Robust Design of Sparse Relative Sensing Networks

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Relative Sensing Networks

A collection of dynamic systems that use sensed relative state information to achieve higher level objectives.

Applications

- formation control
- localization
- environmental surveillance
- ...

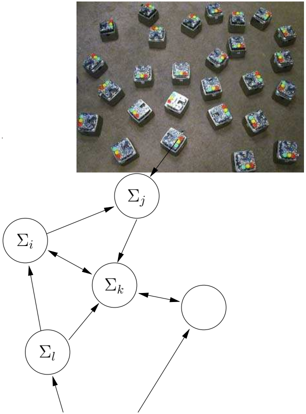
- 'absolute' inertial measurements are often not available (deep space, gps-denied environments $\rightarrow$ "harsh" environments)
- however, relative measurements are available and can be very accurate

Relative Sensing Networks

implicit presence of a 'network' induced by sensing structure

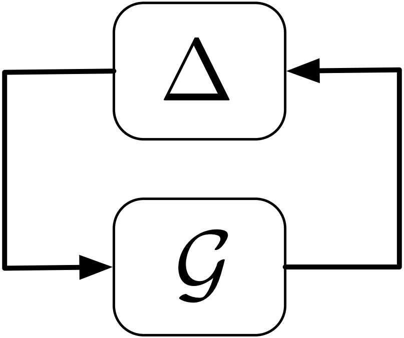
Performance and design of networks:

- Influence of topology on performance
- Optimal topologies
- Sparsity vs connectivity
- Heterogeneity of dynamics
- Robustness of performance



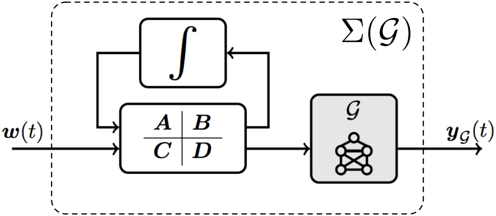
Relative Sensing Networks

combinatorial and dynamic uncertainty



Outline

Modeling of Relative Sensing Networks



State space model

$$\Sigma(\mathcal{G}) : \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{w}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{w}(t) \\ \boldsymbol{y}_{\mathcal{G}}(t) &= (E(\mathcal{G})^T \otimes I)\boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

$(i.e., \boldsymbol{A} = \mathrm{diag}(A_1, A_2, \ldots, A_n))$

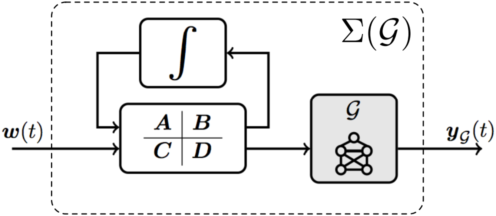
Transfer function

$$T^{\boldsymbol{w}\mapsto\mathcal{G}}(s) = (E(\mathcal{G})^T \otimes I)\boldsymbol{H}(s)$$

with $\boldsymbol{H}(s) = \mathrm{diag}(H_1, H_2, \ldots, H_n)$

and $H_i := C_i(sI - A_i)^{-1}B_i$

Modeling of Relative Sensing Networks



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Performance ? Robustness ?

Performance Analysis of RSNs

$\mathcal{H}_\infty$-norm captures how finite energy exogenous signals are amplified at the monitored outputs.

Theorem ($\mathcal{H}_\infty$-Performance of RSNs)

The $\mathcal{H}_\infty$-norm of a heterogenous RSN is bounded from above by

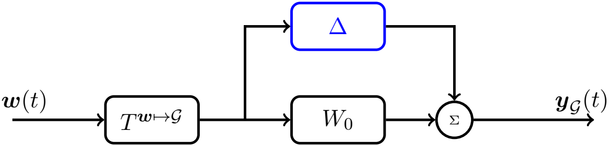
$$\|T^{\boldsymbol{w} \mapsto \mathcal{G}}\|_\infty \leq \|W E(\mathcal{G})^T Q\|_2$$

where $Q = \mathrm{diag}(\|H_1\|_\infty, \ldots, \|H_n\|_\infty)$.

Zelazo and Mesbahi, 2011

- graph-centric characterization of $\mathcal{H}_\infty$-norm
- $\mathcal{H}_\infty$ performance is dependent on *graph structure*
- for SISO systems, this bound is tight

multiplicative output uncertainty



$$W = W_0 + \Delta, \text{ where } \Delta \in \boldsymbol{\Delta}_w$$

$$\boldsymbol{\Delta}_w = \{\mathrm{diag}(\delta_1, \ldots, \delta_{|\mathcal{E}|}) : \delta \in \mathbb{R}^{|\mathcal{E}|}, \|\delta\|_2 \leq 1\}$$

Definition (Robust Connectivity)

A weighted graph is called *robustly connected* under the uncertainty set $\boldsymbol{\Delta}_w$, if and only if the graph stays connected for all $\Delta \in \boldsymbol{\Delta}_w$.

From Analysis to Synthesis

Robust RSN Design

Design the sensing network of an RSN that is at the same time *robustly connected* and *sparse* with good $\mathcal{H}_\infty$ performance.

Tradeoff Between Connectivity and Sparsity

Robust optimization problem (with γ as an upper bound)

$$\min_{w_i \geq 0, \gamma^2 > 0} \max_{\|\delta\|_2 \leq 1} \quad \gamma^2$$

$$\text{subject to} \begin{bmatrix} \gamma^2 I & QE(\mathcal{G}_c)(W+\Delta) \\ (W+\Delta)E(\mathcal{G}_c)^T Q & I \end{bmatrix} \geq 0$$

$$P^T E(\mathcal{G}_c)(W+\Delta)E(\mathcal{G}_c)^T P > 0$$

- robust performance and robust connectivity
- design of nominal edge weights

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Maximization of weighted connectivity

agent dynamics represents node weight

$$\min_{w_i \geq 0, \mu > 0} \max_{\|\delta\|_2 \leq 1} \quad -\mu$$

$$\text{subject to } P^T(E(W_0+\Delta)E^T - \mu Q)P > 0$$

Shafi, Arcak and El Ghaoui, 2010

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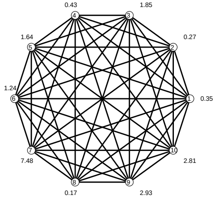
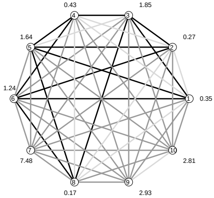
Example

complete graph



$\longrightarrow$

optimal performance



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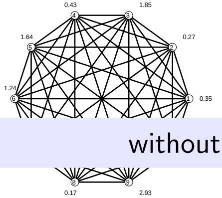

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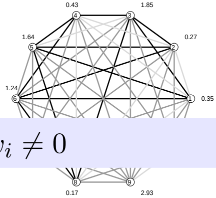
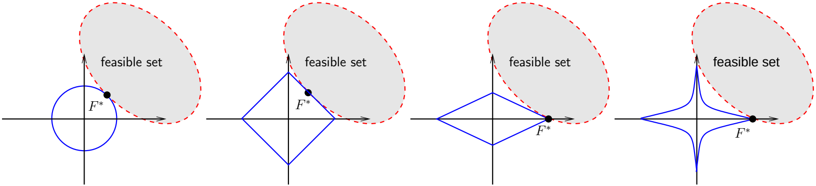


without sparsity constraint, all $w_i \neq 0$

Sparsity Promoting Optimization

weighted ℓ_1-minimization



$$\min \sum_{i=1}^{n} m_i |x_i|$$

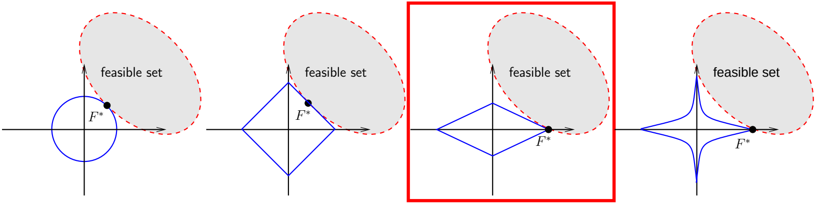
subject to $\boldsymbol{x} \in$ feasible set

- ℓ_1-norm is the *convex envelope* of the cardinality function
- convex optimization problem
- delivers sparse solutions for semidefinite programs

(Candés et al. 2008)

Sparsity Promoting Optimization

weighted ℓ_1-minimization



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Tradeoff Between Connectivity and Sparsity

Sparsity vs connectivity

$$\min_{w_i \geq 0, \mu > 0} \max_{\|\delta\|_2 \leq 1} \quad (1-\alpha) \sum_{i=1}^{n} m_i w_i - \alpha \mu, \quad \alpha \in [0,1]$$

subject to
$$\begin{bmatrix} \gamma^2 I & Q E(\mathcal{G})(W_0 + \Delta) \\ (W_0 + \Delta) E(\mathcal{G})^T Q & I \end{bmatrix} \geq 0$$

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$$\text{subject to} \quad \begin{bmatrix} S^j & F_1^j & \cdots & F_{|\mathcal{E}|}^j \\ F_1^j & T^j & & \\ \vdots & & \ddots & \\ F_{|\mathcal{E}|}^j & & & T^j \end{bmatrix} \geq 0, \quad j = 1, 2$$

$$S^j + T^j \leq 2F_0^j, \quad j = 1, 2$$

$$w_i \geq 0.$$

Rewrite constraints by robust counterpart → SDP

Ben-Tal, El Ghaoui and Nemirovski, 2000

Optimization Algorithm

Algorithm 1 Sparse Topology Design

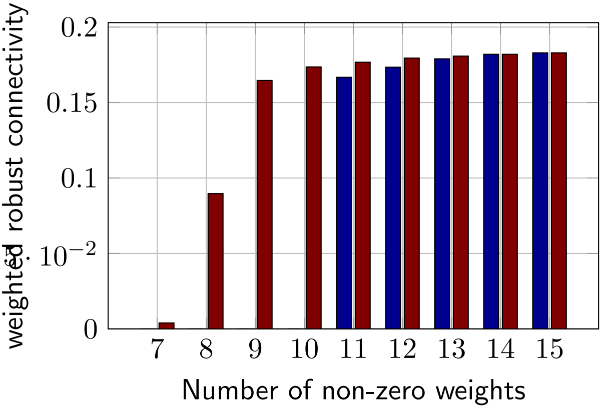
1. Set $h = 0$ and choose $m_i^{(0)}$ for $i = 1, \dots, |\mathcal{E}|$ and $\nu > 0$.
2. Solve the minimization problem to find the optimal solution $w_i^{(h)}$.
3. Update the weights

$$m_i^{(h+1)} = (w_i^{(h)} + \nu)^{-1}.$$

4. Terminate on convergence, otherwise set $h = h + 1$ and go to Step 2.
5. Solve optimization problem for fixed structure obtained in Step 3 (polishing step).

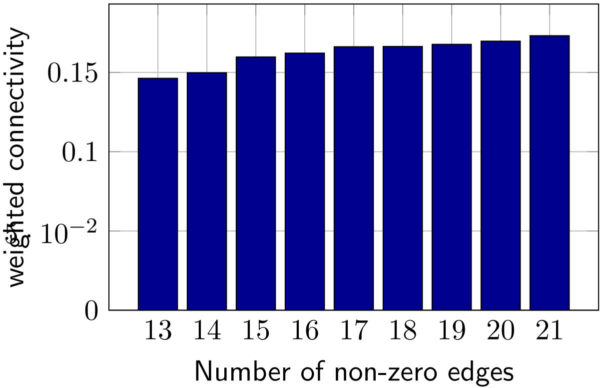
Weights: initial weights $m_i^{(0)}$ can promote desired sub-graphs

Exhaustive Search vs Sparse Design



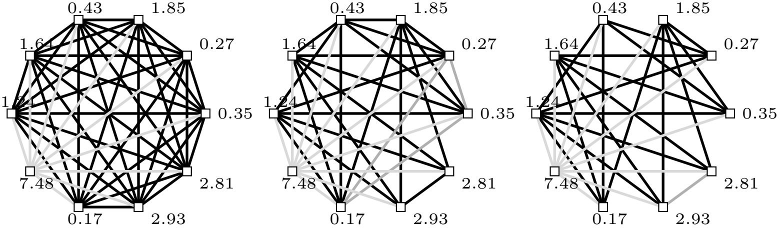
- 6 random agents (15 possible edges)
- $\mathcal{H}_\infty$-norm $\|H_i\|_\infty \in [0.62, 6.72]$, $\gamma = 18$
- $26,704$ possibilities for nominally connected graph topologies

Connectivity maximization



- 7 random agents (21 possible edges)
- $\|H_i\|_\infty \in [0.44, 3.88]$, $\gamma = 10$
- 1.86×10^6 possibilities of nominally connected graph topologies

Topology Optimization with Optimal Performance



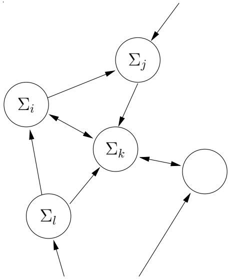
(a) $\gamma = 19.2$, 45 edges. (b) $\gamma = 19.39$, 34 edges. (c) $\gamma = 19.45$, 29 edges.

- 10 random agents (45 possible edges)
- $\|H_i\|_\infty \in [0.17, 7.48]$
- $2^{45} = 3.52 \cdot 10^{13}$ possible graphs topologies

Conclusion and Outlook

design of sparse relative sensing networks

- approximation of exhaustive search by weighted ℓ_1-minimization
- consideration of performance, connectivity and *sparsity* constraints in face of *uncertain* edge weights
- fast convergence of algorithm
- promotion of sub-graphs



Next steps: Alternative to robust counterpart to allow larger networks.

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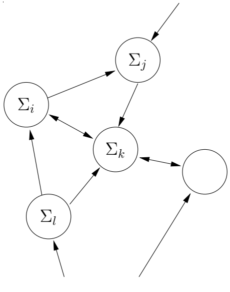

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Thank you!