CLUSTER ASSIGNMENT IN MULTI-AGENT SYSTEMS

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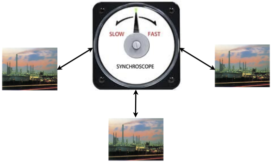
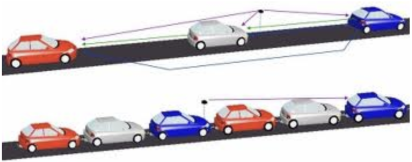
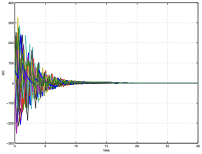


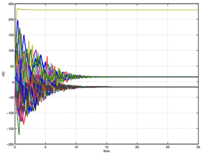


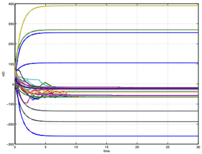
Synchroscope for A/C electrical power systems

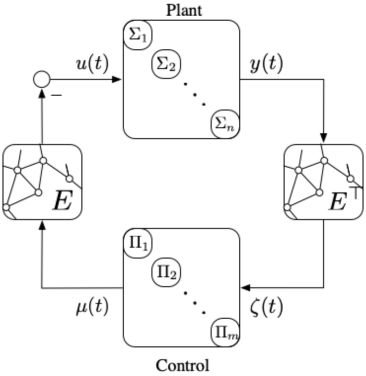
Car following

Agreement v. Clustering









Node dynamics

$$\Sigma_i : \quad \dot{x}_i(t) = f_i(x_i(t), u_i(t)),$$
$$y_i(t) = h_i(x_i(t), u_i(t)).$$

Edge dynamics

$$\Pi_e : \quad \dot{\eta}_e(t) = \phi_e(\eta_e(t), \zeta_e(t)),$$
$$\mu_e(t) = \psi_e(\eta_e(t), \zeta_e(t)).$$

The network is denoted by the triple $(\mathcal{G}, \Sigma, \Pi)$.

- steady-state I/O relation: $\mathrm{y}_i \in k_i(\mathrm{u}_i)$, $\mu_e \in \gamma_e(\zeta_e)$
- Assumption: The agents Σ_i are output-strictly MEI-Passive and the controllers Π_e are MEI-Passive

- We want to understand how symmetries in a multi-agent system affect the steady-state of system

Weakly Equivalent Systems

Two systems are called weakly equivalent if they have the same steady-state relation

Examples of WES:

$$\Upsilon_1 : \; y = u$$

$$\Upsilon_2 : \begin{cases} \dot{x} = -x + u, \\ y = x \end{cases}$$

$$\Upsilon_3 : \begin{cases} \dot{x} = -10x + u, \\ y = 10x \end{cases}$$

$$\Upsilon_4 : \begin{cases} \dot{x} = -x + \sinh(u), \\ y = \operatorname{arcsinh}(x) \end{cases}$$

$$\Upsilon_5 : \begin{cases} \dot{x} = -x + u, \\ y = 0.5(x + u) \end{cases}$$

all have $k(\mathrm{u}) = (u)$

WEAKLY EQUIVALENT SYSTEMS

- We want to understand how symmetries in a multi-agent system affect the steady-state of system

Weakly Equivalent Systems

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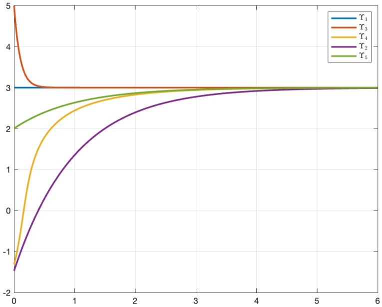
Examples of WES:



Weak Symmetries

Let $(\mathcal{G}, \Sigma, \Pi)$ be a multi-agent system. A weak automorphism of the MAS is a map $\psi : \mathcal{V} \to \mathcal{V}$ such that

1. If $i \to j$ is an edge, then so is $\psi(i) \to \psi(j)$ (graph automorphism)
2. Σ_i is weakly equivalent to $\Sigma_{\psi(i)}$
3. Π_e is weakly equivalent to $\Pi_{\psi(e)}$
4. The map ψ preserves edge orientations

The group of weak automorphisms is denoted $\mathrm{Aut}(\mathcal{G}, \Sigma, \Pi)$.

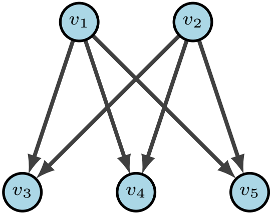
- note $\mathrm{Aut}(\mathcal{G}, \Sigma, \Pi)$ is a subgroup of $\mathrm{Aut}(\mathcal{G})$ (automorphism group of $\mathcal{G}$)

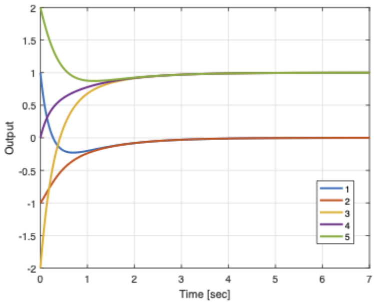
Theorem [S&Z2016]

Consider the diffusively-coupled system $(\mathcal{G}, \Sigma, \Pi)$, and suppose the passivity assumption holds. Then for any steady-state output y of the closed-loop and any weak automorphism $\psi \in \mathrm{Aut}(\mathcal{G}, \Sigma, \Pi)$, it follows that $P_\psi \mathrm{y} = \mathrm{y}$, where P_ψ is the permutation matrix representation of ψ.

- shows such a MAS converges to a cluster configuration (when $0 \notin \gamma_e(0)$)
- clusters determined by the orbits of the weak automorphism group

- Given 5 LTI agents: $G(s) = \frac{1}{s+1}$
- Goal: cluster of size 3 at $\mathrm{y} = 1$ and cluster of size 2 at $\mathrm{y} = 0$.





- orient edges from $1, 2$ to $3, 4, 5$
- choose controllers equal to $\mu_e = -1.2 + \zeta_e$.

Problem 1

Consider a collection of n homogeneous agents $\{\Sigma_i\}_{i\in\mathcal{V}}$, and let $r_1,\ldots,r_k$ be positive integers which sum to n. Find a graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ and homogeneous edge controllers $\{\Pi_e\}_{e\in\mathcal{E}}$ such that the closed loop network converges to a clustered steady-state, with a total of k clusters of sizes $r_1,\ldots,r_k$.

- Assumption: Controllers Π_e are chosen such that they are MEI-Passive and $0\notin\gamma_e(0)$

Problem 2

Given positive integers $r_1,\ldots,r_k$, determine if there exists an oriented graph $\mathcal{G}$ such that it is weakly connected and the action of $\mathrm{Aut}(\mathcal{G})$ on $\mathcal{G}$ has orbits of sizes $r_1,\ldots,r_k$. If a graph exists, then construct it.

- graphs satisfying above properties are denoted as of type $OS(r_1,\ldots,r_k)$

HOW MANY EDGES ARE NEEDED?

Theorem

Let $r_1, \dots r_k$ be any positive integers, and let $n = r_1 + \cdots + r_k$

i) Any directed graph of type OS$(r_1, \dots, r_k)$ has at least m edges, where

$$m = \min_{\mathcal{T} \text{ tree on } k \text{ vertices}} \sum_{\{i,j\} \in \mathcal{T}} \operatorname{lcm}(r_i, r_j). \tag{1}$$

ii) There exist a directed graph of type $OS(r_1, \dots, r_k)$ with at most M edges, where

$$M = \min_{\substack{\mathcal{T} \text{ path on} \\ k \text{ vertices}}} \left(\sum_{\{i,j\} \in \mathcal{T}} \operatorname{lcm}(r_i, r_j) \right) + \min_{i \in \mathcal{V}} r_i. \tag{2}$$

- lower bound can be found using any algorithm finding a minimal spanning tree, e.g. Kruskal's algorithm (polynomial time)
- upper bound requires solving variant of the traveling salesman problem (**NP**-hard)

ALGORITHM

Algorithm 1 Building OS-type graphs

Input: A collection $r_1, \ldots, r_k$ of positive integers summing to n, and a path $\mathcal{T}$ on k vertices.

Output: A graph $\mathcal{G}$ of type $OS(r_1, \ldots, r_k)$.

```
1: Let 𝒢 = (𝕍, ℰ) be an empty graph.
2: for j = 1, ..., k and p = 1, ..., r_j do
3:     Add a node with label v_p^j to 𝕍.
4: for any edge {i, j} in 𝒯 do
5:     for p = 1, ..., lcm(r_i, r_j) do
6:         Add the edge v^i_{p mod r_i} → v^j_{p mod r_j} to 𝔼.
7: Compute i* = arg min{r_i}. If r_{i*} = 1, go to step 10.
8: for p = 1, ..., r_{i*} do
9:     Add the edge v_p^{i*} → v^{i*}_{(p+1) mod r_{i*}} to 𝔼.
10: Return 𝒢 = (𝕍, 𝔼).
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Corollary

Suppose that all cluster sizes $r_1, \cdots, r_k$ are equal, and bigger than 1. Then there exists a graph of type OS($r_1, \ldots, r_k$) with at most $n = r_1 + \ldots + r_k$ edges.

Corollary

Let $r_1, \cdots, r_k$ be positive integers such that $k \geq 2$ and that for every j, l, either r_j divides r_l or vice versa. Then there exists a graph of type OS($r_1, \ldots, r_k$) with $n = r_1 + \ldots + r_k$ edges.

Corollary

Let $r_1, \cdots, r_k$ be positive integers such that $r_j \leq q$ for all j, and let $n = r_1 + \cdots + r_k$. Then there exists a graph of type OS($r_1, \ldots, r_k$) with at most $n + O(q^3)$ edges.

consider collection of $n = 15$ identical agents

$$\dot{x} = -x + u + \alpha,\ y = x$$

(α is random variable) identical controllers

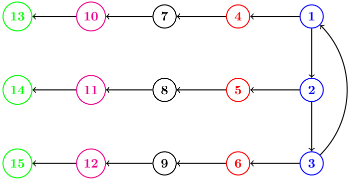
$$\mu = a_1 + a_2(\zeta + \cos(\zeta)),$$

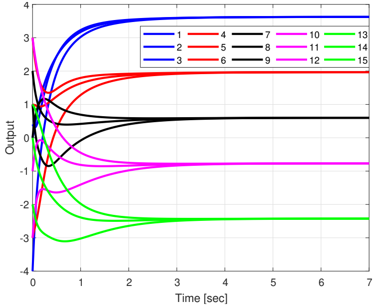
(a_1, a_2 random variables)

construct graph with five equally-sized clusters

EXAMPLE

$$r_1 = r_2 = r_3 = r_4 = r_5 = 3$$





consider collection of $n = 10$ identical agents

$$\dot{x} = -x + u + \alpha, \; y = x$$

(α is random variable) identical controllers

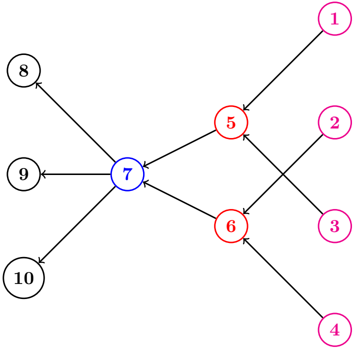
$$\mu = a_1 + a_2(\zeta + \cos(\zeta)),$$

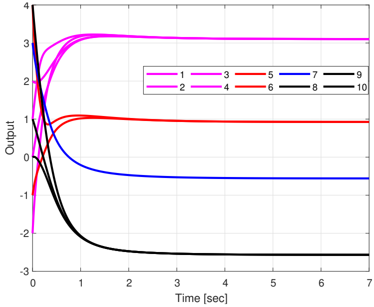
(a_1, a_2 random variables)

construct graph with $r_1 = 1, r_2 = 2, r_3 = 3, r_4 = 4$

EXAMPLE

$r_1 = 1, r_2 = 2, r_3 = 3, r_4 = 4$





- explored the cluster assignment problem for MAS
- provide a characterization of graphs with an automorphism group containing orbits of a prescribed size
- provide a constructive algorithm for to find such graphs

Questions?