\psi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
- 180° rotation: $\psi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
- 270° rotation: $\psi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

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Automorphisms encode graph **symmetries**



- **reflection:** $\psi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
- **reflection:** $\psi_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
- **reflection:** $\psi_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$
- **reflection:** $\psi_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

Definition

Let X be a set, and let Γ be a collection of invertible functions $X \rightarrow X$. Then Γ is called a **group** if the identity map, Id , belongs to Γ , and for any $\Gamma \ni f, g : X \rightarrow X$, both the composite function $f \circ g$ and the inverse function f^{-1} belong to Γ .

Automorphisms of a graph form a *group* - $\text{Aut}(\mathcal{G})$

$$- \text{Aut}(\mathcal{G}) = \{\text{Id}, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\}$$

A **subgroup** is a subset of a group, and also satisfies all properties of a group

$$- \{\text{Id}, \psi_1, \psi_2, \psi_3\}$$

$$- \{\text{Id}, \psi_2, \psi_4, \psi_5\}$$

$$- \{\text{Id}, \psi_2\}$$

$$- \{\text{Id}, \psi_6\}$$

$$- \{\text{Id}, \psi_7\}$$

Γ -SYMMETRIC GRAPHS

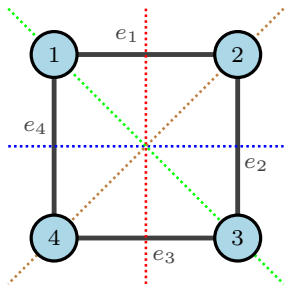
- Subgroups of $\text{Aut}(\mathcal{G})$ define specific symmetries in \mathcal{G}
- for any subgroup $\Gamma \subseteq \text{Aut}(\mathcal{G})$, we say that \mathcal{G} is Γ -symmetric

Γ -SYMMETRIC GRAPHS

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Definition

For a Γ -symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and vertex $i \in \mathcal{V}$, the set $\Gamma_i = \{\gamma(i) \mid \gamma \in \Gamma\}$ is called the **vertex orbit** of i . Similarly, for an edge $e = ij \in \mathcal{E}$, the set $\Gamma_e = \{\gamma(i)\gamma(j) \mid \gamma \in \Gamma\}$ is termed the **edge orbit** of e .



Consider $\Gamma = \{\text{Id}, \psi_2\}$ (ψ_2 is the 180° rotation)

- **Vertex Orbit:**

$$\Gamma_1 = \Gamma_3 = \{1, 3\}, \quad \Gamma_2 = \Gamma_4 = \{2, 4\}$$

vertices inside a vertex orbit are equivalent

representative vertex set: $\mathcal{V}_0 = \{1, 2\}$

- **Edge Orbit:**

$$\Gamma_{e_1} = \Gamma_{e_3} = \{e_1, e_3\},$$

$$\Gamma_{e_2} = \Gamma_{e_4} = \{e_2, e_4\}$$

representative edge set: $\mathcal{E}_0 = \{e_1, e_2\}$

combine notions of graph symmetries with point groups

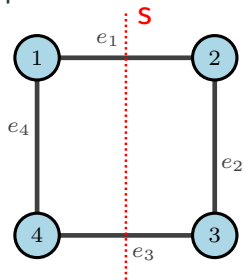
- let \mathcal{G} be a Γ -symmetric graph
- Γ also represented as a *point group*
 - a set of isometries that preserve symmetries
 - homomorphism $\tau : \Gamma \rightarrow O(\mathbb{R}^d)$
 - τ assigns an orthogonal matrix (describing an isometry of \mathbb{R}^d such as a rotation or reflection) to each element of Γ

Definition

A framework (\mathcal{G}, p) in \mathbb{R}^d is called $\tau(\Gamma)$ -symmetric if

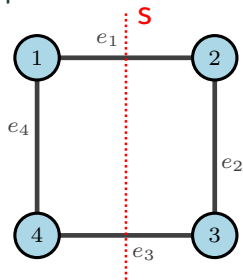
$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}.$$

example...



- consider $\Gamma = \{\text{Id}, \psi_4\} \subseteq \text{Aut}(\mathcal{G})$
 - $\gamma = \psi_4 \in \Gamma$ (reflection about mirror S)
 - isometry $\tau(\gamma) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\tau(\gamma) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix}$
- satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$.
- **note:** for a $\tau(\Gamma)$ -symmetric framework (\mathcal{G}, p) and for every $j \in \Gamma_i$, there is a $\gamma_j \in \Gamma$ such that $\tau(\gamma_j)p_j = p_i$ for all $j \in \Gamma_i$

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isometries of configuration p coincide with symmetries of the automorphisms of \mathcal{G}

- in $\tau(\Gamma)$ -symmetric frameworks, the configurations p are in a special geometric position (not necessarily generic)
- symmetry can lead to unexpected infinitesimal flexibility/rigidity

Definition

An infinitesimal motion u of a $\tau(\Gamma)$ -symmetric framework (\mathcal{G}, p) is $\tau(\Gamma)$ -*symmetric* if

$$\tau(\gamma)(u_i) = u_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}.$$

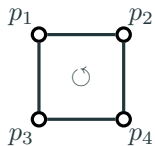
We say that (\mathcal{G}, p) is $\tau(\Gamma)$ -*symmetric infinitesimally rigid* if every $\tau(\Gamma)$ -symmetric infinitesimal motion is trivial.

- recall that infinitesimal motions are in the kernel of the rigidity matrix

$$R(p)\delta p = 0$$

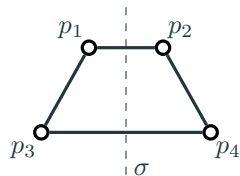
- we can find a subspace of the kernel that is isomorphic to the space of ‘fully-symmetric’ infinitesimal motions
- velocity assignments to the points of (\mathcal{G}, p) that exhibit exactly the same symmetry as the configuration p

SYMMETRIC RIGIDITY



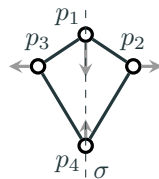
(a)

- C_{4v} -symmetric (and hence $\tau(\Gamma)$ -symmetric for any subgroup $\tau(\Gamma)$ of C_{4v})
- $\tau(\Gamma)$ -symmetric infinitesimally rigid



(b)

- C_s -symmetric (with respect to the reflection σ)
- $\tau(\Gamma)$ -symmetric infinitesimally rigid



(c)

- C_s -symmetric (with respect to the reflection σ) with a non-trivial C_s -symmetric infinitesimal motion
- $\tau(\Gamma)$ -symmetric infinitesimally flexible

Symmetric Formation Control Objective

Consider a group of n integrator agents that interact over the Γ -symmetric sensing graph \mathcal{G} . Let $\mathbf{p} \in \mathbb{R}^{dn}$ be a configuration such that $(\mathcal{G}, \mathbf{p})$ is $\tau(\Gamma)$ -symmetric for some desired point group $\tau(\Gamma)$, and let \mathcal{V}_0 be a set of representatives of the vertex orbits of \mathcal{G} under Γ . Design a control $u_i(t)$ for each agent i such that

- (i) $\lim_{t \rightarrow \infty} \|p_i(t) - p_j(t)\| = \|\mathbf{p}_i - \mathbf{p}_j\| = \mathbf{d}_{ij}$ for all $ij \in \mathcal{E}$; (distance constraints)
- (ii) $\lim_{t \rightarrow \infty} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\| = 0$ for all $u, v \in \Gamma_i, i \in \mathcal{V}_0$. (symmetry constraints)

- the **formation potential**

$$F_f(p(t)) = \frac{1}{4} \sum_{ij \in \mathcal{E}} (\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^2)^2$$

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- the **symmetry potential**

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u, v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the **formation potential**

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Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the **symmetric formation potential**

$$F(p(t)) = F_f(p(t)) + F_s(p(t))$$

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- closed-loop dynamics

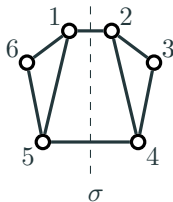
$$\dot{p}(t) = -R(p(t))^T (R(p(t))p(t) - \mathbf{d}^2) - Qp(t)$$

where Q is symmetric and a block-diagonal matrix with

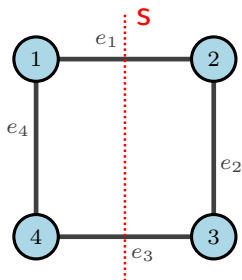
$$[Q_i]_{uv} = \begin{cases} d_{\Gamma_i}(u)I, & u = v, u \in \Gamma_i \\ -\tau(\gamma_{uv}), & uv \in \mathcal{E}, u, v \in \Gamma_i \\ 0, & \text{o.w.} \end{cases} \quad \begin{aligned} & \bullet Q_i \in \mathbb{R}^{|\Gamma_i|d \times |\Gamma_i|d} \\ & \bullet [Q]_{uv} \in O(\mathbb{R}^d) \text{ (orthogonal group)} \\ & \bullet \tau(\gamma_{uv})^{-1} = \tau(\gamma_{vu})^T \end{aligned}$$

- Q_i has a decomposition $Q_i = E(\Gamma_i)E(\Gamma_i)^T$
- $Q = \bar{E}(\Gamma)\bar{E}(\Gamma)^T$
- any p in a symmetric position satisfies $Qp = 0$

- symmetric formation potential makes no assumption on relation between the graph \mathcal{G} and the point group $\tau(\Gamma)$
- we restrict our study to graphs where communication required by symmetric potential use same edges as \mathcal{G}



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- we restrict our study to graphs where communication required by symmetric potential use same edges as \mathcal{G}



- $\Gamma = \{\text{Id}, \psi_4\} \subseteq \text{Aut}(\mathcal{G})$
 - $\Gamma_1 = \Gamma_2 = \{1, 2\}$, $\Gamma_3 = \Gamma_4 = \{3, 4\}$
 - $\mathcal{V}_0 = \{1, 4\}$
 - **isometry** $\tau(\gamma) : (a, b) \mapsto (-a, b)$
- satisfies $\tau(\gamma)(p_i) = p_{\gamma(i)}$ for all $i \in \mathcal{V}$ and for each $i \in \mathcal{V}_0$ and $j \in \Gamma_i \setminus \{i\}$, the edge ij is in \mathcal{E} (i.e. $\mathcal{G}(\Gamma_i)$ is connected)

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

- closed-loop dynamics

$$\dot{p}(t) = -R(p(t))^T (R(p(t))p(t) - \mathbf{d}^2) - Qp(t)$$

- dynamics at for each agent

$$\dot{p}_i(t) = \sum_{ij \in \mathcal{E}} (\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^2)(p_j(t) - p_i(t)) + \sum_{\substack{ij \in \mathcal{E} \\ i, j \in \Gamma_u}} (\tau(\gamma_{ij})p_j(t) - p_i(t))$$

Theorem

[Z, Shulze, Tanigawa '23]

Consider a team of n integrator agents interacting over a Γ -symmetric graph \mathcal{G} satisfying Assumption 1 that can be drawn with maximum point group symmetry \mathcal{S} in \mathbb{R}^d , and let

$$\mathcal{F}_f = \{p \in \mathbb{R}^{dn} \mid \|p_i - p_j\| = \mathbf{d}_{ij} \text{ } ij \in \mathcal{E}\}, \text{ and } \mathcal{F}_s = \{p \in \mathbb{R}^{dn} \mid \tau(\gamma)(p_i) = p_{\gamma(i)} \forall \gamma \in \Gamma, i \in \mathcal{V}\}.$$

Then for initial conditions $p_i(0)$ satisfying

$$\sum_{ij \in \mathcal{E}} (\|p_i(0) - p_j(0)\| - \mathbf{d}_{ij})^2 \leq \epsilon_1, \text{ and } \|p_i(0) - \tau(\gamma_{ij})p_j(0)\|^2 \leq \epsilon_2$$

for all $i, j \in \Gamma_u$ and $u \in \mathcal{V}_0$, for a sufficiently small and positive constant ϵ_1 and ϵ_2 , the control

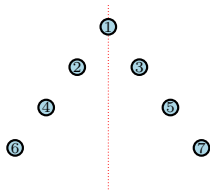
$$u = -\nabla F(p(t)),$$

renders the set $\mathcal{F}_f \cap \mathcal{F}_s$ exponentially stable, i.e.

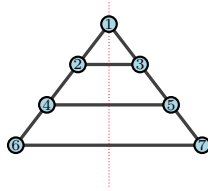
$$\lim_{t \rightarrow \infty} \|p_i(t) - p_j(t)\| = \mathbf{d}_{ij} \text{ and } \lim_{t \rightarrow \infty} \tau(\gamma)(p_i(t)) = \lim_{t \rightarrow \infty} p_{\gamma(i)}(t) \text{ for all } \gamma \in \Gamma, i \in \mathcal{V}.$$

EXAMPLE: THE VIC FORMATION

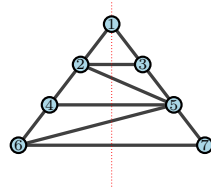
- formation flight for aircraft originated in WWI
- **Vic** formation used by pilots to improve visual communication and defensive advantages



Vic formation with symmetry
mirror

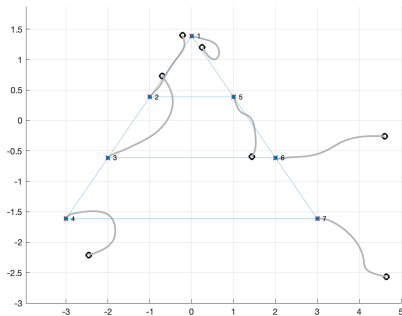


Flexible framework (9 edges;
satisfies Assumption 1)

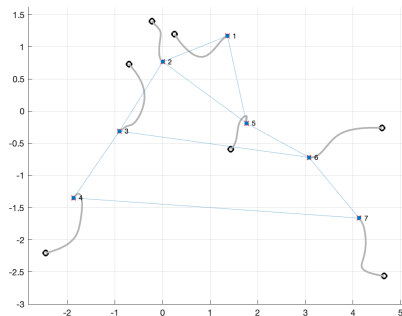


Minimally Rigid framework
(11 edges)

EXAMPLE: THE VIC FORMATION



- symmetry constraints force agents to correct formation
- requires less agent communication than standard formation control with MIR requirement



- with flexible framework and only formation potential can not guarantee convergence to correct shape

- proposed strategy does not take advantage of the full power of symmetry

- proposed strategy does not take advantage of the full power of symmetry
- can we find redundant information between the symmetry constraints and the distance constraints?

Definition

An infinitesimal motion u of a $\tau(\Gamma)$ -symmetric framework (\mathcal{G}, p) is $\tau(\Gamma)$ -symmetric if

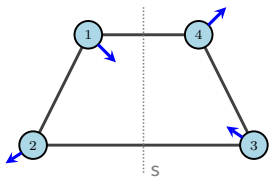
$$\tau(\gamma)(u_i) = u_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}. \quad (1)$$

We say that (\mathcal{G}, p) is $\tau(\Gamma)$ -symmetric infinitesimally rigid if every $\tau(\Gamma)$ -symmetric infinitesimal motion is trivial.

infinitesimal motions can also be studied in this framework

- $\tau(\gamma)(u_i) = u_{\gamma(i)}$
- understanding symmetry structure means we only need to find infinitesimal motion for one representative vertex in each vertex orbit

EXAMPLE



Rigidity matrix

$$R(p) = \begin{bmatrix} (a - c \ b - d) & (c - a \ d - b) & (0 \ 0) & (0 \ 0) \\ (2a \ 0) & (0 \ 0) & (0 \ 0) & (-2a \ 0) \\ (0 \ 0) & (2c \ 0) & (-2c \ 0) & (0 \ 0) \\ (0 \ 0) & (0 \ 0) & (a - c \ d - b) & (c - a \ b - d) \end{bmatrix}$$

- 4-dimensional kernel - flexible framework
- 3 trivial motions

(\mathcal{G}, p)

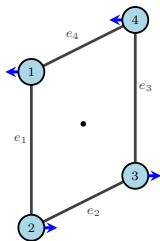
- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-c, d)^T$
- $p_4 = (-a, b)^T$

1-dimensional flex spanned by

$$(1 \ -1 \ -1 \ \frac{2(c-a)+b-d}{d-b} \ -1 \ -\frac{2(c-a)+b-d}{d-b} \ 1 \ 1)^T$$

flex is **not** symmetric with respect to s

EXAMPLE



$$R(p) = \begin{bmatrix} (a - c \ b - d) & (c - a \ d - b) & (0 \ 0) & (0 \ 0) \\ (a + c \ b + d) & (0 \ 0) & (0 \ 0) & (-a - c \ -b - d) \\ (0 \ 0) & (0 \ 0) & (c - a \ d - b) & (a - c \ b - d) \\ (0 \ 0) & (a + c \ b + d) & (-a - c \ -b - d) & (0 \ 0) \end{bmatrix}$$

- 4-dimensional kernel - flexible framework
- 3 trivial motions

(\mathcal{G}, p)

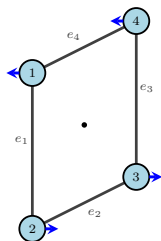
- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-a, -b)^T$
- $p_4 = (-c, -d)^T$

1-dimensional flex spanned by

$$\left(-1 \ 0 \ \frac{cd-ab}{ad-bc} \ \frac{a^2-c^2}{ad-bc} \ 1 \ 0 \ -\frac{cd-ab}{ad-bc} \ -\frac{a^2-c^2}{ad-bc}\right)^T$$

flex is symmetric with respect to 180° rotation
 (\mathcal{C}_2)

EXAMPLE



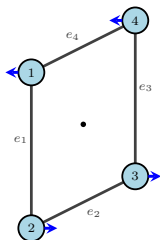
(\mathcal{G}, p)

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-a, -b)^T$
- $p_4 = (-c, -d)^T$

$$R(p) = \begin{bmatrix} (a - c \ b - d) & (c - a \ d - b) & (0 \ 0) & (0 \ 0) \\ (a + c \ b + d) & (0 \ 0) & (0 \ 0) & (-a - c \ -b - d) \\ (0 \ 0) & (0 \ 0) & (c - a \ d - b) & (a - c \ b - d) \\ (0 \ 0) & (a + c \ b + d) & (-a - c \ -b - d) & (0 \ 0) \end{bmatrix}$$

- 180° rotation of points corresponds to $\psi_2 \in \text{Aut}(\mathcal{G})$
- recall: vertex orbits : $\{1, 3\}$, $\{2, 4\}$, edge orbits: $\{e_1, e_3\}$, $\{e_2, e_4\}$

EXAMPLE



$$R(p) = \begin{bmatrix} (a - c \ b - d) & (c - a \ d - b) & (0 \ 0) & (0 \ 0) \\ (a + c \ b + d) & (0 \ 0) & (0 \ 0) & (-a - c \ -b - d) \\ (0 \ 0) & (0 \ 0) & (c - a \ d - b) & (a - c \ b - d) \\ (0 \ 0) & (a + c \ b + d) & (-a - c \ -b - d) & (0 \ 0) \end{bmatrix}$$

(\mathcal{G}, p)

- $p_1 = (a, b)^T$
- $p_2 = (c, d)^T$
- $p_3 = (-a, -b)^T$
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symmetries make certain rows and columns of the rigidity matrix **redundant**

symmetries make certain rows and columns of the rigidity matrix **redundant**

$$R(p) = \begin{array}{l} e_1 \\ e_4 \\ \psi_2(e_1) \\ \psi_2(e_4) \end{array} \begin{array}{cccc} 1 & 2 & 3 = \psi_2(1) & 4 = \psi_2(2) \\ \left(\begin{array}{cccc} (a - c \ b - d) & (c - a \ d - b) & (0 \ 0) & (0 \ 0) \\ (a + c \ b + d) & (0 \ 0) & (0 \ 0) & (-a - c \ -b - d) \\ (0 \ 0) & (0 \ 0) & (c - a \ d - b) & (a - c \ b - d) \\ (0 \ 0) & (a + c \ b + c) & (-a - c \ -b - d) & (0 \ 0) \end{array} \right) \end{array}$$

symmetries make certain rows and columns of the rigidity matrix **redundant**

$$R(p) = \begin{matrix} & & 1 & 2 & 3 = \psi_2(1) & 4 = \psi_2(2) \\ \begin{matrix} e_1 \\ e_4 \\ \psi_2(e_1) \\ \psi_2(e_4) \end{matrix} & \left(\begin{array}{cccc} (a-c & b-d) & (c-a & d-b) & (0 & 0) & (0 & 0) \\ (a+c & b+d) & (0 & 0) & (0 & 0) & (-a-c & -b-d) \\ (0 & 0) & (0 & 0) & (c-a & d-b) & (a-c & b-d) \\ (0 & 0) & (a+c & b+c) & (-a-c & -b-d) & (0 & 0) \end{array} \right) \end{matrix}$$

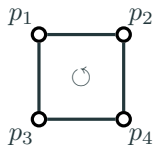
Orbit Rigidity Matrix

$$\begin{matrix} & & 1 & 2 \\ \begin{matrix} e_1 \\ e_4 \end{matrix} & \left(\begin{array}{cc} (p_1 - p_2)^T & (p_2 - p_1)^T \\ (p_1 - \psi_2(p_2))^T & (p_2 - \psi_2^{-1}(p_1))^T \end{array} \right) = \left(\begin{array}{cc} (a-c, b-d) & (c-a, d-b) \\ (a+c, b+d) & (c+a, d+b) \end{array} \right) \end{matrix}$$

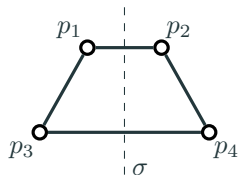
- 2 rows - one for each representative of edge orbits under action of ψ_2
- 4 columns - nodes p_1, p_2 each have two dof; nodes $p_3 = \psi_2(p_1)$ and $p_4 = \psi_2(p_2)$ are uniquely determined by the symmetries

- relation between vertices within vertex orbits and between vertex orbits (through edge orbits) captured by **quotient gain graph** of a Γ -symmetric graph
 - node set is representative vertex set \mathcal{V}_0
 - edge set is representative edge set \mathcal{E}_0 : choose edge of form $i\gamma(j)$ with $i, j \in \mathcal{V}_0$
 - it is ok for $i = j$
 - edges are directed with 'edge gain' being the group action $\gamma \in \Gamma$

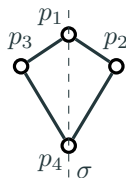
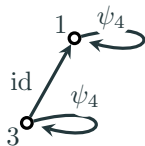
QUOTIENT GAIN GRAPHS



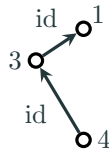
- $\Gamma = \{\text{Id}, \psi_1\}$ (rotation)
- $\Gamma_i = \{1, 2, 3, 4\}$
- $\mathcal{V}_0 = \{1\}, \mathcal{E}_0 = \{e_1\}$



- $\Gamma = \{\text{Id}, \psi_4\}$ (reflection)
- $\Gamma_{1,2} = \{1, 2\}, \Gamma_{3,4} = \{3, 4\}$
- $\mathcal{V}_0 = \{1, 3\}, \mathcal{E}_0 = \{12, 13, 24\}$



- $\Gamma = \{\text{Id}, \psi_6\}$ (reflection)
- $\Gamma_1 = \{1\}, \Gamma_4 = \{4\}, \Gamma_{2,3} = \{2, 3\}$
- $\mathcal{V}_0 = \{1, 3, 4\}, \mathcal{E}_0 = \{13, 14\}$



Definition [Shulze 2011]

The **orbit rigidity matrix** $\mathcal{O}(\mathcal{G}_0, \bar{p})$ of (\mathcal{G}, p) is the $|\mathcal{E}_0| \times d|\mathcal{V}_0|$ matrix defined as follows. The row corresponding to an edge $((i, j); \gamma)$, where $i \neq j$, has the form:

$$\left(0 \cdots 0 \quad (\bar{p}_i - \tau(\gamma)\bar{p}_j)^T \quad 0 \cdots 0 \quad (\bar{p}_j - \tau(\gamma)^{-1}\bar{p}_i)^T \quad 0 \cdots 0 \right),$$

with the d -dimensional entries $(\bar{p}_i - \tau(\gamma)\bar{p}_j)^T$ and $(\bar{p}_j - \tau(\gamma)^{-1}\bar{p}_i)^T$ being in the columns corresponding to vertex i and j , respectively. The row corresponding to a loop $((i, i); \gamma)$ has the form:

$$\left(0 \cdots 0 \quad (2\bar{p}_i - \tau(\gamma)\bar{p}_i - \tau(\gamma)^{-1}\bar{p}_i)^T \quad 0 \cdots 0 \right),$$

with the d -dimensional entry $(2\bar{p}_i - \tau(\gamma)\bar{p}_i - \tau(\gamma)^{-1}\bar{p}_i)^T$ being in the columns corresponding to vertex i .

Theorem [Shulze 2011]

Let (\mathcal{G}, p) be a $\tau(\Gamma)$ -symmetric framework with orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, \bar{p})$. Then,

- (i) the kernel of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ is isomorphic to the space of $\tau(\Gamma)$ -symmetric infinitesimal motions of (\mathcal{G}, p) , and
- (ii) the cokernel of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ is isomorphic to the space of $\tau(\Gamma)$ -symmetric self-stresses of (\mathcal{G}, p) .

- Orbit rigidity matrix can be used to identify symmetric infinitesimal flexes
- full-rank $\mathcal{O}(\mathcal{G}_0, \bar{p})$ implies none exist
- size of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ does not depend on p , but only the graph and symmetry constraints
- $\tau(\Gamma)$ -isostatic frameworks have orbit rigidity matrices with full row-rank

key point: quotient gain graph and orbit rigidity matrix suggests a further way to exploit symmetry in formation control

- representative edges used to maintain distances
- symmetry within vertex orbits have no need for distance constraints

- the **representative edge formation** potential

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij \in \mathcal{E}_0} \left(\|p_i - \tau(\gamma)p_j\|^2 - \mathbf{d}_{i\gamma(j)}^2 \right)^2$$

- γ is label of edge in quotient gain graph

A MODIFIED FORMATION POTENTIAL

- the **representative edge formation** potential

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij \in \mathcal{E}_0} \left(\|p_i - \tau(\gamma)p_j\|^2 - \mathbf{d}_{i\gamma(j)}^2 \right)^2$$

- γ is label of edge in quotient gain graph

- the **symmetry potential**

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

A MODIFIED FORMATION POTENTIAL

- the **representative edge formation potential**

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij \in \mathcal{E}_0} \left(\|p_i - \tau(\gamma)p_j\|^2 - \mathbf{d}_{i\gamma(j)}^2 \right)^2$$

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- the **symmetry potential**

$$F_s(p(t)) = \frac{1}{2} \sum_{i \in \mathcal{V}_0} \sum_{\substack{u,v \in \Gamma_i \\ uv \in \mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\|^2$$

Assumption 1

The sub-graph induced by each vertex orbit Γ_i is connected.

- the **symmetric formation potential**

$$F(p(t)) = F_e(p(t)) + F_s(p(t))$$

- node relabeling - representative vertices first

$$\tilde{p} = Pp = \begin{bmatrix} p_o^T & p_f^T \end{bmatrix}^T$$

- propose the gradient control

$$u(t) = -\nabla F(p(t))$$

Then the control for each agent $i \in \mathcal{V}_0$ can be expressed as

$$u_i(t) = u_i^{(a)}(t) + u_i^{(b)}(t) + u_i^{(c)}(t), \quad (2)$$

where

$$\begin{aligned} u_i^{(a)}(t) &= \sum_{\substack{i\gamma(j) \in \mathcal{E}_0 \\ j \in \mathcal{V}_0, i \neq j}} (\|p_i(t) - \tau(\gamma)p_j(t)\|^2 - \mathbf{d}_{ij}^2)(\tau(\gamma)p_j(t) - p_i(t)) \\ u_i^{(b)}(t) &= \sum_{i\gamma(i) \in \mathcal{E}_0} (\|(I - \tau(\gamma))p_i\|^2 - \mathbf{d}_{i\gamma(i)}^2)(2I - \tau(\gamma) - \tau(\gamma)^{-1})p_i \\ u_i^{(c)}(t) &= \sum_{ij \in \mathcal{E}(\Gamma_i)} (\tau(\gamma_{ij})p_j(t) - p_i(t)). \end{aligned}$$

The control for the agents in $\mathcal{V} \setminus \mathcal{V}_0$ is simply

$$u_i(t) = \sum_{ij \in \mathcal{E}(\Gamma_u)} (\tau(\gamma_{ij})p_j(t) - p_i(t)), \quad (3)$$

for each $u \in \mathcal{V}_0$.

in state-space form

$$\begin{bmatrix} \dot{p}_0(t) \\ \dot{p}_f(t) \end{bmatrix} = \begin{bmatrix} -\mathcal{O}^T(\mathcal{G}_0, p_0(t)) \left(\mathcal{O}(\mathcal{G}_0, p_0(t)) p_0(t) - \mathbf{d}_0^2 \right) \\ 0 \end{bmatrix} - P Q P^T \begin{bmatrix} p_0(t) \\ p_f(t) \end{bmatrix}$$

recall our earlier idea

$$\dot{p}(t) = -R(p(t))^T (R(p(t))p(t) - \mathbf{d}^2) - Qp(t)$$

we can define an error system with

$$e = \begin{bmatrix} \sigma \\ q \end{bmatrix} = \begin{bmatrix} \mathcal{O}(\mathcal{G}_0, p_0(t))p_0(t) - \mathbf{d}_0^2 \\ \bar{E}(\Gamma)^T P^T p(t) \end{bmatrix}$$

orbit error dynamics

$$\begin{aligned} \begin{bmatrix} \dot{\bar{\sigma}}(t) \\ \dot{\bar{q}}(t) \end{bmatrix} &= - \begin{bmatrix} \mathcal{O}\mathcal{O}^T & \mathcal{O}\bar{E}_0(\Gamma) \\ \bar{E}_0^T(\Gamma)\mathcal{O}^T & \bar{E}^T(\Gamma)\bar{E}(\Gamma) \end{bmatrix} \begin{bmatrix} \bar{\sigma}(t) \\ \bar{q}(t) \end{bmatrix} \\ &= - \begin{bmatrix} \mathcal{O} & 0 \\ \bar{E}^T(\Gamma)P^T \end{bmatrix} \begin{bmatrix} \mathcal{O}^T \\ 0^T \end{bmatrix} P\bar{E}(\Gamma) \begin{bmatrix} \bar{\sigma}(t) \\ \bar{q}(t) \end{bmatrix}. \end{aligned}$$

Theorem

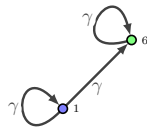
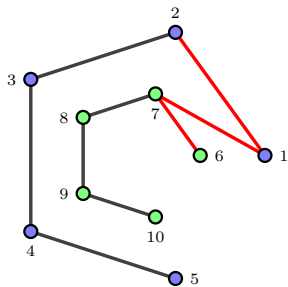
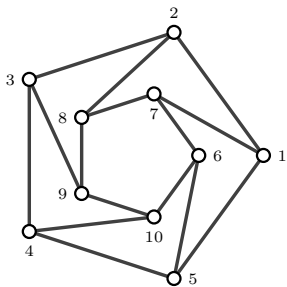
Let \mathbf{p} be the target formation satisfying conditions (i) and (ii) of the Symmetry-Forced Formation Control Problem, and assume that $(\mathcal{G}, \mathbf{p})$ is a $\tau(\Gamma)$ -symmetric isostatic framework. Then the origin is a locally exponentially stable equilibrium of the orbit error dynamics.

Theorem

The orbit rigidity control uses at most $(1 + 1/|\Gamma|)|\mathcal{V}|$ edges.

- can be significantly less than $2|\mathcal{V}| - 3$

EXAMPLE

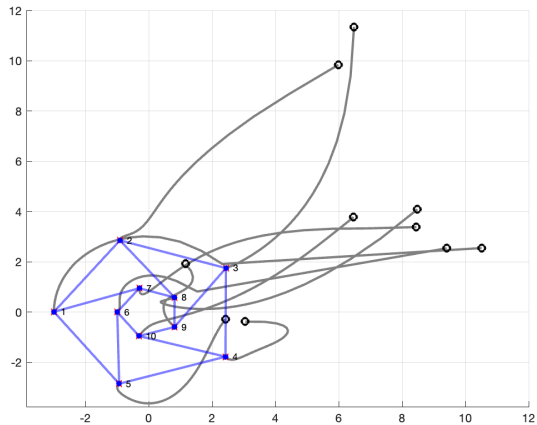


- graph has 15 edges
- at least 17 edges required for infinitesimal rigidity
- flexible framework

- $2\pi/5$ rotational symmetry
- can use only spanning tree subgraph for each vertex orbit
- only 3 distances required

- quotient gain graph

EXAMPLE



- nice...but symmetries are defined with respect to a global origin

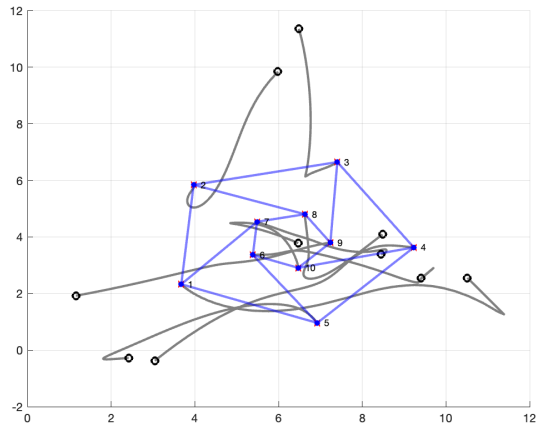
idea: augment a virtual consensus dynamics

$$\begin{bmatrix} \dot{p}_0(t) \\ \dot{p}_f(t) \end{bmatrix} = \begin{bmatrix} -\mathcal{O}^T(\mathcal{G}_0, c_0(t)) \left(\mathcal{O}(\mathcal{G}_0, c_0(t))c_0(t) - \mathbf{d}_0^2 \right) \\ 0 \end{bmatrix} - P Q P^T \begin{bmatrix} c_0(t) \\ c_f(t) \end{bmatrix}$$
$$\dot{r} = -L(\mathcal{G})r$$

with $c(t) = p(t) - r(t)$

- cascade structure
- same analysis idea

CENTROID CONSENSUS



Summary

- $\tau(\Gamma)$ -symmetric graphs captures symmetry of configurations and graphs
- symmetric formation potential used to design distributed control law with less edges compared to “traditional” formation control strategies
- opportunities for more sophisticated motion coordination

Zelazo, Tanigawa and Shulze, *Forced Symmetric Formation Control*, arXiv 2024.

Future Work

- formation maneuvering requires time-varying point group symmetries
- is it possible to distributedly decide on certain symmetries?
- can we eliminate need for requiring self-state in protocol?
- more?

Questions?