# Fekete Points, Formation Control and the Balancing Problem

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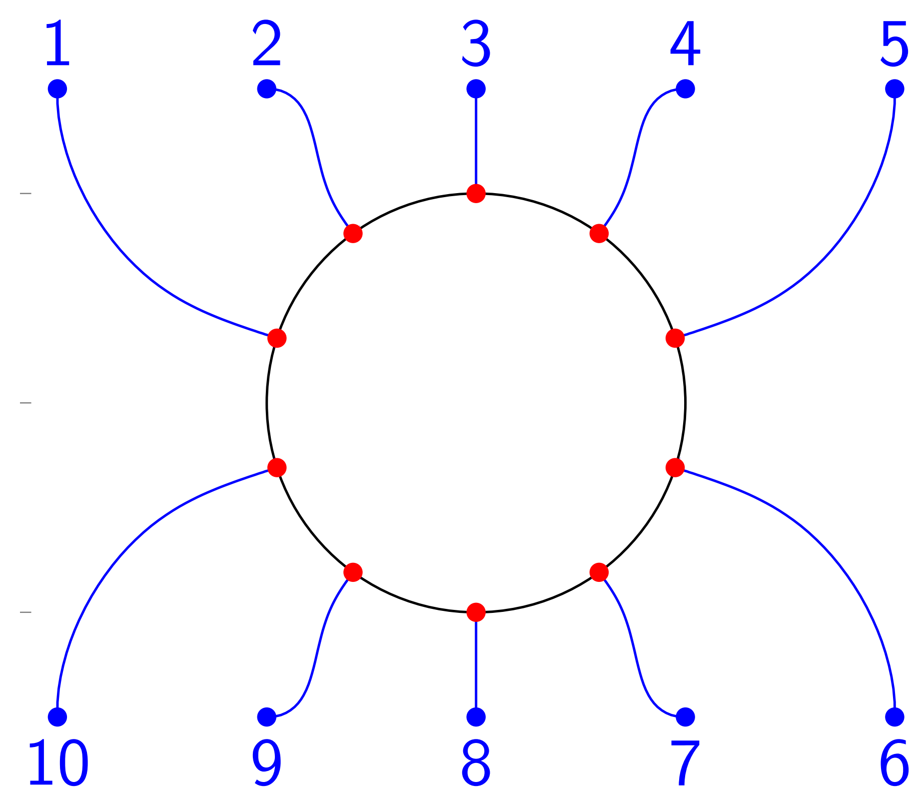


German-Israeli Foundation for Scientific Research and Development

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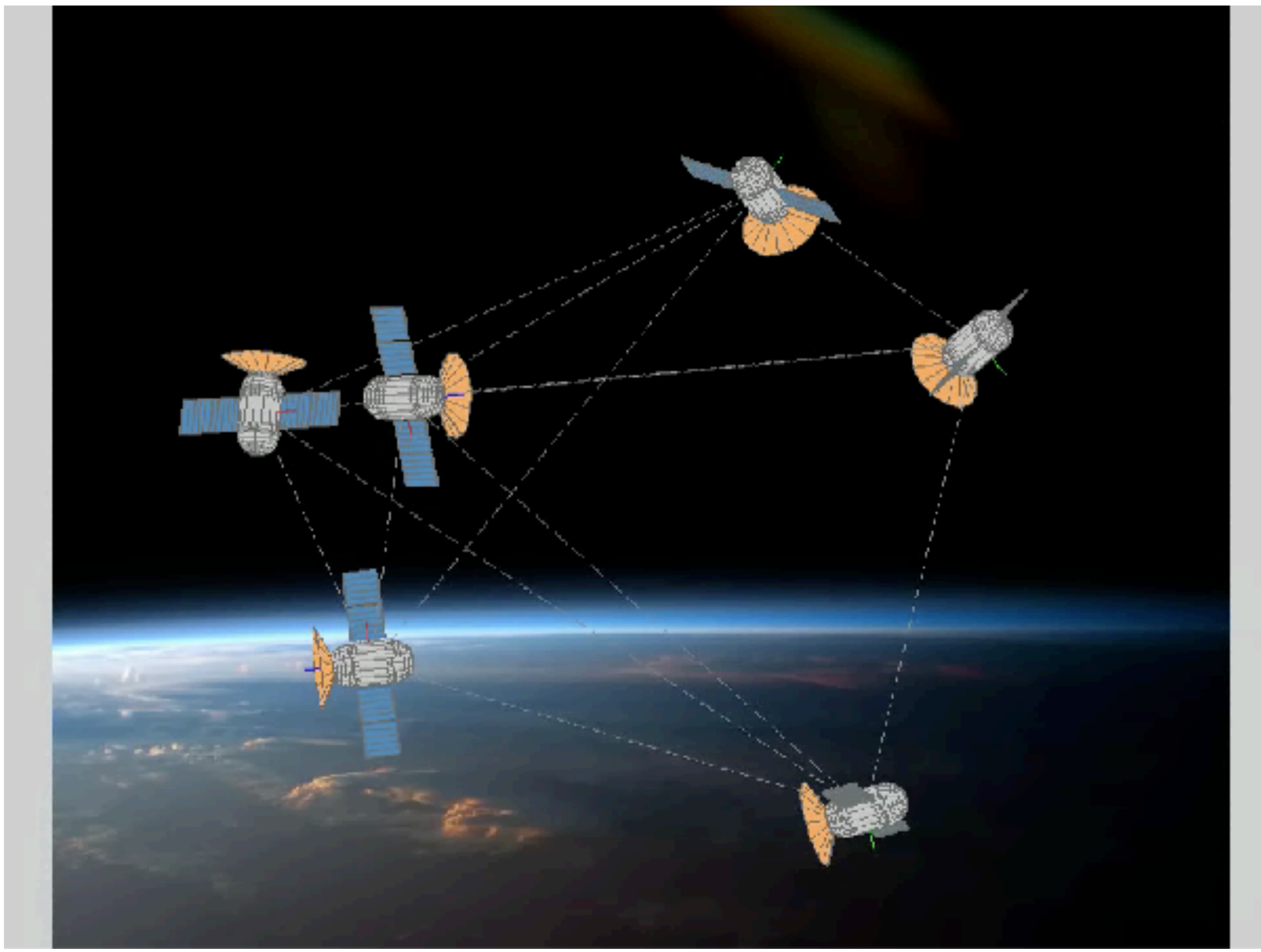
**Rennes, FR 23.02.2017**

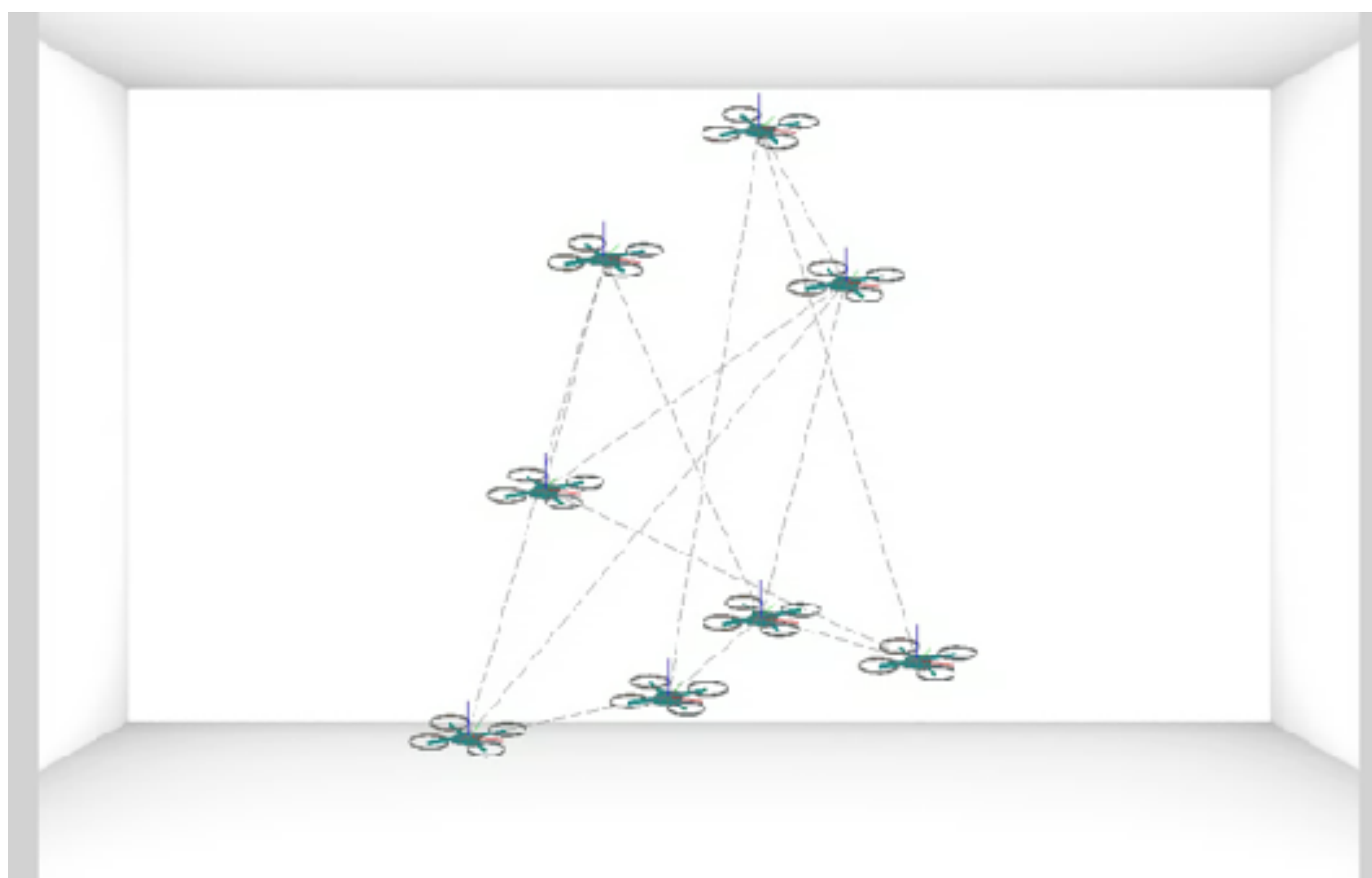
# formation control

**Formation Control** is one of the canonical problems in multi-agent and multi-robot coordination

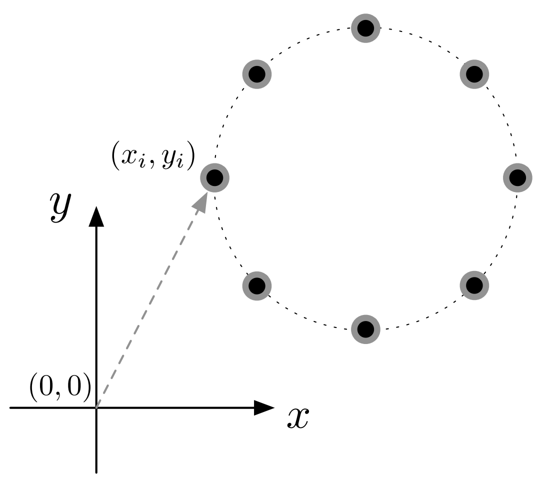


## The Formation Control Problem

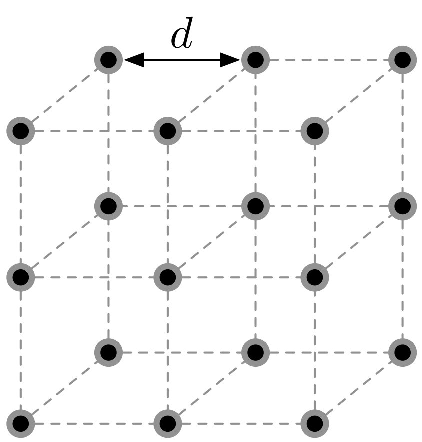
Given a team of robots endowed with the ability to sense relative state information to neighboring robots, design a control for each robot using only *local information* that asymptotically stabilizes the team to a desired formation shape.



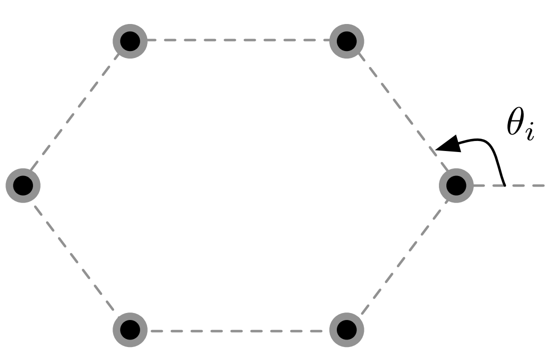
# The Formation Control Problem




Formation specified
in global coordinates




Formation specified
by inter-agent distances

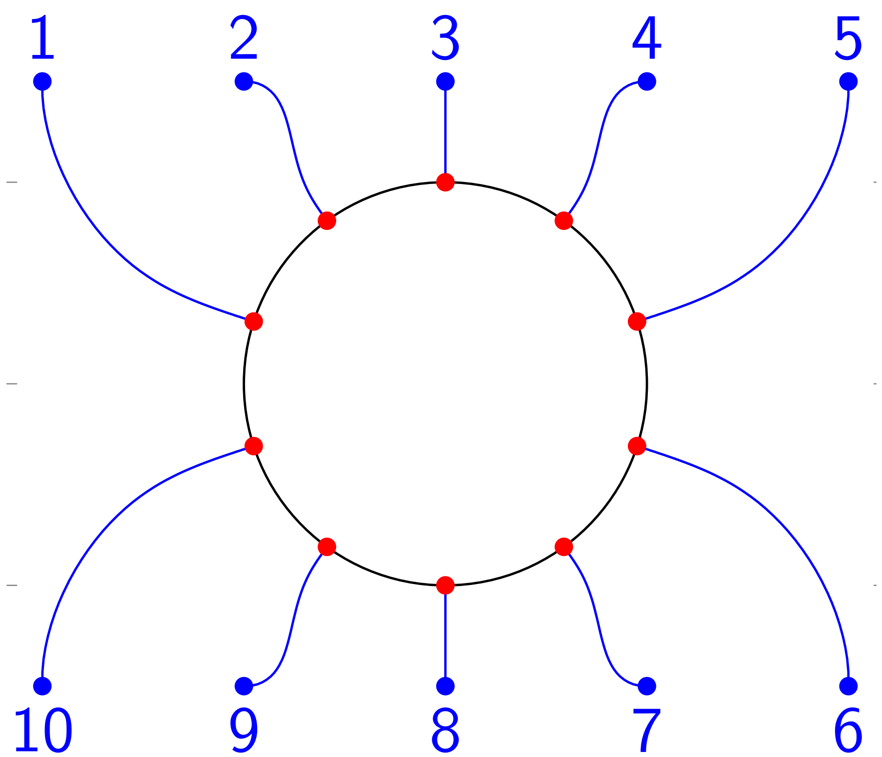



Formation specified
by inter-agent bearings

## Rigidity Theory

a combinatorial theory for characterizing the "stiffness" or "flexibility" of structures formed by rigid bodies connected by flexible linkages or hinges.

# formation control…another approach




***formation shape*** is specified by a compactly embedded submanifold of the ambient Euclidean space

$$M \subset \mathbb{R}^d$$

design a **decentralized** control that drives each agent to the desired submanifold, and a **distributed** control that arranges their configuration on the submanifold in a ***balanced*** fashion

$$\begin{aligned} \max \quad & \sum_{j>i} d(x_i, x_j)^2 \\ s.t. \quad & x_i \in M \end{aligned}$$

# formation control…another approach

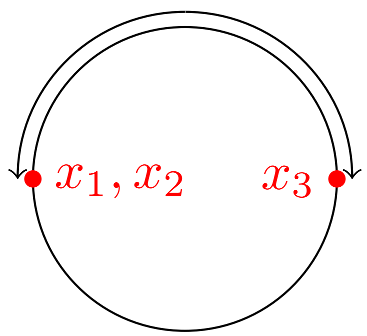
an example…

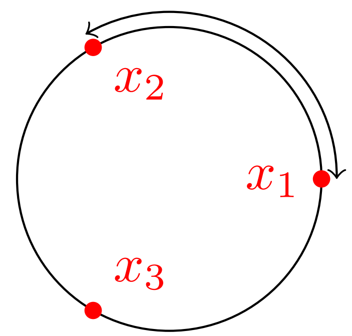
$M \subset \mathbb{R}^2$ is unit circle in the plane

$n = 3$ agents

$$\max \quad \sum_{j>i} d(x_i, x_j)^2$$
$$s.t. \quad x_i \in M$$

# formation control…another approach




$$\begin{aligned} \max \quad & \sum_{j>i} d(x_i, x_j)^2 \\ s.t. \quad & x_i \in M \end{aligned}$$

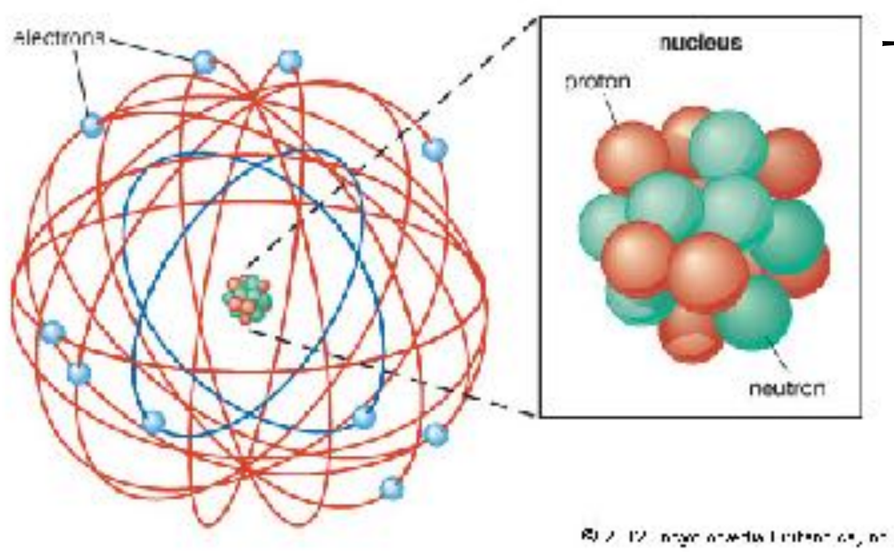
a modification…

chose cost function that is "small"
when agents are close to each other

$$\sum_{j>i} \ln d(x_i, x_j) = \ln \left( \prod_{j>i} d(x_i, x_j) \right)$$

# formation control and Fekete points

$$\sum_{j>i} \ln d(x_i, x_j) = \ln \left( \prod_{j>i} d(x_i, x_j) \right)$$




Thomson Atomic Model
(1904)

Föppl
(1912)

*Stabile Anordnungen von Elektronen im Atom*

$$V_n = \prod_{1 \leq i < j \leq n} (x_j - x_i)^2$$

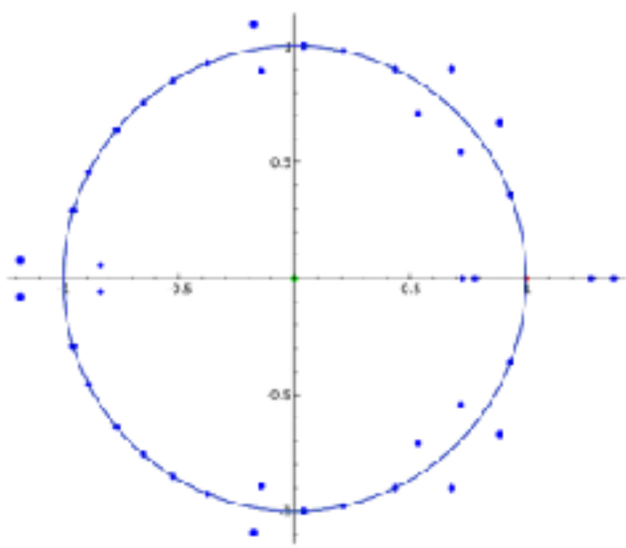
Vandermode polynomial

Schur
(1918)

*Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*

# formation control and Fekete points

$$\sum_{j>i} \ln d(x_i, x_j) = \ln \left( \prod_{j>i} d(x_i, x_j) \right)$$



roots of Fekete polynomial


STEVE SMALE

Mathematical Problems for the Next Century[1]

Fekete (1923) *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*

Smale (1998) ***Problem 7:*** *Distribution of Points on the 2-Sphere (Fekete points)*

# formation control and Fekete points

$$\sum_{j>i} \ln d(x_i, x_j) = \ln\left(\prod_{j>i} d(x_i, x_j)\right)$$

**Problem 7: Distribution of Points on the 2-Sphere**

Let $V_N(x) = \sum_{1 \le i < j \le N} \log \frac{1}{\|x_i - x_j\|}$, where $x = (x_1, \ldots, x_N)$, the $x_i$ are distinct points on the 2-sphere $S^2 \subset \mathbb{R}^3$, and $\|x_i - x_j\|$ is the distance in $\mathbb{R}^3$. Denote $\min_x V_N(x)$ by $V_N$.

*Find* $(x_1, \ldots, x_N)$ *such that*

$$V_N(x) - V_N \le c \log N, \quad c \text{ a universal constant.} \quad (2)$$

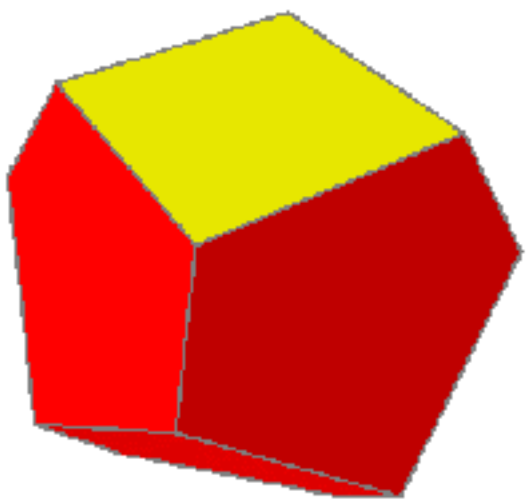
To "find" means to give an algorithm which on input $N$ outputs distinct $x_1, \ldots, x_N$ on the 2-sphere satisfying (2). To be precise one could take a real number algorithm in the sense of BCSS (adjoining a square root computation) with halting time polynomial in $N$.

This problem emerged from complexity theory, jointly with Mike Shub [Shub and Smale, 1993]. It is motivated by finding a good starting polynomial for a homotopy algorithm for realizing the Fundamental Theorem of Algebra.

# formation control and Fekete points

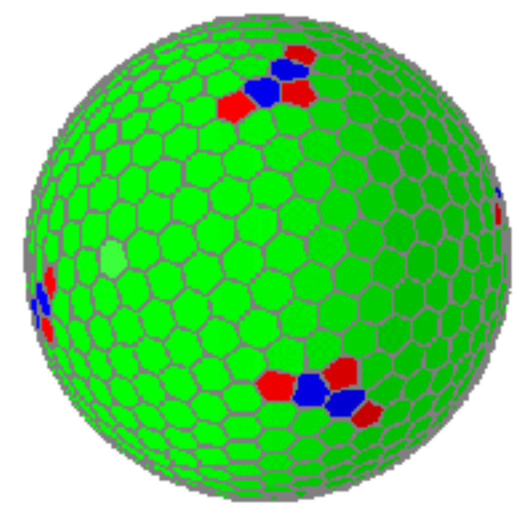
Global Minima for the Thomson Problem
David J. Wales and Sidika Ulker
Structure and Dynamics of Spherical Crystals Characterised for the Thomson Problem, Phys. Rev. B, 74, 212101 (2006).

http://www-wales.ch.cam.ac.uk/~wales/CCD/Thomson/table.html



N = 10

N = 50



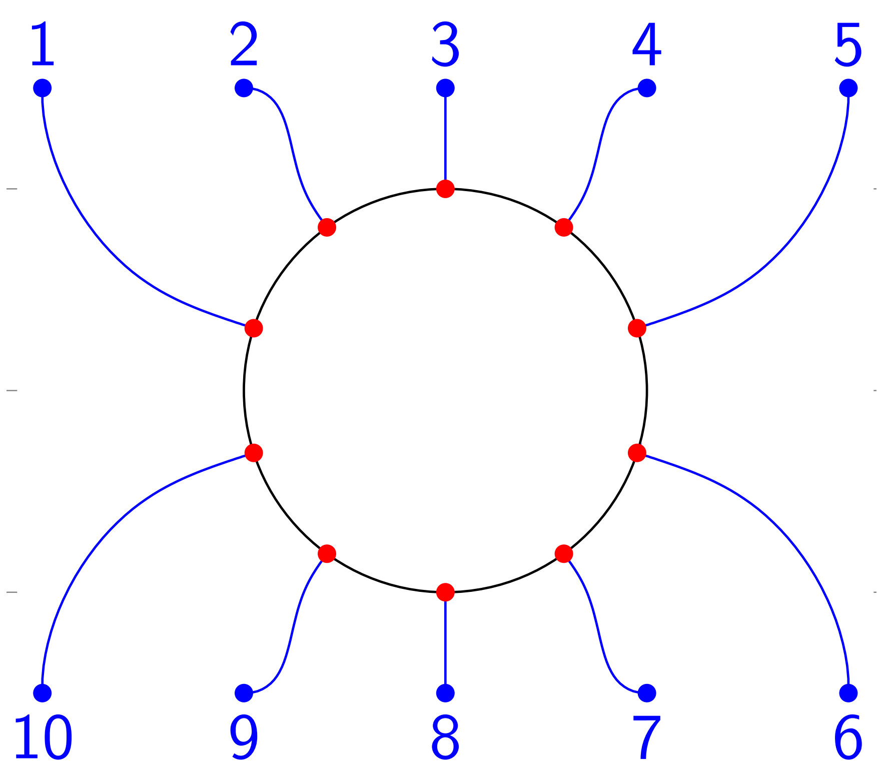
N = 972

# formation control…another approach

***formation shape*** is specified by a compactly embedded submanifold of the ambient Euclidean space

$$M \subset \mathbb{R}^d$$

design a **decentralized** control that drives each agent to the desired submanifold, and a **distributed** control that arranges their configuration on the submanifold in a ***balanced*** fashion

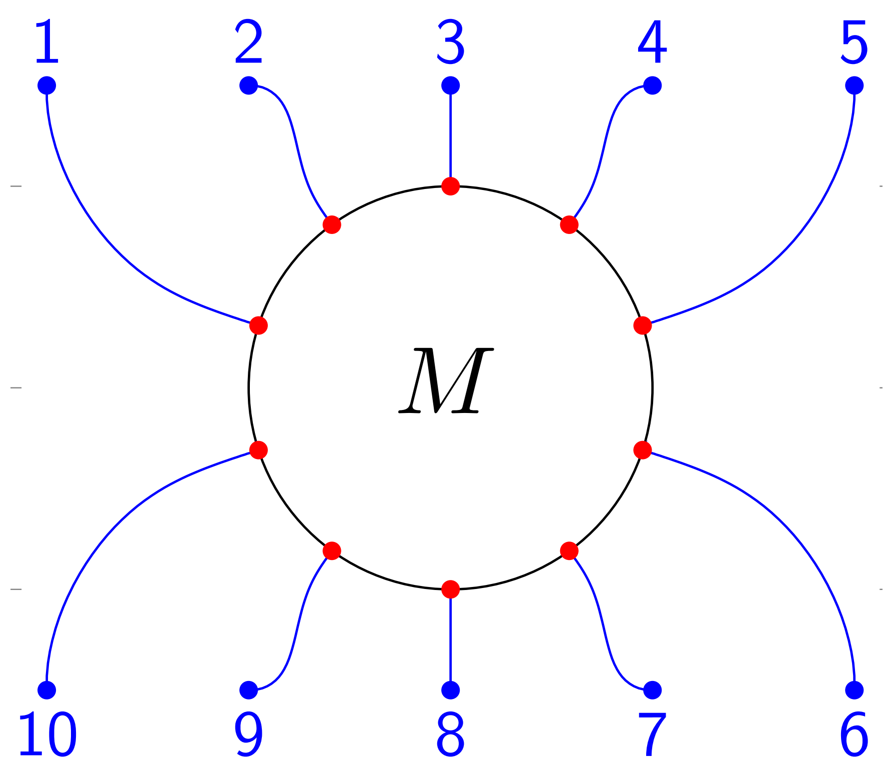



$$\max \quad \ln\left(\prod_{j>i} d(x_i, x_j)\right)$$
$$s.t. \quad x_i \in M$$

**Fekete Points**

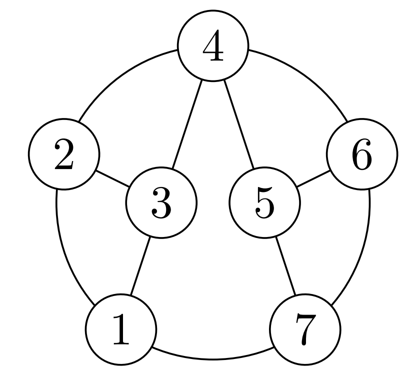
# asymptotic stability of Fekete points






$x_i \in \mathbb{R}^d$

"information exchange" network




$$W_{ij} = \begin{cases} w_{ij}, & i \sim j \\ 0, & o.w. \end{cases}$$

$$r \,:\, \mathbb{R}^d \to M$$

smooth retraction onto the submanifold

$$\phi \,:\, M \to \mathbb{R}$$

$$\phi(x) = \sum_{j>i} W_{ij} \ln(d(x_i, x_j))$$

"balancing" potential

# asymptotic stability of Fekete points

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**Theorem**

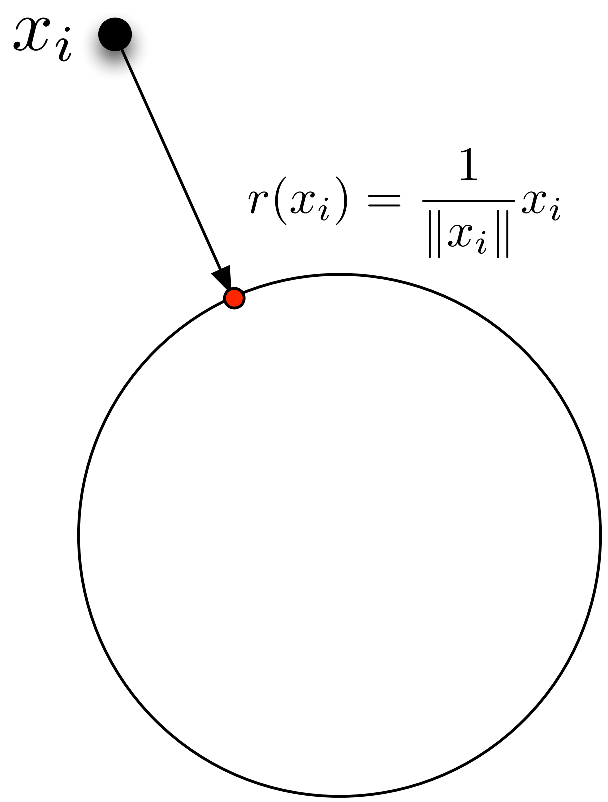
The solutions of

$$\dot{x} = (r(x) - x) + \mathrm{grad}\, \phi(r(x))$$

asymptotically approach the maximizers of $\phi$ in a stable fashion.

| | |
|---|---|
| $r(x) - x$ | a *decentralized control* that asymptotically stabilizes our formation shape |
| $\mathrm{grad}\, \phi(r(x))$ | a *distributed control* that stabilizes the maximizers of potential function |

# an example - the unit circle




retraction onto unit circle

$$r(x_i) = \frac{1}{\|x_i\|} x_i$$

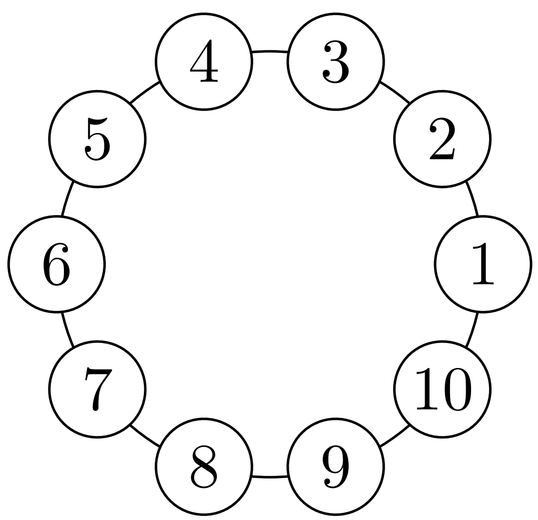
$$\phi(x) = \sum_{j>i} W_{ij} \ln(d(x_i, x_j))$$

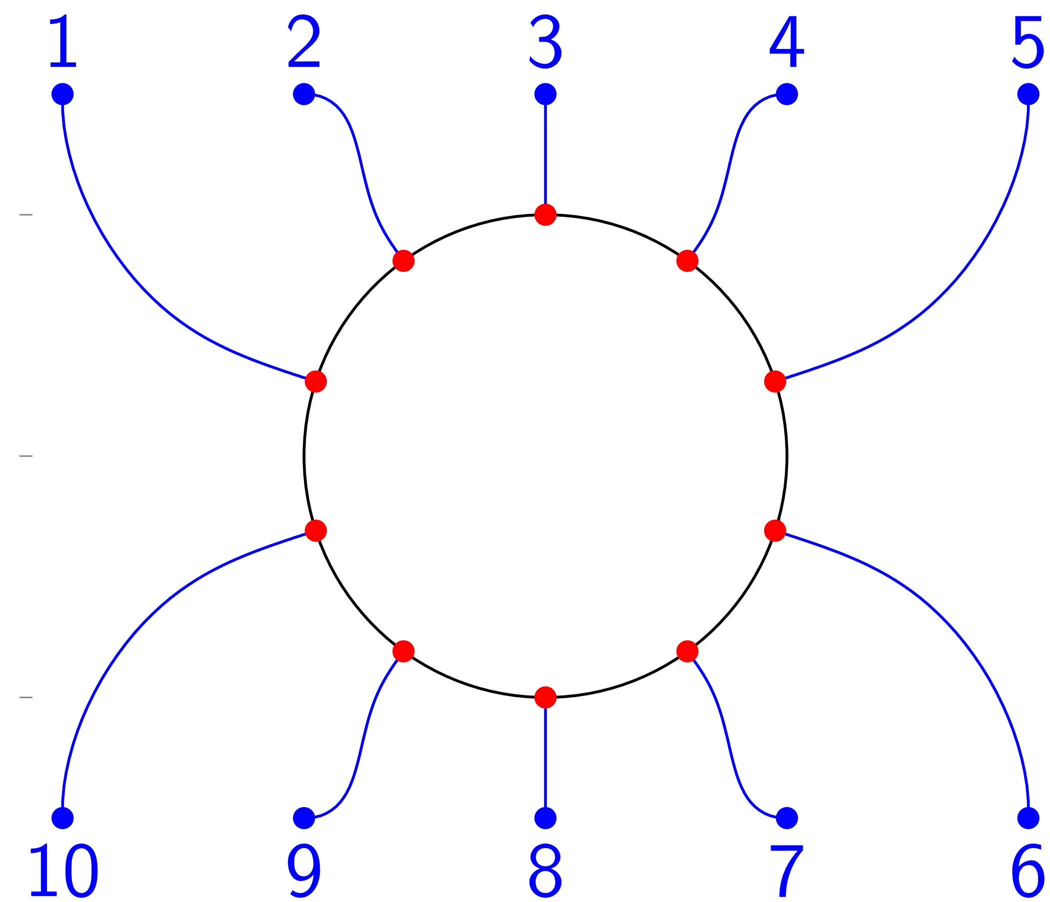
$$[\mathrm{grad}\, \phi(r(x))]_i = \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|} \left( \log \left( \frac{1}{\|x_i\| \|x_j\|} \begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_i \\ x_j \cdot \Omega x_j & x_i \cdot x_j \end{bmatrix} \right) \right)^{-1} x_i$$

$$\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

# an example - the unit circle

"information exchange" network




$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right) x_i + \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|} \left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} x_i$$

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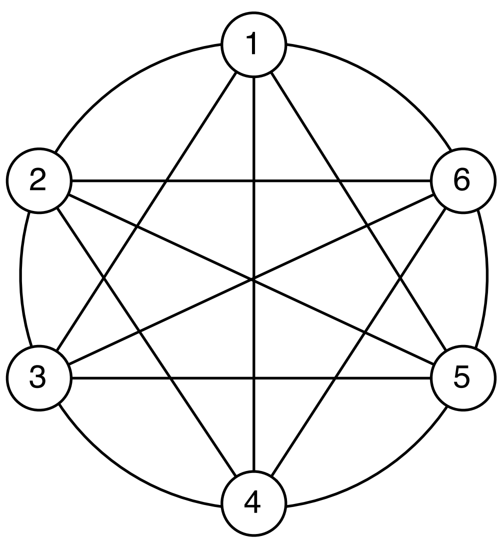
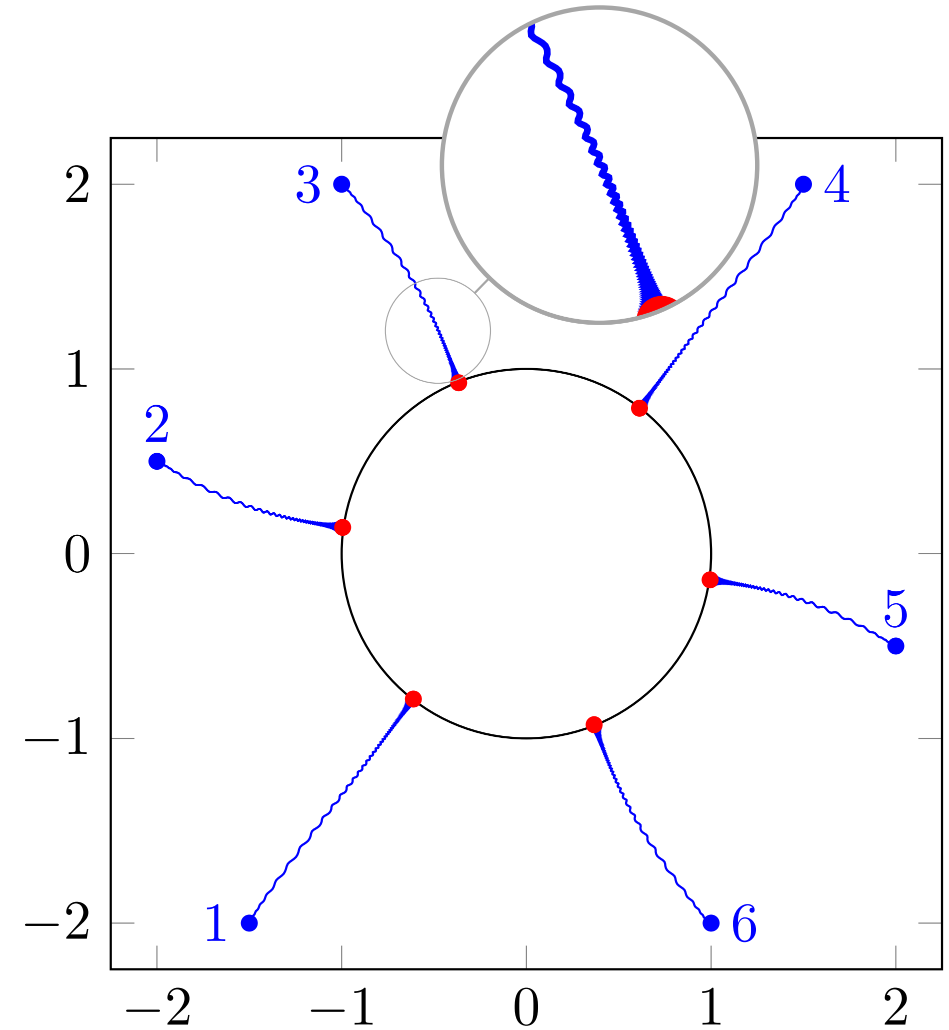
# an example - the unit circle

“information exchange”
network

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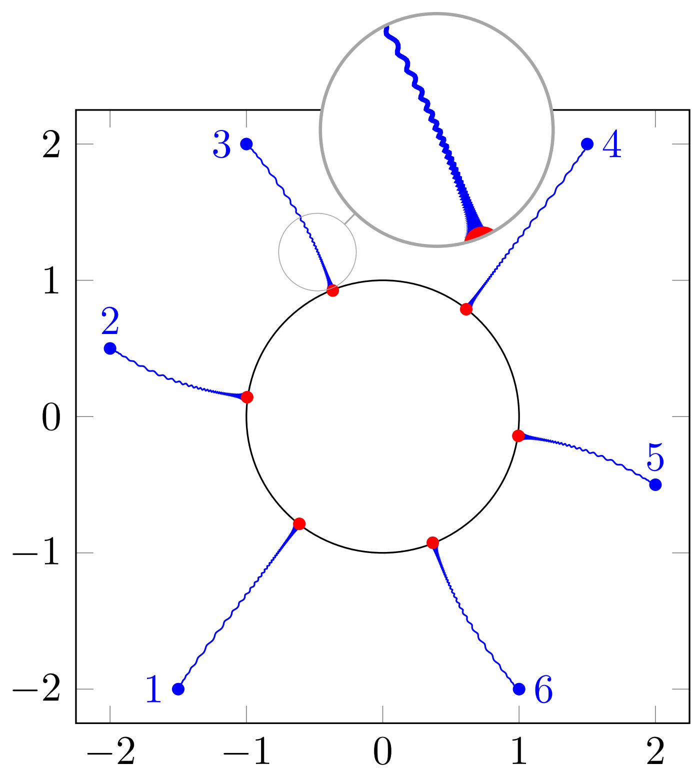



**do evenly spaced configuration correspond to equilibrium?**

*directed angles:* $\alpha_{ij}\Omega = \log\left(\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)$

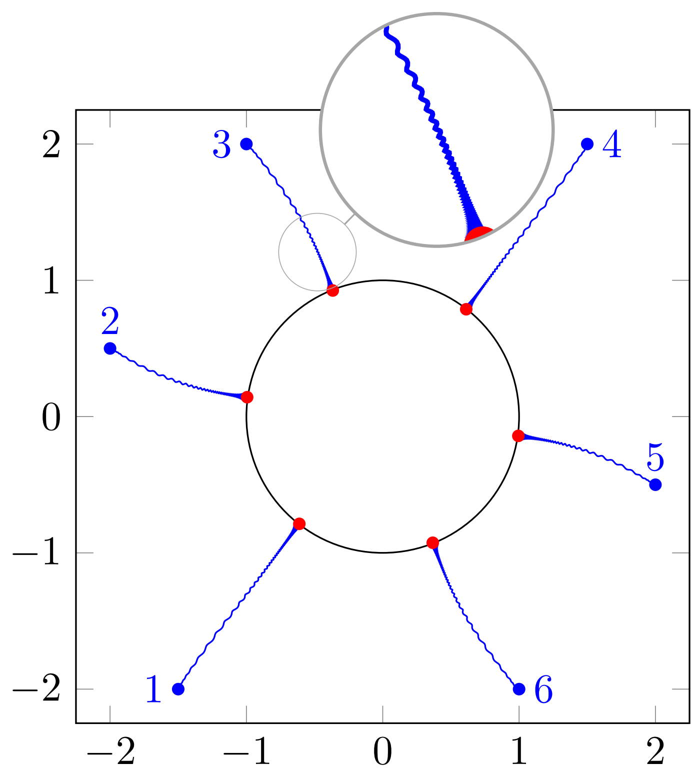
$$\left(\log\left(\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_i \\ x_j \cdot \Omega x_j & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} = -\frac{1}{\alpha_{ij}}\Omega$$

equilibrium: $$\sum_{i \sim j} \frac{1}{\alpha_{ij}} = 0$$

# an example - the unit circle

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right)x_i + \sum_{j=1}^{n}\frac{W_{ij}}{\|x_i\|}\left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i\cdot x_j & x_i\cdot\Omega x_j \\ x_j\cdot\Omega x_i & x_i\cdot x_j\end{bmatrix}\right)\right)^{-1} x_i$$

$$\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$




does evenly spaced configuration correspond to equilibrium?

angles between red points

$$\alpha_{12} = -\frac{2\pi}{6},\ \alpha_{13} = -\frac{2\pi}{3},\ \alpha_{14} = \pm\pi,$$

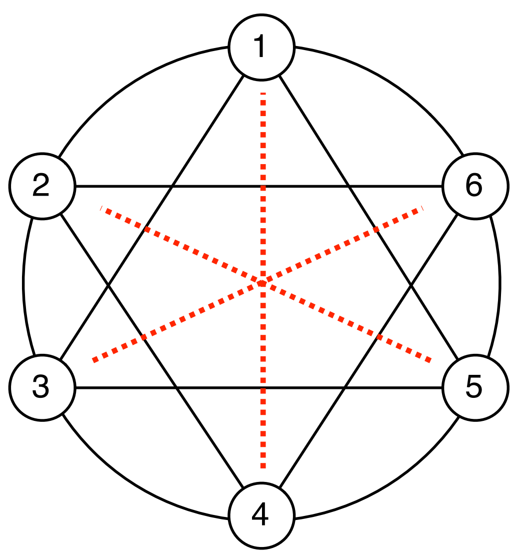
$$\alpha_{15} = \frac{2\pi}{3},\ \alpha_{16} = \frac{2\pi}{6}$$

sum of reciprocals $\frac{1}{\alpha_{12}} + \frac{1}{\alpha_{13}} + \frac{1}{\alpha_{14}} + \frac{1}{\alpha_{15}} + \frac{1}{\alpha_{16}} \neq 0$

# an example - the unit circle

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right)x_i + \sum_{j=1}^{n}\frac{W_{ij}}{\|x_i\|}\left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i\cdot x_j & x_i\cdot\Omega x_j \\ x_j\cdot\Omega x_i & x_i\cdot x_j\end{bmatrix}\right)\right)^{-1}x_i$$

$$\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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*directed angles:* $\alpha_{ij}\Omega = \log\left(\begin{bmatrix} x_i\cdot x_j & x_i\cdot\Omega x_j \\ x_j\cdot\Omega x_i & x_i\cdot x_j\end{bmatrix}\right)$

angles between red points

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$$\alpha_{15} = \frac{2\pi}{3},\ \alpha_{16} = \frac{2\pi}{6}$$

sum of reciprocals $\frac{1}{\alpha_{12}} + \frac{1}{\alpha_{13}} + \frac{1}{\alpha_{15}} + \frac{1}{\alpha_{16}} = 0$

# graph-theoretic characterization of equilibria

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right) x_i + \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|} \left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} x_i$$

more generally,
equilibrium must satisfy:

$$\begin{bmatrix} 0 & W_{12}/\alpha_{12} & \cdots & \cdots & W_{1n}/\alpha_{1n} \\ W_{21}/\alpha_{21} & 0 & W_{23}/\alpha_{23} & \cdots & W_{2n}/\alpha_{2n} \\ \vdots & W_{32}/\alpha_{32} & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \\ W_{n1}/\alpha_{n1} & W_{n2}/\alpha_{n2} & \cdots & & 0 \end{bmatrix} \mathbf{1} = 0$$

# graph-theoretic characterization of equilibria

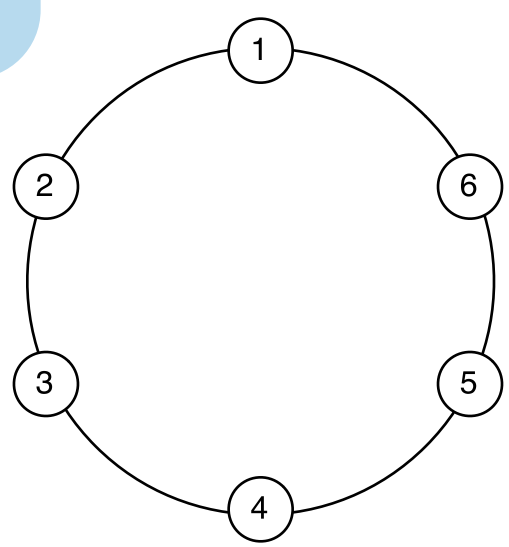
$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right) x_i + \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|}\left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} x_i$$

equivalently...

$$E(\mathcal{G}) \begin{bmatrix} \vdots \\ \frac{1}{\alpha_{ij}} \\ \vdots \end{bmatrix} = 0$$

$E(\mathcal{G})$ *incidence matrix* of a graph

null-space characterizes *cycles* in the graph




$$E(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

# graph-theoretic characterization of equilibria

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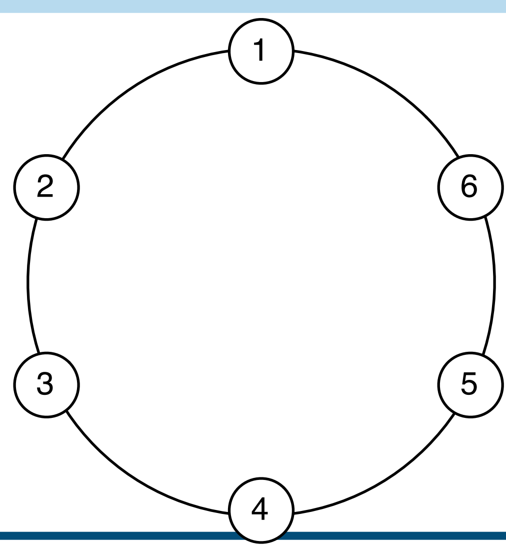
equivalently...

$$E(\mathcal{G})\begin{bmatrix}\vdots\\ \frac{1}{\alpha_{ij}} \\ \vdots\end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix}\frac{1}{\alpha_{12}}\\ \frac{1}{\alpha_{23}}\\ \frac{1}{\alpha_{34}}\\ \frac{1}{\alpha_{45}}\\ \frac{1}{\alpha_{56}}\\ \frac{1}{\alpha_{61}}\end{bmatrix} = 0$$



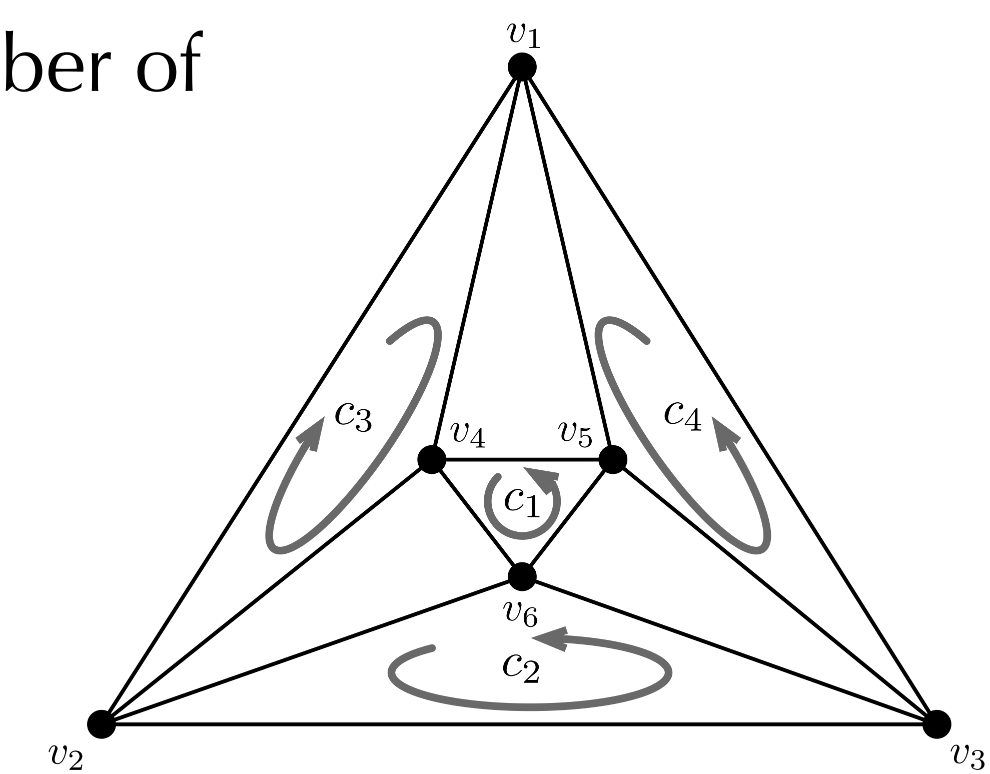

one cycle - one equilibria!

$$\frac{1}{\alpha_{12}} = \frac{1}{\alpha_{23}} = \frac{1}{\alpha_{34}} = \frac{1}{\alpha_{45}} = \frac{1}{\alpha_{56}} = \frac{1}{\alpha_{61}}$$

# incidence matrix and its null-space

$E(\mathcal{G})$ *incidence matrix* of a graph

- incidence matrix of unweighted graph is a matrix over the Galois Field GF(3) $\{0, 1, -1\}$
- the null space of the incidence matrix over GF(3) is called the *cycle space of the graph*
- dimension of the cycle space is the number of *linearly independent cycles over GF(3)*

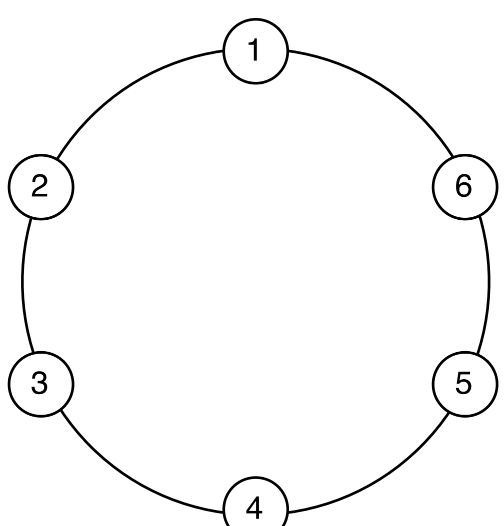


# graph-theoretic characterization of equilibria

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right) x_i + \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|} \left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} x_i$$

a "balanced" configuration should have all directed angles with the same magnitude!

need graphs with "special" null-space

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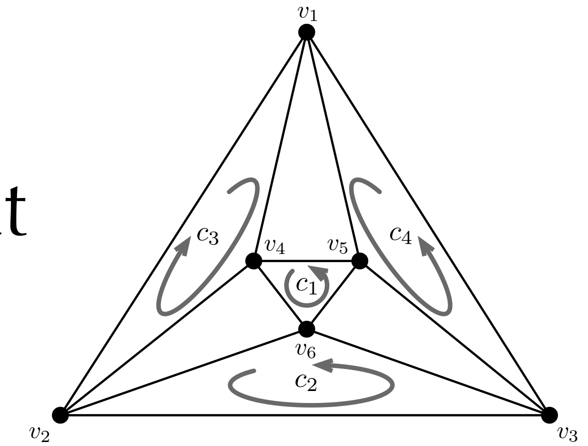
# Corollary

The solutions of

$$\dot{x} = (r(x) - x) + \operatorname{grad} \phi(r(x))$$

for M the unit circle, asymptotically converges to a balanced formation if and only if the graph possesses an Eulerian cycle (iff every vertex has even degree)

An *Eulerian Cycle* is a walk on a graph beginning and ending at the same node that traverses each edge only once.



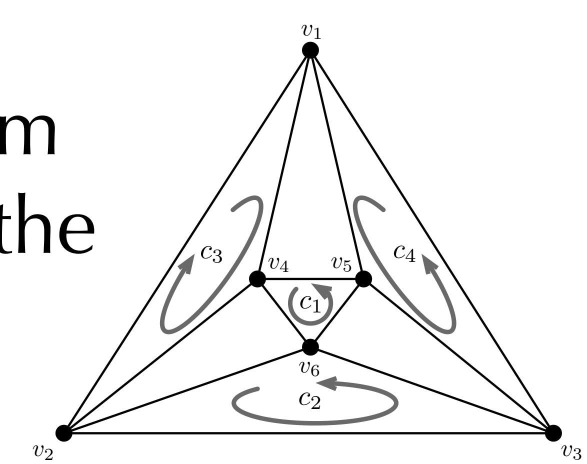
## Corollary

The solutions of

$$\dot{x} = (r(x) - x) + \text{grad}\,\phi(r(x))$$

for M the unit circle, asymptotically converges to a balanced formation if and only if the graph possesses an Eulerian cycle (iff every vertex has even degree)

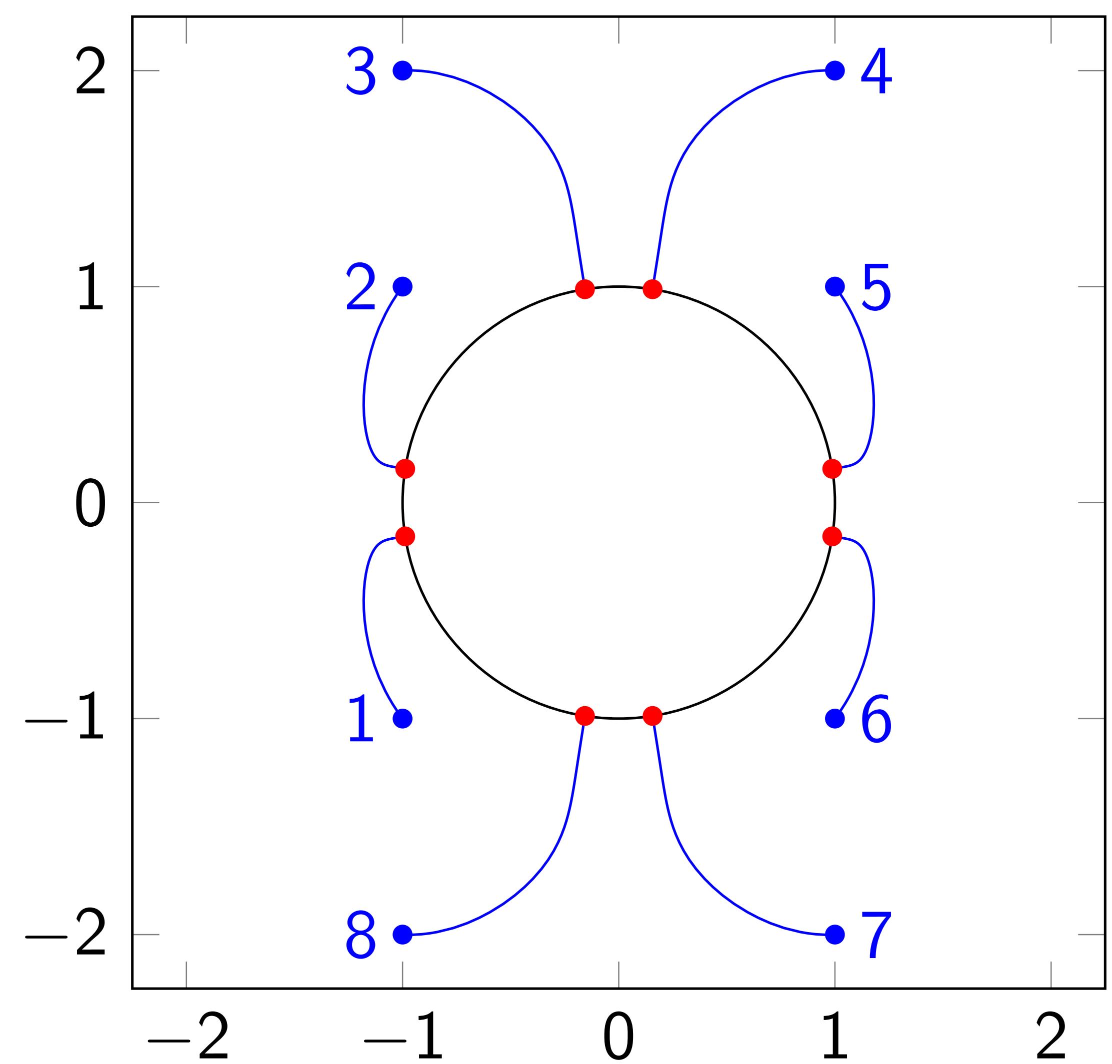
An Eulerian cycle is a vector over GF(3) from the nullspace of the incidence matrix with the property that all entries are 1 or -1.



# exploiting knowledge of equilibria

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right) x_i + \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|} \left(\log\left(\frac{1}{\|x_i\|\|x_j\|} \begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} x_i$$

weights on the graph can be used
"redefine" balanced configuration

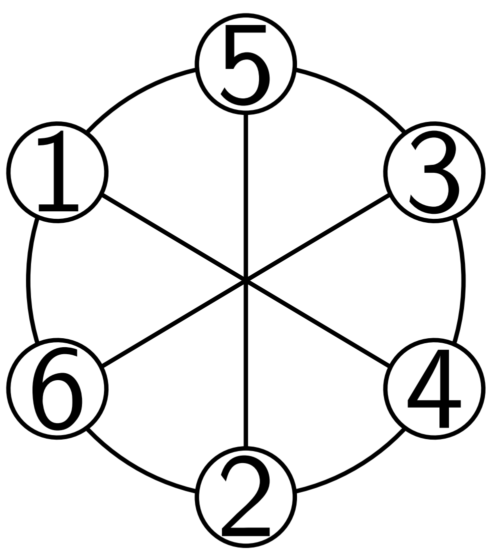


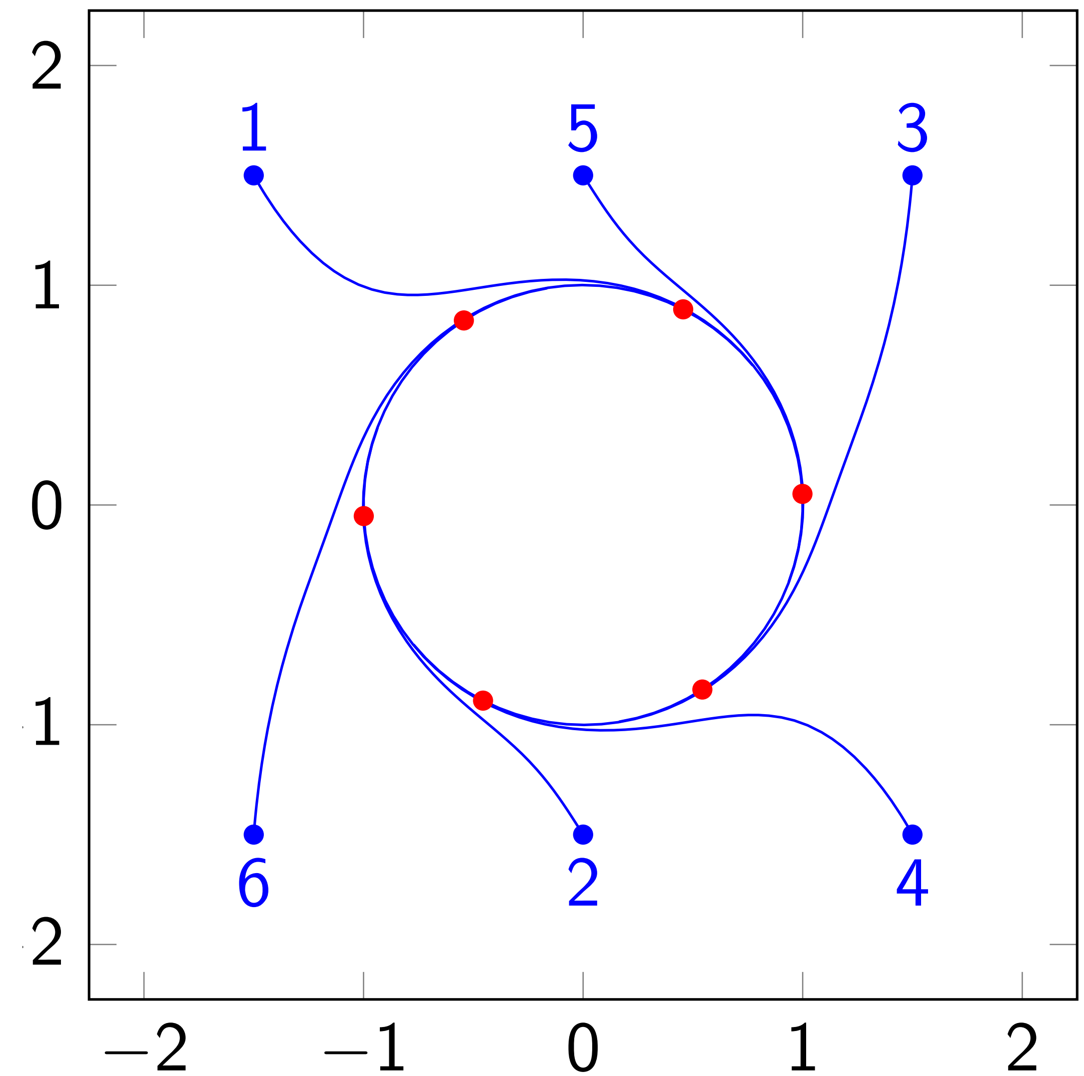
# exploiting knowledge of equilibria

$$\dot{x}_i = \left(\frac{1-\|x_i\|}{\|x_i\|}\right) x_i + \sum_{j=1}^{n} \frac{W_{ij}}{\|x_i\|}\left(\log\left(\frac{1}{\|x_i\|\|x_j\|}\begin{bmatrix} x_i \cdot x_j & x_i \cdot \Omega x_j \\ x_j \cdot \Omega x_i & x_i \cdot x_j \end{bmatrix}\right)\right)^{-1} x_i$$

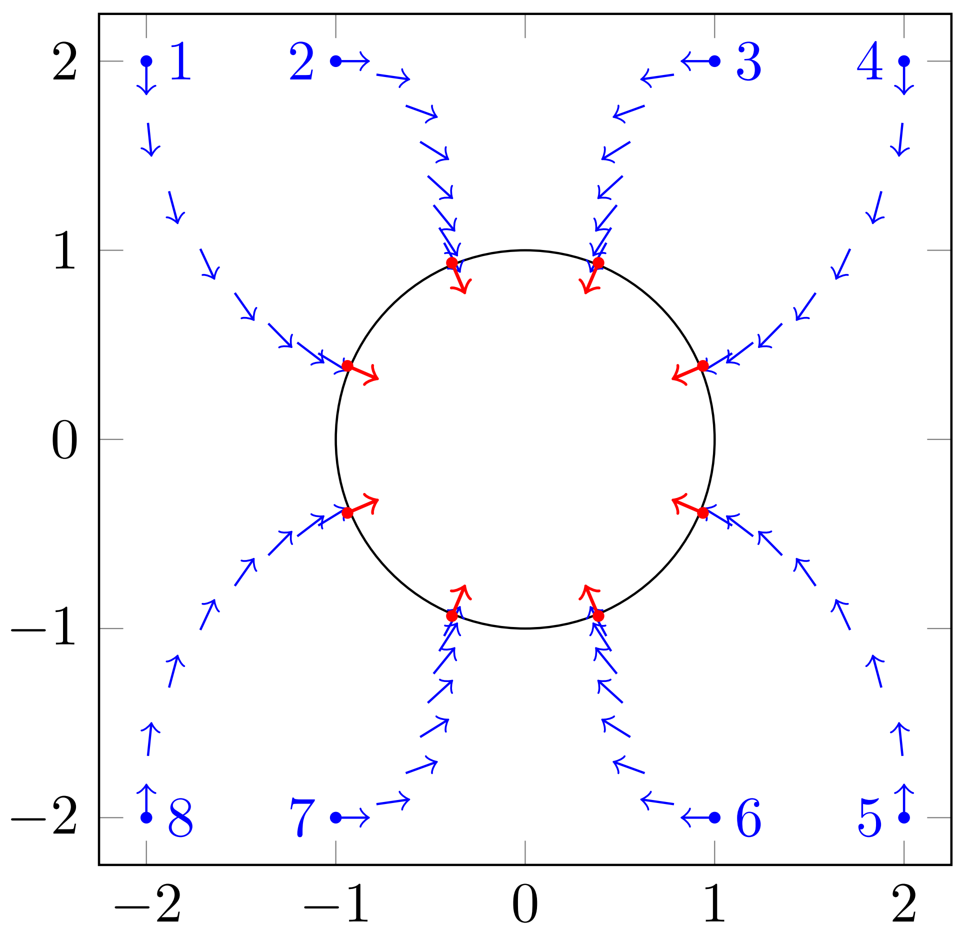
weights on the graph can be used
prevent equilibria

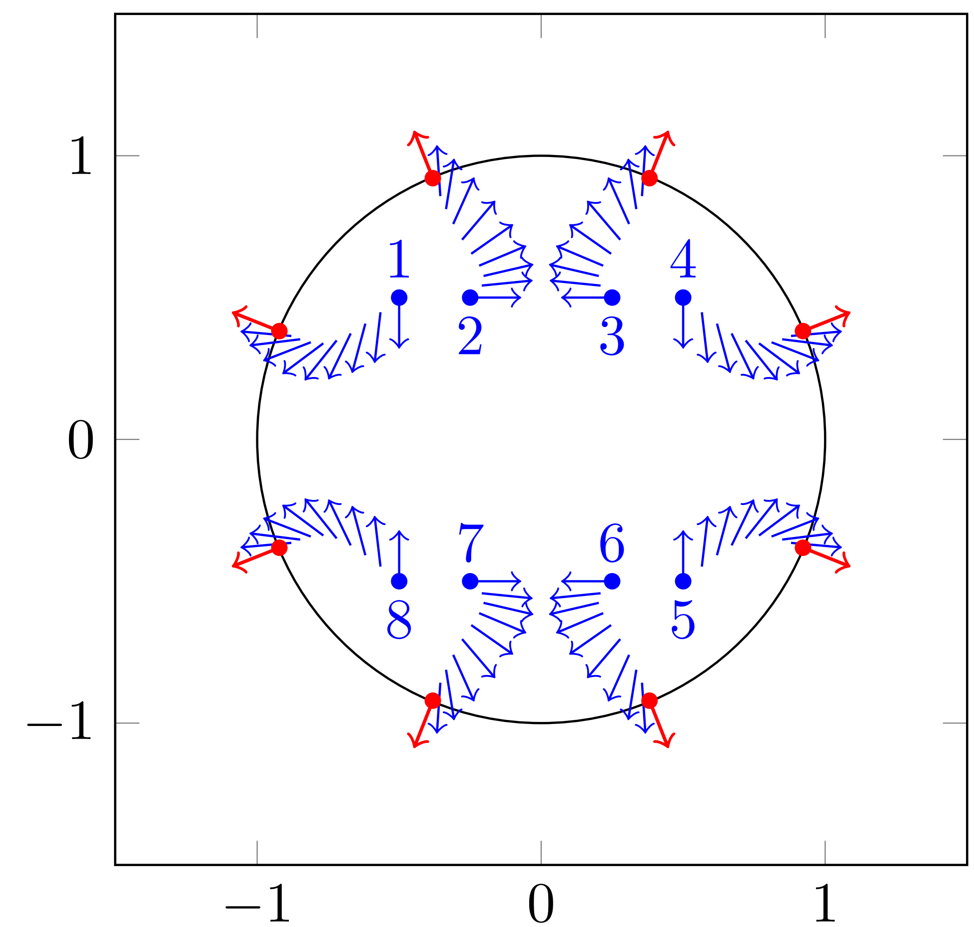
:



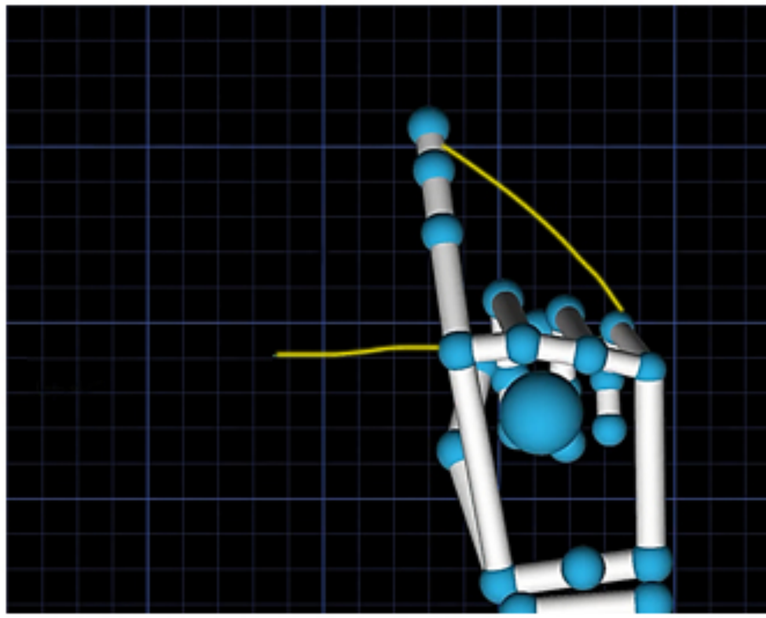
# extensions to SE(2)



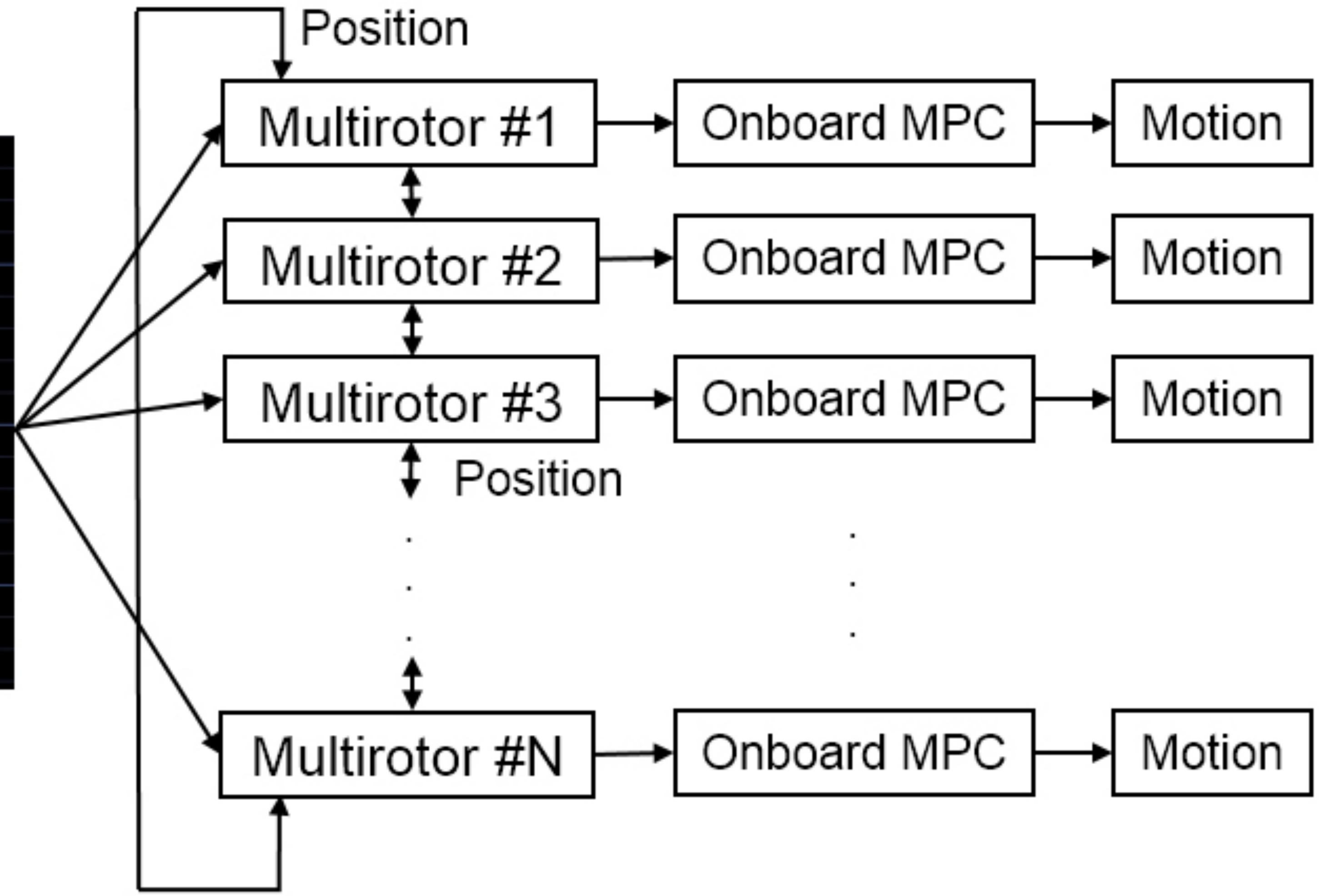



# multi-rotor UAV implementation





Drawing a target formation shape, and deciding radius or vertex positions




- user “draws” desired formation shape (Leap Motion)
- UAV implement retraction balancing algorithm
  - MPC to control position
  - backstepping for attitude regulation
  - wrench observer (FDI) for disturbance estimation

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## A Distributed Control Approach to Formation Balancing and Maneuvering of Multiple Multirotor UAVs

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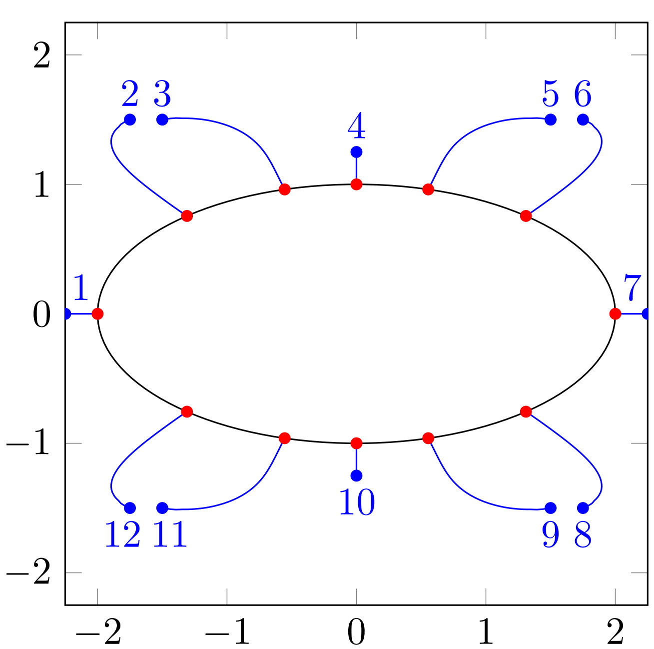
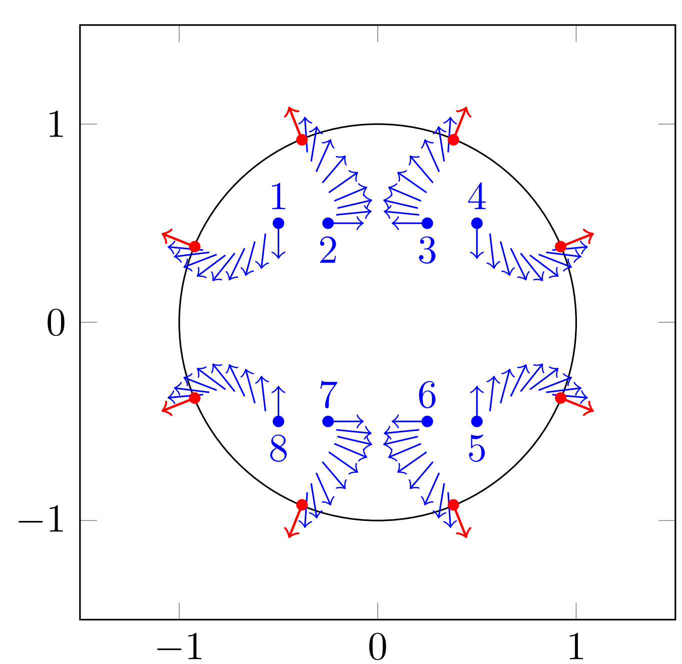
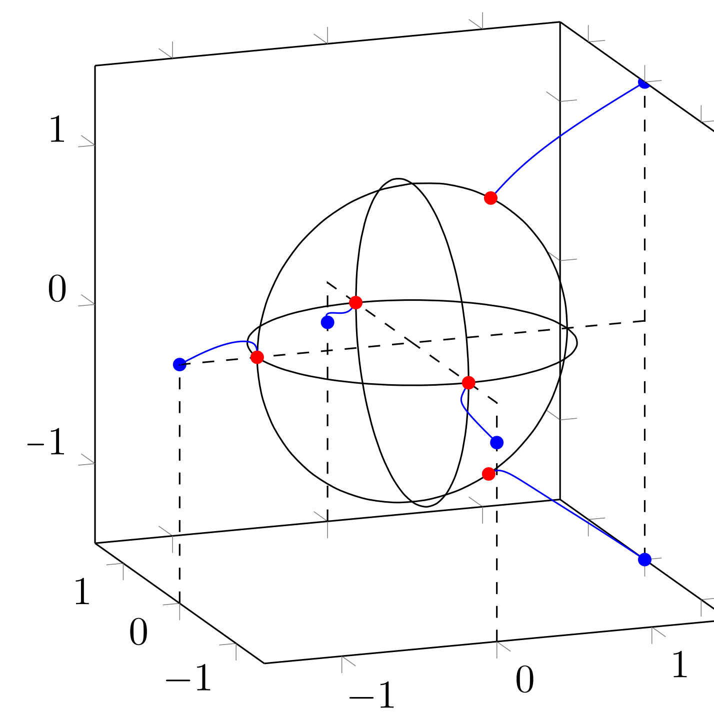
# Conclusions

- Fekete points leads to a novel approach for formation control
- decentralized and distributed implementation
- graph-theoretic interpretations
- extensions:
  - balancing on special Euclidean group
  - time-varying information exchange network
  - formation tracking



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