Design of Sparse Relative Sensing Networks

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Motivation

A collection of dynamic systems that use sensed relative state information to achieve higher level objectives.

Applications

- formation control
- localization
- environmental surveillance
- ...

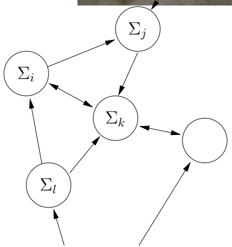
- 'absolute' inertial measurements are often not available (deep space, gps-denied environments)
- however, relative measurements are available and can be very accurate

Relative Sensing Networks

implicit presence of a 'network' induced by sensing structure

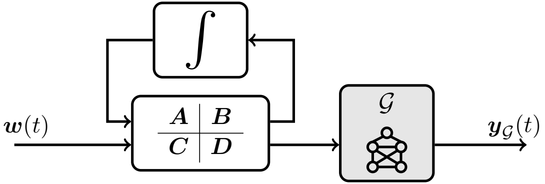
Performance and design of networks:

- Influence of topology on performance
- Optimal topologies
- Sparsity vs connectivity
- Heterogeneity of dynamics



Outline

Modeling of Relative Sensing Networks

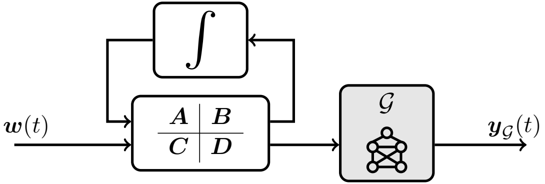


RSNs couple all agents through their outputs, described by an underlying sensing graph $\mathcal{G}$.

$$G = (\mathcal{V}, \mathcal{E}),$$

$\mathcal{V}$ node set (i.e. agents)
$\mathcal{E}$ edge set

Modeling of Relative Sensing Networks



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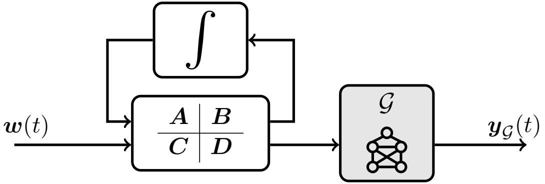
$G = (\mathcal{V}, \mathcal{E})$, $\mathcal{V}$ node set (i.e. agents)
$\mathcal{E}$ edge set

$$\begin{aligned}[y_{\mathcal{G}}(t)]_k =& y_i(t) - y_j(t) \\ \boldsymbol{y}_{\mathcal{G}}(t) =& (E(\mathcal{G})^T \otimes I)\boldsymbol{y}(t), \quad E(\mathcal{G}) \in \mathbb{R}^{n \times |\mathcal{E}|}\end{aligned}$$

incidence matrix captures difference



State space model

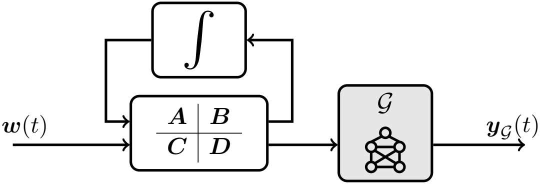
$$\Sigma(\mathcal{G}) : \begin{cases} \dot{\boldsymbol{x}}(t) & = & \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{w}(t) \\ \boldsymbol{y}(t) & = & \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{w}(t) \\ \boldsymbol{y}_{\mathcal{G}}(t) & = & (E(\mathcal{G})^T \otimes I)\boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

Transfer function

$T^{\boldsymbol{w}\mapsto\mathcal{G}}(s) = (E(\mathcal{G})^T \otimes I)\boldsymbol{H}(s)$ with $\boldsymbol{H}(s) = \mathrm{diag}(H_1, H_2, \ldots, H_n)$

and $H_i := C_i(sI - A_i)^{-1}B_i$

Modeling of Relative Sensing Networks



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Performance ?

Performance Analysis of RSNs

$\mathcal{H}_\infty$-norm captures how finite energy exogenous signals are amplified at the monitored outputs.

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Theorem ($\mathcal{H}_\infty$-Performance of RSNs)

The $\mathcal{H}_\infty$-norm of a heterogenous RSN is bounded from above by

$$\|T^{\boldsymbol{w}\mapsto\mathcal{G}}\|_\infty \leq \|E(\mathcal{G})^T Q\|_2$$

where $Q = \mathrm{diag}(\|H_1\|_\infty, \ldots, \|H_n\|_\infty)$.

Zelazo and Mesbahi, 2011

- graph-centric characterization of $\mathcal{H}_\infty$-norm
- $\mathcal{H}_\infty$ performance is dependent on *graph structure*
- for SISO systems, this bound is tight

From Analysis to Synthesis

Open Question:

Given n agents, how can we design RSNs with an *optimal* sensing structure?

Design of Relative Sensing Networks

Optimization problem

minimization of the spectral norm of the weighted incidence matrix

$$\min_{\mathcal{G}} \quad \|QE(\mathcal{G})\|_2$$

subject to: $\mathcal{G}$ is connected

- mixed integer problem
- minimizing the upper bound

Design of Relative Sensing Networks

Optimization problem

minimization of the spectral norm of the weighted incidence matrix

$$\min_{w_i \geq 0} \quad \|QE(\mathcal{G}_c)W\|_2$$

subject to: $\mathcal{G}$ is connected

- mixed integer problem
- minimizing the upper bound
- consider edge weights $w_i \geq 0$ ($w_i = 0 \rightarrow$ no edge)

$$W = \mathrm{diag}(w_1, \ldots, w_{|\mathcal{E}|})$$

- $\mathcal{G}_c$ complete graph

semidefinite optimization problem

Reformulation as SDP

Performance constraint

$$\|WE(\mathcal{G}_c)^T Q\|_2^2 \leq \gamma^2$$

$$\begin{bmatrix} \gamma^2 I & QE(\mathcal{G}_c)W \\ WE(\mathcal{G}_c)^T Q & I \end{bmatrix} \geq 0.$$

Connectivity constraint

- $\lambda_2(\mathcal{G}) > 0$
- eigenvector associated with $\lambda_1(\mathcal{G}) = 0$ is $\mathbf{1}$

$$P^T E(\mathcal{G}_c) W E(\mathcal{G}_c)^T P > 0, \quad \mathbf{Im(P)} = \mathrm{span}(\mathbf{1}^\perp)$$

Boyd, 1998

Optimization Problem

Semidefinite optimization problem (with γ as an upper bound)

$$\min_{w_i \geq 0, \gamma^2 > 0} \quad \gamma^2$$

$$\text{subject to} \begin{bmatrix} \gamma^2 I & QE(\mathcal{G}_c)W \\ WE(\mathcal{G}_c)^T Q & I \end{bmatrix} \geq 0 \qquad \text{performance constraint}$$

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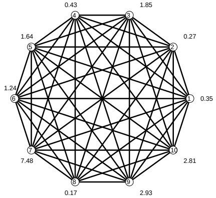
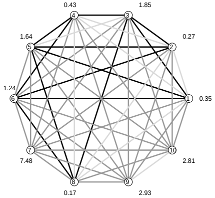
Example

complete graph → optimal performance





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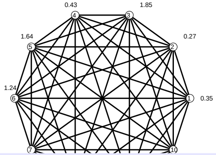
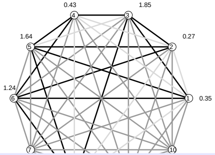


$\rightarrow$

optimal performance



without sparsity constraint, all $w_i \neq 0$

Definition (0-norm)

The 0-norm of a vector $\boldsymbol{w} \in \mathbb{R}^n$ with $\boldsymbol{w} = [w_1^T, \dots, w_{|\mathcal{E}|}^T]^T$ is defined as

$$\|\boldsymbol{w}\|_0 = \{\text{number of } w_i | w_i \neq 0\}.$$

$$\begin{aligned} \min_{w_i \geq 0} \quad & \|\boldsymbol{w}\|_0 \\ \text{subject to} \quad & \begin{bmatrix} \gamma^2 I & QEW(\mathcal{G}_c) \\ WE(\mathcal{G}_c)^T Q & I \end{bmatrix} \geq 0 \\ & P^T E(\mathcal{G}_c) W E(\mathcal{G}_c)^T P > 0 \\ & \text{and } \gamma \text{ fixed} \end{aligned}$$

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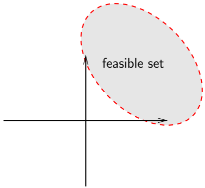
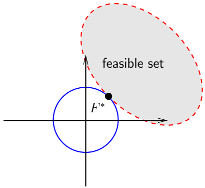
- objective function is non-convex $\rightarrow$ combinatorial problem
- convex relaxation by re-weighted ℓ_1-optimization

Sparsity Promoting Optimization



see also Candès, Wakin and Boyd, 2008
Lin, Fardad and Jovanoviċ, 2011

Sparsity Promoting Optimization



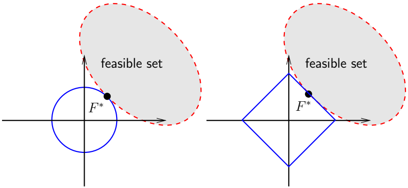
$$\min \; \|\boldsymbol{x}\|_2$$

subject to $\boldsymbol{x} \in$ feasible set

- convex optimization problem
- does not deliver sparse solutions

see also Candès, Wakin and Boyd, 2008
Lin, Fardad and Jovanovič, 2011

Sparsity Promoting Optimization



$$\min \|\boldsymbol{x}\|_1$$
$$\text{subject to } \boldsymbol{x} \in \text{feasible set}$$

- convex optimization problem
- delivers sparse solutions for linear programs

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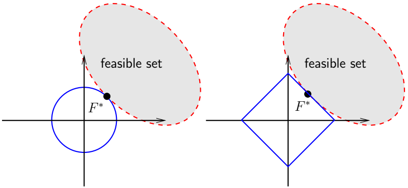
Sparsity Promoting Optimization



"ℓ_1-magic"

$$\min \; \|\boldsymbol{x}\|_1$$

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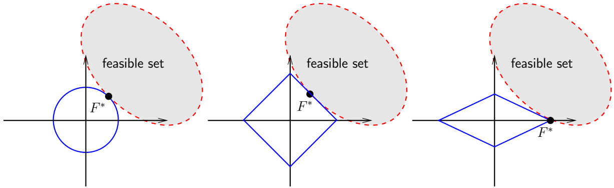

$$\min \sum_{i=1}^{n} m_i |x_i|$$

subject to $\boldsymbol{x} \in$ feasible set

- convex optimization problem
- delivers sparse solutions for semidefinite programs

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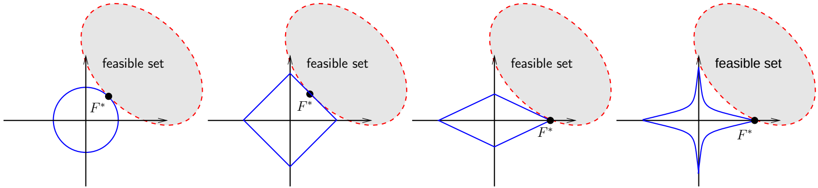
Sparsity Promoting Optimization



$$\min \|\boldsymbol{x}\|_p, \quad 0 < p < 1$$

subject to $\boldsymbol{x} \in$ feasible set

- non-convex optimization problem
- delivers sparse solutions

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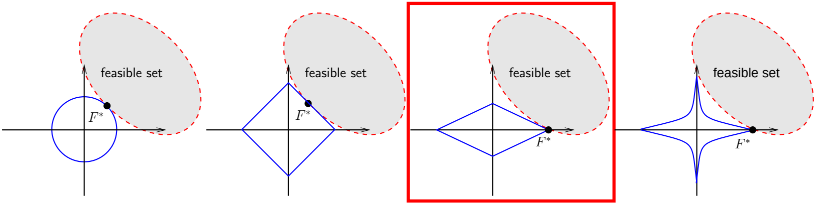
Sparsity Promoting Optimization



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Semidefinite optimization problem

$$\min_{w_i \geq 0} \quad \sum_{i=1}^{n} m_i w_i$$

$$\text{subject to} \begin{bmatrix} \gamma^2 I & QE(\mathcal{G}_c)W \\ WE(\mathcal{G}_c)^T Q & I \end{bmatrix} \geq 0$$

$$P^T E(\mathcal{G}_c) W E(\mathcal{G}_c)^T P > 0$$

Maximum weight on each edge

$$0 \leq w_i \leq w_{i,\max}$$

resulting graph is always a tree

Tradeoff Between Connectivity and Sparsity

Maximization of weighted connectivity

agent dynamic represents node weight

$$\max_{w_i \geq 0, \mu > 0} \quad \mu$$

$$\text{subject to } P^T(EWE^T - \mu Q)P > 0$$

Shafi, Arcak and El Ghaoui, 2010

Tradeoff Between Connectivity and Sparsity

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Shafi, Arcak and El Ghaoui, 2010

Sparsity vs connectivity

$$\min_{w_i \geq 0, \mu > 0} \quad (1-\alpha)\sum_{i=1}^{n} m_i w_i - \alpha\mu, \quad \alpha \in (0,1)$$

$$\text{subject to } \begin{bmatrix} \gamma^2 I & QE(\mathcal{G})W \\ WE(\mathcal{G})^T Q & I \end{bmatrix} \geq 0$$

$$P^T(EWE^T - \mu Q)P > 0$$

Optimization Algorithm

Algorithm 1 Sparse Topology Design

1. Set $h = 0$ and choose $m_i^{(0)}$ for $i = 1, \ldots, |\mathcal{E}|$ and $\nu > 0$.
2. Solve the minimization problem to find the optimal solution $w_i^{(h)}$.
3. Update the weights

$$m_i^{(h+1)} = (w_i^{(h)} + \nu)^{-1}.$$

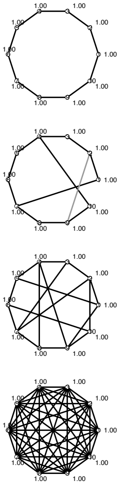
4. Terminate on convergence, otherwise set $h = h + 1$ and go to Step 2.

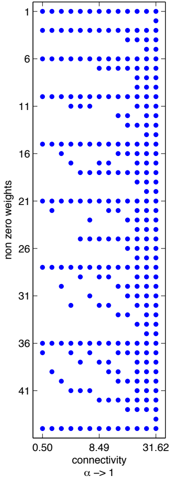
Weights: initial weights $m_i^{(0)}$ can promote desired sub-graphs

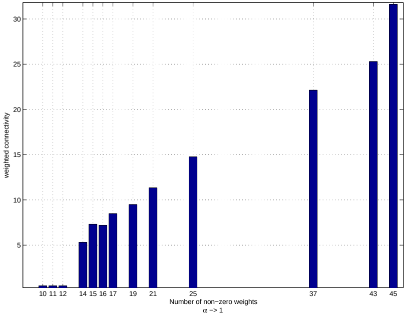
Example: Homogenous RSN

Promotion of sub-graphs (e.g. path)

10 agents, $\gamma = 10$, $m^0_{\text{path}} = 10^{-4}$



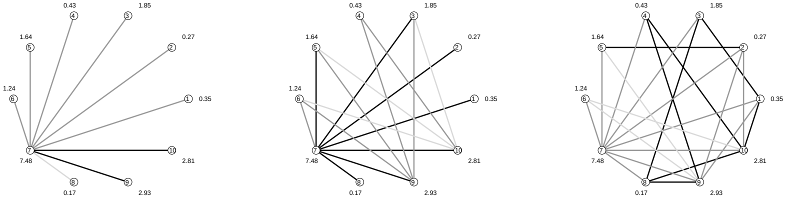


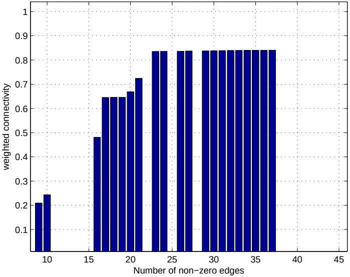


Example: Heterogenous RSN

Topology optimization

10 random agents with $\|H_i\|_\infty \in [0.17, 7.48]$, $\gamma = 10$

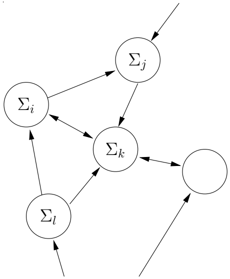




Conclusion and Outlook

design of sparse relative sensing networks

- consideration of performance, connectivity and *sparsity* constraints
- homogenous and heterogenous agent dynamics
- fast convergence of algorithm
- suitable for large networks
- promotion of sub-graphs



Next steps: Extensions to design of *robust* relative sensing networks.

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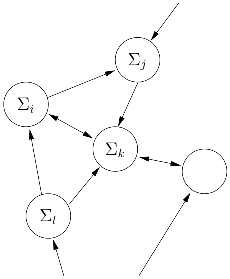

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