Consensus over Networks with Merging Agents

Daniel Zelazo

***Abstract*—We study consensus among agents that physically merge when they come close. The protocol is a hybrid system. During flows, the agents run the consensus protocol over the current graph. A jump occurs when two agents that are not yet merged come within a docking distance: the states of their two clusters are reset to their common center of mass (*state merge*), and every agent of the new cluster is connected to the other agents of the cluster and to all of their neighbors (*neighborhood merge*). Merged agents thus have equal states and identical neighborhoods, so the flow keeps them together. We show that the resulting graph is a blow-up of a contraction of the initial graph, in which each cluster is contracted to a node and then expanded back into a clique. The network therefore evolves as a consensus network over the contracted graph with cluster masses, and the protocol needs no sensing links beyond the original ones. Using this structure, we show that all agents merge at the initial centroid in finite time, with an explicit bound that holds for every order of the merges. Extensions to formations, collision avoidance, and leader–follower networks are given, and the results are illustrated by modular self-assembly and cooperative transport.**

***Index Terms*—Multi-agent systems, consensus, hybrid dynamical systems, formation control, graph contractions, collision avoidance.**

I. Introduction

THE consensus protocol is one of the fundamental distributed algorithms employed in multi-agent systems [1], [2], and is used as a subroutine for higher-level objectives such as rendezvous [3] and formation control [4]. Typically, consensus algorithms assume simple integrator dynamics for the agents. While such models are justified when determining high-level quantities, there is a modeling conflict when the agents are physical bodies: two bodies cannot occupy the same point, so the consensus state is physically unrealizable, and the agents approach it only asymptotically, without ever coming into contact.

On the other hand, there is a growing interest in *assembly* protocols for robotic systems, where simple robots physically couple with other robots in order to achieve more sophisticated tasks, as in flying modules [5] and floating platforms [6] (see [7] for a survey). A closely related scenario is cooperative transport, where a team of robots must gather around an object, for instance around the legs of a table, attach to it, and then move it as a single body [8], [9]. In such settings the robots may perform a consensus or formation protocol, and when they are sufficiently close they perform some physical docking, after which they move together as a monolithic system. Docking, however, changes the system that the coordination protocol acts on. Two agents that were free to move independently become one rigid body that inherits the sensing and communication links of both [10], and a protocol that ignores this may, for instance, command different velocities to robots that are rigidly attached.

This motivates the *aggregating consensus protocol*, in which two agents merge their state whenever they are sufficiently close, while continuing to run the consensus algorithm. The idea was originally explored in [11], which focused on the algebraic characterization of node merging and graph contractions. From a network systems perspective, it raises three questions. First, *how should the interaction topology change* when agents physically couple? Second, *what network, and what effective dynamics, emerge* from a sequence of such couplings? Third, do these structural properties lead to guarantees that are *independent of the order* in which the agents happen to merge?

For the first question we introduce the *neighborhood merge*. Graph contractions, the natural graph-theoretic counterpart of merging, reduce the order of the graph, while the number of physical agents does not change. The neighborhood merge instead adds edges, so that merged agents become indistinguishable to the rest of the network and move together because of the structure of the graph and not because of a change of the model order. For the second question, we show that the network generated by any sequence of merges is the *blow-up* of a contraction of the initial graph: each group of merged agents is contracted to a single node, and every node is then expanded back into a clique of its agents, joined to the agents of every adjacent group (Proposition 1). We also show that this network evolves as a consensus network over the contracted graph in which each cluster has a mass equal to its number of agents and two adjacent clusters are coupled by all the links between their agents (Theorem 1). It follows that the protocol needs no sensing links beyond the original ones. For the third question, this structure leads to an invariant centroid and to a disagreement that never increases when agents merge, and from there to aggregation at the initial centroid in finite time, with an explicit bound that holds for every order of the merges (Theorem 2). We then show that the same model describes the assembly of a prescribed formation (Corollary 1), that collisions between the bodies of the agents can be excluded without changing these guarantees (Theorem 3), and that, in leader–follower networks, all agents merge at the position of the leader (Corollary 2).

The results are formalized with the hybrid dynamical systems framework of [12], which has been used for consensus with intermittent or event-based information exchange [13], [14] and for opinion dynamics with state-dependent connectivity [15]. As in these works, we augment the state by $\{0,1\}$ vectors indicating the presence of edges. In [15], however, edges are removed and the limit is a set of disconnected clusters, while here edges are added, the continuous state itself

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jumps, and the limit is a single cluster. We emphasize that the stability tools we use are standard, and that the $\{0,1\}$ vectors, of dimension $N(N-1)/2$, serve only the analysis and are not stored by the agents (Section V-A). The contribution lies in the network model and its structural characterization, which make these tools applicable. The results are illustrated by two examples: a team of robots that gathers around the legs of a table, attaches to it, and carries it to a goal, and the self-assembly of 19 hexagonal modules into a prescribed structure, with and without collision avoidance.

This work is also related to finite-time consensus, usually obtained with non-Lipschitz protocols [16], whereas here the protocol is linear and exact agreement is produced by the merging events. The structural results are related to model reduction by graph partitions [17], [18]. In these works the partition is chosen by the designer, while here it is generated by the closed-loop system.

The paper is organized as follows. Sections II–IV introduce the neighborhood merge and the hybrid model, and Section V presents the main results. Section VI extends them to formations, collision avoidance, and leader–follower networks, and Section VII illustrates them with numerical examples.

Notation: For $N \in \mathbb{N}$, $\mathbb{N}_N := \{1,\dots,N\}$. The vector of all ones is $\mathbf{1}$, the identity matrix is I_ν, $\operatorname{diag}(v)$ is the diagonal matrix with the entries of the vector v on its diagonal, and $\otimes$ denotes the Kronecker product. For column vectors $x_1,\dots,x_N$, $(x_1,\dots,x_N)$ denotes their stacking. For a vector v, $\|v\|$ is its Euclidean norm and, for a closed set S, $\|v\|_S := \inf_{w\in S}\|v-w\|$. The cardinality of a finite set S is $|S|$. For $x=(x_1,\dots,x_N)$ with $x_i\in\mathbb{R}^\nu$, $\bar{x} := \frac{1}{N}\sum_i x_i$ and $\mathcal{A}_x := \{x\colon x_1=\cdots=x_N\}$ is the *agreement subspace*. The orthogonal projection of x onto $\mathcal{A}_x$ is $\mathbf{1}\otimes\bar{x}$, and thus

$$\|x\|^2_{\mathcal{A}_x} = \|x-\mathbf{1}\otimes\bar{x}\|^2 = \sum_{i=1}^{N}\|x_i-\bar{x}\|^2. \tag{1}$$

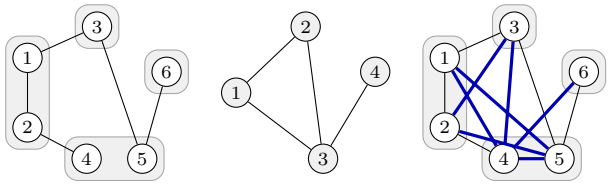
II. Preliminaries

In this section we briefly provide background on graphs and hybrid systems.

A. Graph theory

A graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ has vertex set $\mathcal{V}=\mathbb{N}_N$ and edge set $\mathcal{E}$, a set of unordered pairs of distinct vertices, and we write $uw=wu$ for the pair $\{u,w\}$. All graphs are undirected and without self-loops. The neighborhood of u is $\mathcal{N}_\mathcal{G}(u) := \{w\colon uw\in\mathcal{E}\}$, its closed neighborhood is $\mathcal{N}_\mathcal{G}[u] := \mathcal{N}_\mathcal{G}(u)\cup\{u\}$, and $\mathcal{N}_\mathcal{G}[S] := \bigcup_{u\in S}\mathcal{N}_\mathcal{G}[u]$ for $S\subseteq\mathcal{V}$. Two distinct vertices u,w are *true twins* if $\mathcal{N}_\mathcal{G}[u]=\mathcal{N}_\mathcal{G}[w]$ and they are adjacent [19]. A set $S\subseteq\mathcal{V}$ is a *clique* if any two distinct vertices of S are adjacent. The adjacency matrix $A_\mathcal{G}$ of $\mathcal{G}$ has $[A_\mathcal{G}]_{uw}=1$ if $uw\in\mathcal{E}$ and $[A_\mathcal{G}]_{uw}=0$ otherwise. The Laplacian of $\mathcal{G}$ is $L_\mathcal{G} := \operatorname{diag}(A_\mathcal{G}\mathbf{1})-A_\mathcal{G}$, with eigenvalues $0=\lambda_1(\mathcal{G})\le\lambda_2(\mathcal{G})\le\cdots\le\lambda_N(\mathcal{G})$, and

$$x^\top(L_\mathcal{G}\otimes I_\nu)x = \sum_{uw\in\mathcal{E}}\|x_u-x_w\|^2 \;\ge\; \lambda_2(\mathcal{G})\|x\|^2_{\mathcal{A}_x}. \tag{2}$$



(a) $\mathcal{G}$ and π (b) $\mathcal{G}/\!\!/\pi$ (c) $\mathcal{B}(\mathcal{G},\pi)$

Fig. 1. (a) A graph $\mathcal{G}$ and a partition π with cells $\mathcal{C}_1=\{1,2\}$, $\mathcal{C}_2=\{3\}$, $\mathcal{C}_3=\{4,5\}$, $\mathcal{C}_4=\{6\}$ (shaded). (b) The contraction $\mathcal{G}/\!\!/\pi$, with node a representing $\mathcal{C}_a$. (c) The blow-up $\mathcal{B}(\mathcal{G},\pi)$: each cell becomes a clique and adjacent cells are completely joined (thick edges are those not in $\mathcal{G}$). Vertices of the same cell become true twins: 4 and 5 have different closed neighborhoods in $\mathcal{G}$, $\mathcal{N}_\mathcal{G}[4]=\{2,4\}$ and $\mathcal{N}_\mathcal{G}[5]=\{3,5,6\}$, but $\mathcal{N}[4]=\mathcal{N}[5]=\{1,\dots,6\}$ in $\mathcal{B}(\mathcal{G},\pi)$.

The graph is connected if and only if $\lambda_2(\mathcal{G})>0$ [2]. Adding edges to a graph adds a positive semidefinite matrix to its Laplacian, and therefore, if $\mathcal{E}\subseteq\mathcal{E}'$ then $\lambda_2(\mathcal{V},\mathcal{E})\le\lambda_2(\mathcal{V},\mathcal{E}')$.

A partition $\pi=\{\mathcal{C}_1,\dots,\mathcal{C}_r\}$ of $\mathcal{V}$ has characteristic matrix $P\in\{0,1\}^{N\times r}$, $P_{ua}=1$ if and only if $u\in\mathcal{C}_a$, and cell sizes $n := P^\top\mathbf{1}=(n_1,\dots,n_r)$. The *contraction* $\mathcal{G}/\!\!/\pi$ is the graph on $\mathbb{N}_r$ in which $a\neq b$ are adjacent if and only if some vertex of $\mathcal{C}_a$ is adjacent in $\mathcal{G}$ to some vertex of $\mathcal{C}_b$ [18]. Conversely, the *blow-up* $\mathcal{B}(\mathcal{G},\pi)$ is the graph on $\mathcal{V}$ in which two distinct vertices are adjacent if and only if they belong to the same cell, or to cells that are adjacent in $\mathcal{G}/\!\!/\pi$.[1] Thus in $\mathcal{B}(\mathcal{G},\pi)$ every cell is a clique, any two cells are either completely joined or not joined at all, and $\mathcal{B}(\mathcal{G},\pi)/\!\!/\pi=\mathcal{G}/\!\!/\pi$ (see Fig. 1).

B. Hybrid systems

We use the framework of [12]. A hybrid system $\mathscr{H}=(C,f,D,G)$ with state $\xi\in\mathbb{R}^n$ is

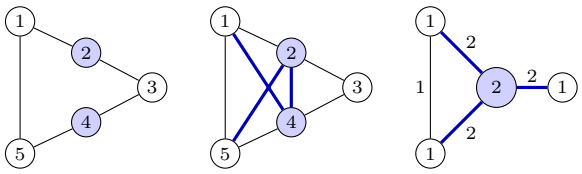
$$\mathscr{H}\colon\quad \dot{\xi}=f(\xi),\quad \xi\in C, \qquad \xi^+\in G(\xi),\quad \xi\in D, \tag{3}$$

with flow set C, flow map f, jump set D and set-valued jump map G. A solution ϕ is defined on a hybrid time domain $\operatorname{dom}\phi\subset\mathbb{R}_{\ge0}\times\mathbb{N}_{\ge0}$, where (t,j) records the ordinary time t elapsed and the number j of jumps performed [12, Def. 2.6]. We write t_j for the time of the j-th jump and $t_0:=0$, so that $\phi(\cdot,j)$ flows on $[t_j,t_{j+1}]$, and $\phi(t,j)\in C$ for all t in the interior of this interval. A solution is maximal if it cannot be extended and complete if its domain is unbounded. A closed set $\mathcal{A}\subset\mathbb{R}^n$ is *uniformly globally asymptotically stable* (UGAS) for $\mathscr{H}$ if all maximal solutions are complete and [12, Def. 3.6]: (a) there is a class-$\mathcal{K}_\infty$ function α such that $\|\phi(t,j)\|_\mathcal{A}\le\alpha(\|\phi(0,0)\|_\mathcal{A})$ for every solution and all $(t,j)\in\operatorname{dom}\phi$, and (b) for each $\varepsilon,\rho>0$ there is $\tau>0$ such that $\|\phi(0,0)\|_\mathcal{A}\le\rho$ implies $\|\phi(t,j)\|_\mathcal{A}\le\varepsilon$ for all $(t,j)\in\operatorname{dom}\phi$ with $t+j\ge\tau$.

III. Graph Contractions and Neighborhood Merging

When studying dynamic systems over graphs, each vertex is associated with the state of an agent, and the merging

[1]The graph $\mathcal{B}(\mathcal{G},\pi)$ is the H-join of [20] with $H=\mathcal{G}/\!\!/\pi$ and complete graphs on the cells. In extremal graph theory the cells of a blow-up are usually independent sets. Here they are cliques, because merged agents must be adjacent to each other.



(a) $\mathcal{G}$ (b) $\mathcal{G}^S$, $S = \{2, 4\}$ (c) weighted $\mathcal{G}/\!\!/\pi$

Fig. 2. (a) A graph in which agents 2 and 4 are about to dock. (b) The merge of $S = \{2, 4\}$: the two agents become true twins by acquiring each other's neighbors (thick edges). (c) The reduced model of Theorem 1: the contraction of $\mathcal{G}$ by the clusters, with node labels equal to the cluster masses n_a and edge labels equal to the weights $n_a n_b$.

of agents corresponds, from the graph-theoretic perspective, to a *vertex contraction*. A contraction, however, reduces the order of the graph, while the total state dimension of the system corresponds to the number of physical agents. We must therefore consider an operation on the graph that preserves the number of nodes and ensures that, once agents have merged, they follow the same trajectories.

Definition 1 (Neighborhood merge)**.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and let $S \subseteq \mathcal{V}$ be nonempty. The *neighborhood merge of S in $\mathcal{G}$* is the graph $\mathcal{G}^S = (\mathcal{V}, \mathcal{E}^S)$ with

$$\mathcal{E}^S := \mathcal{E} \cup \big\{uw \colon u \in S,\ w \in \mathcal{N}_{\mathcal{G}}[S],\ w \neq u\big\}.$$

In other words, every vertex of S is joined to every other vertex of S and to every neighbor of any vertex of S (see Fig. 2). For $S = \{u, w\}$, the nodes u and w become adjacent and each acquires the neighbors of the other. Hence, the neighborhood merge can be seen as an edge-addition operator. Physically, it captures that the agents of a merged cluster can share the relative measurements that each of them has of the rest of the network. In the sequel, we refer to this operation simply as a *merge*.

Lemma 1 (Properties of the neighborhood merge)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\emptyset \neq S \subseteq \mathcal{V}$.*

(i) Monotonicity: *$\mathcal{E} \subseteq \mathcal{E}^S$. Hence $\lambda_2(\mathcal{G}^S) \geq \lambda_2(\mathcal{G})$, and $\mathcal{G}^S$ is connected if $\mathcal{G}$ is.*

(ii) Twin creation: *$\mathcal{N}_{\mathcal{G}^S}[u] = \mathcal{N}_{\mathcal{G}}[S]$ for all $u \in S$. Hence any two distinct vertices of S are true twins in $\mathcal{G}^S$.*

(iii) Twin preservation: *if u, w are true twins in $\mathcal{G}$ and either $\{u, w\} \subseteq S$ or $\{u, w\} \cap S = \emptyset$, then u, w are true twins in $\mathcal{G}^S$.*

Proof. (i) is immediate from the definition and Section II. (ii) Let $u \in S$. By definition u is adjacent in $\mathcal{G}^S$ to every $w \in \mathcal{N}_{\mathcal{G}}[S] \setminus \{u\}$, and $u \in S \subseteq \mathcal{N}_{\mathcal{G}}[S]$, so $\mathcal{N}_{\mathcal{G}^S}[u] \supseteq \mathcal{N}_{\mathcal{G}}[S]$. Conversely, every edge of $\mathcal{E}^S$ incident to u is either an edge of $\mathcal{E}$, whose other endpoint lies in $\mathcal{N}_{\mathcal{G}}[u] \subseteq \mathcal{N}_{\mathcal{G}}[S]$, or a new edge, whose endpoints both lie in $\mathcal{N}_{\mathcal{G}}[S]$. (iii) If $\{u, w\} \subseteq S$ the claim follows from (ii). Let $u, w \notin S$. A vertex $\ell \notin S$ gains edges in $\mathcal{G}^S$ only if $\ell \in \mathcal{N}_{\mathcal{G}}[S]$, that is, only if $\mathcal{N}_{\mathcal{G}}[\ell] \cap S \neq \emptyset$, in which case it becomes adjacent to all of S. Therefore $\mathcal{N}_{\mathcal{G}^S}[\ell] = \mathcal{N}_{\mathcal{G}}[\ell] \cup S$ if $\mathcal{N}_{\mathcal{G}}[\ell] \cap S \neq \emptyset$ and $\mathcal{N}_{\mathcal{G}^S}[\ell] = \mathcal{N}_{\mathcal{G}}[\ell]$ otherwise. Since $\mathcal{N}_{\mathcal{G}}[u] = \mathcal{N}_{\mathcal{G}}[w]$, the same alternative applies to u and to w, and $\mathcal{N}_{\mathcal{G}^S}[u] = \mathcal{N}_{\mathcal{G}^S}[w]$. □

A key concept we require is to identify when two nodes running the consensus protocol will follow the same trajectory. The next result shows that this is the case for true twins with equal states.

Lemma 2 (Twins move together)**.** *Let u, w be true twins in $\mathcal{G}$ and let $\kappa := |\mathcal{N}_{\mathcal{G}}[u]|$. Let x be a solution of $\dot{x} = -(L_{\mathcal{G}} \otimes I_\nu)x$. Then $y(t) := x_u(t) - x_w(t)$ satisfies $\dot{y}(t) = -\kappa\, y(t)$, so that $y(t) = e^{-\kappa(t-s)}y(s)$ for all $t \geq s$. In particular, if $x_u(s) = x_w(s)$, then $x_u(t) = x_w(t)$ for all $t \geq s$.*

Proof. Let $T := \mathcal{N}_{\mathcal{G}}[u] = \mathcal{N}_{\mathcal{G}}[w]$, so that $|T| = \kappa$. Since $T = \mathcal{N}_{\mathcal{G}}(u) \cup \{u\}$ and the term $x_u - x_u$ is zero,

$$\dot{x}_u = \sum_{\ell \in \mathcal{N}_{\mathcal{G}}(u)} (x_\ell - x_u) = \sum_{\ell \in T} (x_\ell - x_u) = \sum_{\ell \in T} x_\ell - \kappa\, x_u,$$

and in the same way $\dot{x}_w = \sum_{\ell \in T} x_\ell - \kappa\, x_w$. Subtracting the two gives $\dot{y}(t) = -\kappa\, y(t)$, whose unique solution from time s is $y(t) = e^{-\kappa(t-s)}y(s)$, $t \geq s$. If $x_u(s) = x_w(s)$, then $y(s) = 0$, hence $y(t) = 0$, that is, $x_u(t) = x_w(t)$ for all $t \geq s$. □

When agents merge repeatedly, the vertices of each cluster become twins of each other. The next two results show that this is exactly the structure of a blow-up, and what it implies for the Laplacian.

Lemma 3 (Blow-ups and twins)**.** *Let π be a partition of $\mathcal{V}$ and $\mathcal{G}$ a graph on $\mathcal{V}$. Then $\mathcal{G} = \mathcal{B}(\mathcal{G}, \pi)$ if and only if any two distinct vertices of the same cell of π are true twins in $\mathcal{G}$.*

Proof. If $\mathcal{G} = \mathcal{B}(\mathcal{G}, \pi)$, the closed neighborhood of a vertex of $\mathcal{C}_a$ is the union of $\mathcal{C}_a$ and of the cells adjacent to $\mathcal{C}_a$ in $\mathcal{G}/\!\!/\pi$, which is the same for all vertices of $\mathcal{C}_a$. Conversely, let any two vertices of the same cell be true twins. Since true twins are adjacent, every cell is a clique. Now let $\mathcal{C}_a$ and $\mathcal{C}_b$ be adjacent in $\mathcal{G}/\!\!/\pi$, so that some $u \in \mathcal{C}_a$ is adjacent in $\mathcal{G}$ to some $w \in \mathcal{C}_b$, and take any $u' \in \mathcal{C}_a$ and $w' \in \mathcal{C}_b$. Since $\mathcal{N}_{\mathcal{G}}[u'] = \mathcal{N}_{\mathcal{G}}[u]$ contains w, the vertex u' is adjacent to w. Since $\mathcal{N}_{\mathcal{G}}[w'] = \mathcal{N}_{\mathcal{G}}[w]$ contains u', the vertex u' is adjacent to w'. Hence adjacent cells are completely joined. Cells that are not adjacent in $\mathcal{G}/\!\!/\pi$ are not joined, by definition of the contraction. Therefore $\mathcal{G} = \mathcal{B}(\mathcal{G}, \pi)$. □

Lemma 4 (Laplacian of a blow-up)**.** *Let $\pi = \{\mathcal{C}_1, \dots, \mathcal{C}_r\}$ be a partition of $\mathcal{V}$ with characteristic matrix P and sizes n, let $A_\pi := A_{\mathcal{G}/\!\!/\pi}$, and let $L := L_{\mathcal{B}(\mathcal{G},\pi)}$. Then*

$$LP = PL^\pi, \quad L^\pi := \operatorname{diag}(A_\pi n) - A_\pi \operatorname{diag}(n), \tag{4}$$

and $P^\top LP = \operatorname{diag}(n)L^\pi$ is the Laplacian of $\mathcal{G}/\!\!/\pi$ with weight $n_a n_b$ on the edge ab.

Proof. Let $u \in \mathcal{C}_a$. By definition of the blow-up, u is adjacent to the other $n_a - 1$ vertices of $\mathcal{C}_a$ and to all n_b vertices of each cell $\mathcal{C}_b$ with $[A_\pi]_{ab} = 1$. Hence $L_{uu} = n_a - 1 + \sum_c [A_\pi]_{ac} n_c$, and $L_{uw} = -1$ for every neighbor w of u. We use these entries to compute LP row by row. The entry $[LP]_{ub} = \sum_{w \in \mathcal{C}_b} L_{uw}$ sums row u of L over the cell $\mathcal{C}_b$. For $b \neq a$ this gives $-n_b[A_\pi]_{ab}$. For $b = a$ it gives $L_{uu} - (n_a - 1) = \sum_c [A_\pi]_{ac} n_c = [A_\pi n]_a$. On the other hand, row u of P is the a-th unit vector, so $[PL^\pi]_{ub} = [L^\pi]_{ab}$. Since

$[A_\pi]_{aa} = 0$, this equals $-[A_\pi]_{ab} n_b$ for $b \neq a$ and $[A_\pi n]_a$ for $b = a$. Thus LP and PL^π have the same entries, which proves (4). It remains to identify $P^\top LP$. The columns of P are the indicator vectors of disjoint cells, so $P^\top P = \operatorname{diag}(n)$ and $P^\top LP = P^\top PL^\pi = \operatorname{diag}(n)L^\pi$. Its off-diagonal entries are $-n_a n_b [A_\pi]_{ab}$, which are symmetric in a and b, and its row sums vanish because $L^\pi \mathbf{1} = A_\pi n - A_\pi n = 0$. A symmetric matrix with nonpositive off-diagonal entries and zero row sums is the Laplacian of the undirected graph whose edge weights are the negatives of its off-diagonal entries. Here these weights are $n_a n_b$ on each edge ab of $\mathcal{G}/\!\!/\pi$. □

In words, (4) states that the vectors that are constant on each cell form an invariant subspace of L, on which L acts as L^π. The matrix L^π also has zero row sums, but its off-diagonal entries are $-n_b[A_\pi]_{ab}$. It is therefore the Laplacian of a *directed* graph on the contraction, in which cluster a is attracted by cluster b with weight n_b, and it is not symmetric when the cells have different sizes. Scaling its rows by the cluster masses makes it symmetric, and $L^\pi = \operatorname{diag}(n)^{-1}\big(\operatorname{diag}(n)L^\pi\big)$ is similar to the symmetric matrix $\operatorname{diag}(n)^{-1/2}\big(\operatorname{diag}(n)L^\pi\big)\operatorname{diag}(n)^{-1/2}$. Hence its eigenvalues are real and nonnegative, and $n^\top L^\pi = 0$.

IV. A Hybrid System Model

We now explore a consensus protocol that performs neighborhood merges whenever two states are in close proximity. We term this an *aggregating consensus* protocol as in [11]. The basic idea is that after each merge, the states of the merged nodes are discretely updated to have the same value. This results in a hybrid model, which we define here. The model is inspired by that in [15] with respect to the selection of the edges as state variables and the jumping mechanism.

A. State and data

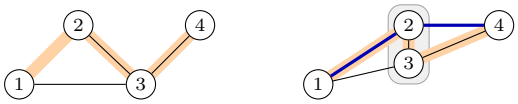
We consider $N \geq 2$ agents with states $x_i \in \mathbb{R}^\nu$, $i \in \mathbb{N}_N$, and $x = (x_1, \ldots, x_N)$. Let $M := N(N-1)/2$ be the number of unordered pairs of agents. The interaction graph is described by a logical state $e \in \{0,1\}^M$ with one component $e_{uw} = e_{wu}$ for each unordered pair of agents. Therefore, $\mathcal{G}(e)$ is the graph in which uw is an edge if and only if $e_{uw} = 1$, and $e(\mathcal{E})$ denotes the vector associated with an edge set $\mathcal{E}$. A second logical state $m \in \{0,1\}^M$ records which agents have merged: $m_{uw} = m_{wu} = 1$ if and only if agents u and w have merged with each other, either directly or through a chain of merges. The state of the hybrid system is

$$\xi := (x, e, m) \in \Xi := \mathbb{R}^{N\nu} \times \{0,1\}^M \times \{0,1\}^M. \quad (5)$$

The model has two design parameters. The first is the *merging threshold* $d > 0$ (possibly depending on the physical dimensions of the agents), which we also refer to as the docking tolerance. The second is the *docking graph* $\mathcal{G}_\mathrm{d} = (\mathbb{N}_N, \mathcal{E}_\mathrm{d})$, whose edges are the pairs of agents that are able to dock with each other.

Assumption 1. The docking graph $\mathcal{G}_\mathrm{d}$ is connected.

The two graphs play different roles, as illustrated in Fig. 3(a). The interaction graph $\mathcal{G}(e)$ describes which agents measure each other's relative states, and it changes when agents merge. The docking graph $\mathcal{G}_\mathrm{d}$ describes which pairs can physically latch, which is fixed by the hardware. A docking of i and k requires $ik \in \mathcal{E}_\mathrm{d}$, but not $ik \in \mathcal{E}$: two agents may dock without measuring each other, and may measure each other without being able to dock. Assumption 1 ensures that all agents can eventually belong to one cluster. For rendezvous, where any two agents may dock, $\mathcal{G}_\mathrm{d}$ is the complete graph.



(a) $m_0 = 0$ (b) after the docking of 2 and 3

Fig. 3. (a) Four agents with interaction graph $\mathcal{G}(e_0)$ (black) and docking graph $\mathcal{G}_\mathrm{d}$ (shaded), a path along which the agents are to be assembled. Agents 1 and 2 can dock but do not measure each other. Agents 1 and 3 measure each other but cannot dock. (b) After 2 and 3 merge, they form a cluster (box, drawn apart for readability): $m_{23} = 1$, $x_2 = x_3$, and the merge of $\{2,3\}$ adds the thick edges, so that $\mathcal{N}[2] = \mathcal{N}[3] = \{1,2,3,4\}$. The state satisfies the three conditions of E_2, and $\mathcal{G}_\mathrm{d}$ is unchanged.

Not every value of (x, e, m) describes a physically meaningful configuration. We call *admissible* the states in $\mathcal{K} := E_1 \cap E_2$, where E_1 asks that the interaction graph is connected and E_2 asks that merged agents are consistent with each other:

$$E_1 := \{\xi \in \Xi \colon \mathcal{G}(e) \text{ is connected}\}, \quad (6)$$

$$\begin{aligned} E_2 := \big\{\xi \in \Xi \colon\ & \text{for all distinct } u, w, \ell \in \mathbb{N}_N, \\ & m_{uw} = m_{w\ell} = 1 \implies m_{u\ell} = 1, \\ & m_{uw} = 1 \implies x_u = x_w \text{ and } \mathcal{N}_{\mathcal{G}(e)}[u] = \mathcal{N}_{\mathcal{G}(e)}[w]\big\}. \end{aligned} \quad (7)$$

In words, E_2 contains three requirements, all illustrated in Fig. 3(b). First, if u has merged with w and w with ℓ, then u has merged with ℓ. Second, merged agents are at the same position. Third, merged agents have the same neighbors, that is, they are true twins. The first requirement allows us to group the agents into *clusters*, the sets of agents that have merged with each other. We write $c_u(m)$ for the cluster of agent u, $n_u(m) := |c_u(m)|$ for its size, $\pi(m)$ for the partition of $\mathbb{N}_N$ into clusters, and $r(m)$ for the number of clusters. Typically, no agents have merged initially, $m_0 = 0$. Then E_2 imposes no condition, and the initial state is admissible as soon as $\mathcal{G}(e_0)$ is connected. Lemma 5 below shows that the protocol keeps the state admissible.

B. Flow

During flow, the graph topology does not change (i.e., $\dot{e} = 0$) and agents do not merge (i.e., $\dot{m} = 0$), so that the state x evolves according to the consensus dynamics,

$$\dot{\xi} = f(\xi) := \begin{bmatrix} -(L_{\mathcal{G}(e)} \otimes I_\nu)\, x \\ 0 \\ 0 \end{bmatrix}. \quad (8)$$

C. Jumps

A jump is motivated by a physical merging of agents. In other words, whenever two different agents are close, the two

agents *merge* and effectively act as a single agent. For each $ik \in \mathcal{E}_{\mathrm{d}}$, a jump is enabled in the set

$$D_{ik} := \{\xi \in \Xi \colon m_{ik} = 0,\ \|x_i - x_k\| \le d\}, \tag{9}$$

that is, when the two agents are within the docking tolerance and are not already part of the same cluster. For $\xi \in D_{ik} \cap E_2$, let $S := c_i(m) \cup c_k(m)$ be the set of agents of the two clusters involved. Merging i and k updates the state to $g^{(ik)}(\xi) := (\chi^{(ik)}(\xi), \epsilon^{(ik)}(\xi), \mu^{(ik)}(\xi))$, defined componentwise by

$$\chi_\ell^{(ik)}(\xi) := \begin{cases} \dfrac{n_i(m)\, x_i + n_k(m)\, x_k}{n_i(m) + n_k(m)} & \text{if } \ell \in S, \\ x_\ell & \text{otherwise,} \end{cases} \tag{10a}$$

$$\epsilon^{(ik)}(\xi) := e\big(\mathcal{E}^{S}_{\mathcal{G}(e)}\big), \tag{10b}$$

$$\mu_{uw}^{(ik)}(\xi) := \begin{cases} 1 & \text{if } m_{uw} = 1 \text{ or } \{u, w\} \subseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{10c}$$

The three updates are interpreted as follows. By (10a), all agents of the two clusters are moved to the center of mass of the pair of clusters, each agent counting as a unit mass. Agents outside S are not affected. This is what a latching mechanism does when it closes the residual gap between two free-floating bodies: it exerts only internal forces, so the center of mass of the pair stays where it is and the lighter cluster travels farther. In particular, every jump moves each agent by less than d: for $\xi \in D_{ik}$ and $\ell \in S$,

$$\|\chi_\ell^{(ik)}(\xi) - x_\ell\| \le \frac{\max\{n_i(m), n_k(m)\}}{n_i(m) + n_k(m)} \|x_i - x_k\| < d.$$

By (10b), the graph is updated by the merge of S of Definition 1. By (10c), the two clusters become one. The reset idealizes a docking mechanism that closes the remaining gap, shorter than d, on a time scale much faster than that of the consensus flow.

D. Hybrid system

The flow and jump sets are

$$C := \mathcal{K} \cap \big\{\xi \colon \|x_i - x_k\| \ge d \ \ \forall ik \in \mathcal{E}_{\mathrm{d}} \text{ with } m_{ik} = 0\big\}, \tag{11}$$

$$D := \mathcal{K} \cap \bigcup\nolimits_{ik \in \mathcal{E}_{\mathrm{d}}} D_{ik}, \tag{12}$$

so that $C \cup D = \mathcal{K}$, and the jump map collects all dockings that are enabled:

$$G(\xi) := \big\{g^{(ik)}(\xi) \colon ik \in \mathcal{E}_{\mathrm{d}},\ \xi \in D_{ik}\big\}, \qquad \xi \in D. \tag{13}$$

The aggregating consensus protocol is the hybrid system

$$\mathscr{H} := (C, f, D, G) \tag{14}$$

with data (8), (11), (12), (13). Instead of having all updates performed with a single jump, this modeling choice allows multiple jumps at the same ordinary time (the same choice is used in a similar context in [15]). All results below hold for every solution, that is, for every order of such jumps. On the boundary $\|x_i - x_k\| = d$ both flowing and jumping may be possible, which is the customary way of making the model insensitive to arbitrarily small measurement errors [12].

Lemma 5 (Basic properties of $\mathscr{H}$)**.** *Let Assumption 1 hold.*

(i) *C and D are closed, f is continuous, and G is outer semicontinuous and locally bounded with nonempty values on D, so that $\mathscr{H}$ satisfies the hybrid basic conditions [12, Assumption 6.5].*
(ii) *$G(D) \subset C \cup D = \mathcal{K}$.*
(iii) *From every $\xi_0 = (x_0, e_0, m_0) \in \mathcal{K}$ there exists a nontrivial solution. Every maximal solution is complete and performs at most $r(m_0) - 1 \le N - 1$ jumps.*

Proof. (i) The logical states take finitely many values and, for each of them, the conditions on x defining E_2, D_{ik} and C are equalities or non-strict inequalities, while f and $g^{(ik)}$ are linear in x. (ii) Let $\xi \in D$ and $g = g^{(ik)}(\xi)$. Then $g \in E_1$ by Lemma 1(i). By (10c) the clusters after the jump are those of m with $c_i(m)$ and $c_k(m)$ replaced by S. Two agents in S have equal states by (10a) and are true twins by Lemma 1(ii). Two merged agents outside S keep their states and remain true twins by Lemma 1(iii). Hence $g \in E_2$. (iii) At $\xi \in C \setminus D$ all inequalities in (11) are strict, and the solution of (8) keeps merged agents at equal states by Lemma 2. Hence it remains in C for a short time. By [12, Prop. 6.10], solutions exist and maximal solutions are complete, since (8) is linear and jumps remain in $C \cup D$ by (ii). A jump with $g^{(ik)}$ requires $m_{ik} = 0$ and merges two clusters into one, so each jump decreases the number of clusters by one. □

By Lemma 5(iii), every maximal solution undergoes at most $N-1$ jumps, so there is no Zeno behavior. After the last jump, at time t_J, the solution flows for all $t \ge t_J$, that is, its hybrid time domain ends with $[t_J, \infty) \times \{J\}$.

V. Network Structure and Convergence

This section contains the main results of this work. We first characterize the network generated by the aggregating consensus protocol, and then analyze its convergence.

A. Structure of the Merged Network

We first characterize the graph generated by the protocol, and then the dynamics on it. Both results hold along every solution and for every admissible initial state, including those in which some agents are already merged.

Proposition 1 (Graph generated by the protocol)**.** *Let $\phi = (x, e, m)$ be a solution of $\mathscr{H}$ from $(x_0, e_0, m_0) \in \mathcal{K}$. At every $(t, j) \in \operatorname{dom}\phi$, the interaction graph is the blow-up of the initial graph with respect to the current clusters:*

$$\mathcal{G}\big(e(t,j)\big) = \mathcal{B}\big(\mathcal{G}(e_0),\, \pi(m(t,j))\big).$$

Proof. Fix $(t, j) \in \operatorname{dom}\phi$ and let $\mathcal{G} := \mathcal{G}(e(t,j))$ and $\pi := \pi(m(t,j))$. By Lemma 5(ii), $\phi(t,j) \in C \cup D = \mathcal{K} \subset E_2$, so the agents of each cluster are true twins in $\mathcal{G}$, and $\mathcal{G} = \mathcal{B}(\mathcal{G}, \pi)$ by Lemma 3. Since a blow-up depends on the graph only through its contraction, it remains to show that $\mathcal{G}/\!\!/\pi = \mathcal{G}(e_0)/\!\!/\pi$, which we do by induction on j. For $j = 0$ it holds because $e(t, 0) = e_0$, and e and m do not change during flows. At a jump that merges $S = c_i \cup c_k$, the partition π becomes π^+, in which the two cells are replaced by S, and

$\mathcal{G}$ becomes $\mathcal{G}^S$. The edges added by the merge join vertices of S, which lie in the same cell of π^+, or a vertex of S to a vertex outside S that was already adjacent to some vertex of S. Hence $\mathcal{G}^S/\!\!/\pi^+ = \mathcal{G}/\!\!/\pi^+$. Since π^+ is coarser than π, $\mathcal{G}/\!\!/\pi^+$ is determined by $\mathcal{G}/\!\!/\pi$, which equals $\mathcal{G}(e_0)/\!\!/\pi$ by the induction hypothesis, and therefore $\mathcal{G}^S/\!\!/\pi^+ = \mathcal{G}(e_0)/\!\!/\pi^+$. □

In words, at any time the agents of each cluster form a clique, and two agents in different clusters are adjacent exactly when their clusters were joined by at least one edge of the initial graph. Equivalently, the current graph is obtained by contracting each cluster of the initial graph to a single node, and then replacing each node by a clique of the agents of its cluster, connected to all agents of the neighboring clusters. It therefore depends only on the initial graph and the current clusters, and not on the order in which the merges occurred.

Theorem 1 (Reduced dynamics)**.** *Let $\phi = (x, e, m)$ be a solution of $\mathscr{H}$ from $(x_0, e_0, m_0) \in \mathcal{K}$, fix $(t, j) \in \operatorname{dom}\phi$, and let $\pi = \{\mathcal{C}_1, \dots, \mathcal{C}_r\} := \pi(m(t,j))$, with characteristic matrix P and sizes $n = (n_1, \dots, n_r)$. Then $x(t,j) = (P \otimes I_\nu) z(t,j)$, where $z = (z_1, \dots, z_r)$ collects the positions of the clusters, and $z(\cdot, j)$ satisfies, on the flow interval $\{s\colon (s,j) \in \operatorname{dom}\phi\}$,*

$$n_a \dot{z}_a = \sum_{b\,:\, ab \in \mathcal{E}_{\mathcal{G}(e_0)/\!\!/\pi}} n_a n_b\,(z_b - z_a), \qquad a \in \mathbb{N}_r. \tag{15}$$

Proof. By Lemma 5(ii), $\phi(t,j) \in \mathcal{K} \subset E_2$, so that $x_u(t,j) = x_w(t,j)$ whenever $m_{uw}(t,j) = 1$, which is the first claim. By Proposition 1 and Lemma 4, $L_{\mathcal{G}(e(t,j))} P = P L^\pi$, with L^π defined from $\mathcal{G}(e_0)/\!\!/\pi$. Hence, if z solves $\dot{z} = -(L^\pi \otimes I_\nu) z$, then $(P \otimes I_\nu) z$ solves (8). Multiplying the a-th component by n_a gives (15). □

Proposition 1 and Theorem 1 have several consequences. First, the clusters interact as the vertices of the contracted graph $\mathcal{G}(e_0)/\!\!/\pi$, see Fig. 2(c), but *not* through the unweighted consensus protocol over it: a cluster of n_a agents has inertia n_a and is attracted by a neighboring cluster of n_b agents through $n_a n_b$ unit links, which is what preserves the centroid of the physical agents. Second, the logical state e is a function of e_0 and m, and an implementation does not need to store it. Third, the protocol needs no sensing links beyond those of $\mathcal{G}(e_0)$. By (15), every agent ℓ of cluster a applies the input

$$u_\ell = \sum_{b\,:\, ab \in \mathcal{E}_{\mathcal{G}(e_0)/\!\!/\pi}} n_b\,(z_b - z_a), \tag{16}$$

and $z_b - z_a$ is measured by any original link between the two clusters. It therefore suffices that the agents of a cluster share their measurements, and that each agent knows the size of the cluster at the other end of its links. The edges added by the merge describe this sharing, not new sensing.

B. Stability and Convergence Analysis

We now provide a stability and convergence analysis for the hybrid system representation of the aggregating consensus protocol. We aim at proving uniform global asymptotic stability of the attractor

$$\mathcal{A} := \{\xi \in \Xi\colon x_1 = \cdots = x_N,\ e = \mathbf{1},\ m = \mathbf{1}\}. \tag{17}$$

Thus, $(x, e, m) \in \mathcal{A}$ corresponds to all the states in agreement, all agents merged, and a complete underlying graph. The attractor $\mathcal{A} = \mathcal{A}_x \times \{\mathbf{1}\} \times \{\mathbf{1}\}$ satisfies $\|\xi\|_{\mathcal{A}}^2 = \|x\|_{\mathcal{A}_x}^2 + \|e - \mathbf{1}\|^2 + \|m - \mathbf{1}\|^2$. It is closed but not compact, since $\mathcal{A}_x$ contains the consensus states at every location. For this reason, we establish UGAS directly from its definition, through uniform global stability and uniform global attractivity, as in Theorem 2(iv). The analysis is based on the disagreement function

$$W(\xi) := \tfrac{1}{2}\|x\|_{\mathcal{A}_x}^2 = \tfrac{1}{2}\sum_{i=1}^{N} \|x_i - \bar{x}\|^2. \tag{18}$$

We now state the main result.

Theorem 2 (Aggregating consensus)**.** *Let Assumption 1 hold, let $\phi = (x, e, m)$ be a maximal solution of $\mathscr{H}$ in* (14) *with $\phi(0,0) = (x_0, e_0, m_0) \in \mathcal{K}$, and let $\lambda_0 := \lambda_2(\mathcal{G}(e_0)) > 0$. Then:*

(i) Centroid invariance: $\bar{x}(t,j) = \bar{x}_0$ *for all* $(t,j) \in \operatorname{dom}\phi$.

(ii) Monotone decay: *during flows, $\dot{W} \le -2\lambda_2(\mathcal{G}(e))\, W$, and at a jump with $g^{(ik)}$,*

$$W(\xi^+) - W(\xi) = -\frac{1}{2}\,\frac{n_i n_k}{n_i + n_k}\,\|x_i - x_k\|^2 \le 0, \tag{19}$$

with $n_i = n_i(m)$ and $n_k = n_k(m)$. Moreover, $\lambda_2(\mathcal{G}(e))$ is non-decreasing along ϕ. Consequently,

$$\|x(t,j)\|_{\mathcal{A}_x} \le e^{-\lambda_0 t}\|x_0\|_{\mathcal{A}_x} \qquad \forall (t,j) \in \operatorname{dom}\phi. \tag{20}$$

(iii) Finite-time aggregation: *let $J \le N - 1$ be the number of jumps of ϕ and*

$$T^\star := \max\Big\{0,\ \frac{1}{\lambda_0}\ln\frac{\sqrt{2}\,\|x_0\|_{\mathcal{A}_x}}{d}\Big\} \tag{21}$$

(with $T^\star := 0$ if $\|x_0\|_{\mathcal{A}_x} = 0$). Then $t_J \le T^\star$, and for all $t \ge t_J$,

$$x_i(t,J) = \bar{x}_0 \ \ \forall i \in \mathbb{N}_N, \quad e(t,J) = \mathbf{1}, \quad m(t,J) = \mathbf{1}.$$

(iv) *The set $\mathcal{A}$ in* (17) *is UGAS for $\mathscr{H}$.*

Proof. (i) During flows, $\sum_i \dot{x}_i = -(\mathbf{1}^\top L_{\mathcal{G}(e)} \otimes I_\nu) x = 0$. At a jump with $g^{(ik)}$, only the agents in $S = c_i(m) \cup c_k(m)$ change state. Before the jump the n_i agents of $c_i(m)$ are at x_i and the n_k agents of $c_k(m)$ are at x_k, because $\xi \in E_2$. After the jump, all $n_i + n_k$ agents are at $\chi := (n_i x_i + n_k x_k)/(n_i + n_k)$. Hence $\sum_{\ell \in S} x_\ell = n_i x_i + n_k x_k$ is unchanged.

(ii) By (i), $\bar{x}$ is constant along flows, and with $\tilde{x} := x - \mathbf{1} \otimes \bar{x}$ and $L = L_{\mathcal{G}(e)}$,

$$\dot{W} = -\tilde{x}^\top (L \otimes I_\nu) x = -\tilde{x}^\top (L \otimes I_\nu)\tilde{x} \le -\lambda_2(\mathcal{G}(e))\|\tilde{x}\|^2$$

by $L\mathbf{1} = 0$ and (2), which is the flow inequality in view of (1). At a jump, by (i) and with $y_i := x_i - \bar{x}$, $y_k := x_k - \bar{x}$,

$$\begin{aligned} 2\big(W(\xi^+) - W(\xi)\big) &= \frac{\|n_i y_i + n_k y_k\|^2}{n_i + n_k} - n_i\|y_i\|^2 - n_k\|y_k\|^2 \\ &= -\frac{n_i n_k}{n_i + n_k}\|y_i - y_k\|^2, \end{aligned}$$

which is (19). The graph changes only at jumps, where edges are added (Lemma 1(i)), so $\lambda_2(\mathcal{G}(e))$ is non-decreasing along ϕ and bounded below by λ_0. Therefore

$W(\phi(t,j)) \le e^{-2\lambda_0(t-t_j)}W(\phi(t_j,j))$ on each interval of flow, and $W(\phi(t_{j+1},j+1)) \le W(\phi(t_{j+1},j))$ at each jump. Concatenating these inequalities gives $W(\phi(t,j)) \le e^{-2\lambda_0 t}W(\phi(0,0))$, which is (20).

(iii) We first show that

$$(t,j) \in \operatorname{dom}\phi,\ t > T^\star \implies \|x(t,j)\|_{\mathcal{A}_x} < d/\sqrt{2}. \quad (22)$$

This is trivial if $\|x_0\|_{\mathcal{A}_x} = 0$. Otherwise, by (20) and (21), $\|x(t,j)\|_{\mathcal{A}_x} \le e^{-\lambda_0 t}\|x_0\|_{\mathcal{A}_x} < e^{-\lambda_0 T^\star}\|x_0\|_{\mathcal{A}_x} \le d/\sqrt{2}$. Next, for any x and any i,k,

$$\|x_i - x_k\| \le \|x_i - \bar{x}\| + \|x_k - \bar{x}\| \le \sqrt{2}\,\|x\|_{\mathcal{A}_x}. \quad (23)$$

Suppose that $\xi = (x,e,m) \in \mathcal{K}$ satisfies $\|x\|_{\mathcal{A}_x} < d/\sqrt{2}$ and $m \neq \mathbf{1}$. Then there are at least two clusters and, since $\mathcal{G}_{\mathrm{d}}$ is connected, there is an edge $ik \in \mathcal{E}_{\mathrm{d}}$ joining two different clusters, so $m_{ik} = 0$, while $\|x_i - x_k\| < d$ by (23). Hence

$$\xi \in \mathcal{K},\ \ \|x\|_{\mathcal{A}_x} < d/\sqrt{2},\ \ m \neq \mathbf{1} \implies \xi \notin C. \quad (24)$$

We now show that $t_J \le T^\star$. If $J = 0$ there is nothing to prove. Let $J \ge 1$ and suppose $t_J > T^\star$. Just before the J-th jump, $m \neq \mathbf{1}$, because a jump requires an unmerged pair, and m is constant on the interval of flow $[t_{J-1}, t_J] \times \{J-1\}$. If $t_{J-1} < t_J$, then for $t \in (\max\{t_{J-1}, T^\star\}, t_J)$ the point $\phi(t, J-1)$ belongs to C, being in the interior of an interval of flow, and at the same time violates C by (22) and (24). Hence $t_{J-1} = t_J > T^\star$. Repeating the argument for the jumps $J-1, J-2, \dots, 1$ yields $0 = t_0 = t_J > T^\star \ge 0$, a contradiction.

On the last interval $[t_J, \infty) \times \{J\}$, m is constant, and for $t > \max\{t_J, T^\star\}$ the point $\phi(t,J)$ is in C and satisfies (22). By (24), $m = \mathbf{1}$ on this interval. Since $\phi(t,J) \in E_2$, all agents then have the same state, equal to $\bar{x}_0$ by (i), and any two agents are true twins, hence adjacent, so $e = \mathbf{1}$.

(iv) Let $s := \|\phi(0,0)\|_{\mathcal{A}}$. If $m_0 = \mathbf{1}$ and $e_0 = \mathbf{1}$, then $x_0 \in \mathcal{A}_x$ and ϕ remains in $\mathcal{A}$. Otherwise $s \ge 1$ and, by (20), $\|\phi(t,j)\|_{\mathcal{A}}^2 \le \|x_0\|_{\mathcal{A}_x}^2 + N(N-1) \le (1+N(N-1))s^2$, which is uniform global stability. For uniform global attractivity, fix $\rho > 0$ and let $\tau := N + \max\{0, \underline{\lambda}_N^{-1}\ln(\sqrt{2}\rho/d)\}$, where $\underline{\lambda}_N := 2(1-\cos(\pi/N))$ is the algebraic connectivity of the path, the smallest among connected graphs on N vertices [21]. If $s \le \rho$ and $t + j \ge \tau$, then $t \ge \tau - (N-1) > T^\star \ge t_J$, so $j = J$ and $\phi(t,j) \in \mathcal{A}$ by (iii). □

Remark 1. The conclusions of Theorem 2 hold for every solution, and hence do not depend on which agents merge, or in which order. This is a consequence of Theorem 1: whatever the sequence of merges, the clusters form a consensus network with masses n_a and symmetric weights $n_a n_b$, for which the centroid is invariant and $W = \frac{1}{2}\sum_a n_a\|z_a - \bar{x}\|^2$. The weighted average in (10a) is the only reset of two clusters to a common point that preserves the centroid, and the decrease (19) has the form of the kinetic energy lost in a perfectly inelastic collision of bodies with masses n_i and n_k.

Remark 2. The rate $\lambda_2(\mathcal{G}(e))$ reaches N when all agents have merged, whereas (21) uses only its initial value. The proof of Theorem 2(ii) in fact gives the a posteriori bound $\|x(t,j)\|_{\mathcal{A}_x} \le e^{-\int_0^t \lambda_2(s)\,ds}\|x_0\|_{\mathcal{A}_x}$, where $\lambda_2(s) := \lambda_2(\mathcal{G}(e(s,j)))$ for $(s,j) \in \operatorname{dom}\phi$, with any choice of j at the finitely many jump times, so $T^\star$ is conservative in general. It cannot be improved without further information: for $N = 2$ the agents merge exactly at $t = T^\star$.

Remark 3 (Why the graph is updated at jumps). A simpler alternative would leave the graph unchanged at jumps and let each cluster move rigidly with the average of the velocities commanded by its members. Its clusters then follow (15) with the weight $n_a n_b$ replaced by the number $k_{ab} \le n_a n_b$ of links of $\mathcal{G}(e_0)$ between the two clusters. Since this flow also keeps the centroid and satisfies $\dot{W} \le -2\lambda_2(\mathcal{G}(e_0))W$, the proof of Theorem 2 still gives aggregation in finite time, but more slowly: when two clusters remain, their distance decays at the rate $k_{ab}N/(n_a n_b)$ instead of N.

VI. Formations, Collision Avoidance, and Leaders

An assembly task adds physical requirements to the point model $\mathscr{H}$: a prescribed shape, bodies that do not collide, and a prescribed location known to only a few agents. Each is met by a modification of the flow (8), or of the jump set, that preserves the structure of Section V and its guarantees.

A. Formations

Consider N modules with positions $p_i \in \mathbb{R}^\nu$, $\nu \in \{2,3\}$, that keep a common orientation, and let $\delta = (\delta_1, \dots, \delta_N)$ with pairwise distinct $\delta_i \in \mathbb{R}^\nu$ specify the position of each module in the structure to be built, up to a translation. The structure is assembled when $p_i - p_k = \delta_i - \delta_k$ for all i,k. The docking graph $\mathcal{G}_{\mathrm{d}}$ now contains the pairs of modules that are adjacent in the assembled structure and can be joined by their docking mechanisms. Assumption 1 requires that the structure be in one piece.

Between docking events the modules run the displacement-based formation protocol [4] over the current graph,

$$\dot{p}_i = \sum_{k \in \mathcal{N}_{\mathcal{G}(e)}(i)} \big[(p_k - p_i) - (\delta_k - \delta_i)\big]. \quad (25)$$

Modules i and k, $ik \in \mathcal{E}_{\mathrm{d}}$, dock when their relative position is within the capture tolerance of the desired one,

$$\|\rho_{ik}\| \le d, \qquad \rho_{ik} := (p_k - p_i) - (\delta_k - \delta_i), \quad (26)$$

and they are not yet part of the same partial assembly. The docking mechanism then closes the misalignment ρ_{ik} by translating the two partial assemblies rigidly, about their common center of mass:

$$p_\ell^+ = \begin{cases} p_\ell + \dfrac{n_k}{n_i + n_k}\,\rho_{ik} & \ell \in c_i(m), \\ p_\ell - \dfrac{n_i}{n_i + n_k}\,\rho_{ik} & \ell \in c_k(m), \\ p_\ell & \text{otherwise}, \end{cases} \quad (27)$$

while the graph and the merge state are updated by (10b), (10c). In the coordinates $x_i := p_i - \delta_i$, (25) is the flow (8), (26) is the condition $\|x_i - x_k\| \le d$ in (9), and (27) is the reset (10a). The assembly process is therefore the hybrid system $\mathscr{H}$, and Theorem 2 and Theorem 1 translate as follows.

Corollary 1 (Finite-time self-assembly)**.** *Let Assumption 1 hold, and consider the assembly protocol* (25)–(27) *from*

(p_0, e_0, m_0) *with* $(p_0 - \delta, e_0, m_0) \in \mathcal{K}$. *Then every maximal solution satisfies the following.*

(i) *Once modules* u *and* w *have merged,* $p_u - p_w = \delta_u - \delta_w$ *at all later times.*

(ii) *The last docking occurs at* $t_J \le T^\star$, *with* $T^\star$ *in* (21) *evaluated at* $x_0 = p_0 - \delta$, *and* $p_i(t, J) = \bar{p}_0 + \delta_i - \bar{\delta}$ *for all* $i \in \mathbb{N}_N$ *and all* $t \ge t_J$.

Proof. In the coordinates $x = p - \delta$, item (i) holds because solutions remain in E_2, so that merged modules have equal x. Item (ii) is Theorem 2(iii), with the final position given by the invariant centroid of Theorem 2(i). □

Rendezvous is the special case $\delta = 0$ with $\mathcal{G}_\mathrm{d}$ the complete graph. The other properties of Section V carry over as well. The centroid $\bar{p}$ is invariant, so the location of the completed structure is known in advance from the initial positions. Merged modules are true twins and have equal velocities by Lemma 2, so partial assemblies translate rigidly. Each docking event displaces a module by less than d, by the bound given after (10). In contrast with the formation protocol (25) alone, which reaches the desired shape only asymptotically, the structure is completed exactly and in finite time.

B. Collision avoidance

The results so far do not prevent the bodies of two agents from overlapping while they converge to their positions in the formation. We now show that collisions can be excluded without changing any of the conclusions of Theorem 2, provided that the agents start from a configuration that is *untangled* in a sense made precise below.

Assume that the body of agent u is contained in the closed ball of diameter $\varrho > 0$ centered at p_u. Two agents $u \neq w$ then do not collide if $\|p_u - p_w\| > \varrho$. This constraint is not convex, and the standard remedies, artificial potentials or barrier certificates [22] that repel agents along their line of sight, can oppose the consensus protocol, increase the disagreement, and create deadlocks [23]. We therefore replace it by a stronger but convex constraint, based on a separating direction fixed in advance, which is what makes it compatible with the convergence analysis. To each pair of agents we assign a fixed unit vector $a_{uw} = -a_{wu} \in \mathbb{R}^\nu$, termed the *separating axis* of the pair, and define, with $x = p - \delta$ as in Section VI-A,

$$h_{uw}(x) := a_{uw}^\top (p_u - p_w) - \varrho = a_{uw}^\top (x_u - x_w) + c_{uw}, \quad (28)$$

where $c_{uw} := a_{uw}^\top (\delta_u - \delta_w) - \varrho$. If $h_{uw}(x) > 0$, the two bodies lie on opposite sides of a hyperplane normal to a_{uw}, at a distance $h_{uw}(x)$ along the axis, and in particular $\|p_u - p_w\| > \varrho$ (see Fig. 4(a)). The constant c_{uw} is the *clearance* of the pair along its axis once the formation is completed. The set

$$\Omega(m) := \{x \colon h_{uw}(x) > 0 \ \ \forall u, w \text{ with } m_{uw} = 0\} \quad (29)$$

of collision-free configurations certified by the axes is an open convex polyhedral set.

Assumption 2 (Untangled configuration)**.** There exist $\eta > 0$ and separating axes such that, for all u, w with $m_{0,uw} = 0$, $h_{uw}(x_0) > 0$ and $c_{uw} \ge d + \eta$, and $\|\delta_u - \delta_w\| > \varrho$ for all $u \neq w$ with $m_{0,uw} = 1$.

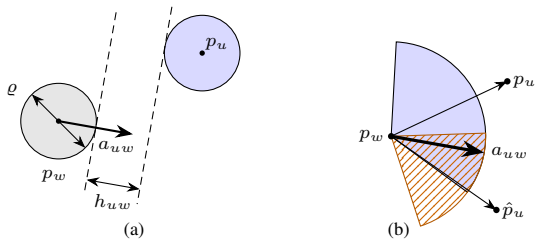


Fig. 4. (a) The separating axis a_{uw} of two agents with bodies of diameter ϱ: if $h_{uw} > 0$, the bodies are separated by a strip of width h_{uw} normal to the axis. (b) The axis must separate the pair both in the initial configuration and in the formation, drawn relative to p_w, with $\hat{p}_u := p_w + \delta_u - \delta_w$ the position of u relative to w in the formation. The shaded sector contains the unit vectors a with $a^\top (p_u - p_w) > \varrho$, and the hatched sector those with $a^\top (\delta_u - \delta_w) \ge \varrho + d + \eta$. The pair is untangled if the two sectors intersect.

Assumption 2 asks that each pair can be separated by parallel hyperplanes of one and the same direction both in the initial configuration and in the formation (see Fig. 4(b)). It excludes pairs that would have to exchange sides in order to reach their positions. It also requires $\|\delta_u - \delta_w\| \ge \varrho + d + \eta$, so that in the formation any two bodies are at least $d + \eta$ apart. Docked agents are therefore not in contact, and are held together by their docking mechanisms. This margin ensures that, once all relative positions are within d of their values in the formation, the condition in (32) cannot block the remaining merges. The next result shows that the assignment of the positions in the formation by a minimum-cost matching [24] produces untangled configurations whenever the agents are initially sufficiently spread out.

Proposition 2 (Matching untangles)**.** *Let* $m_0 = 0$, $\underline{\delta} := \min_{u \neq w} \|\delta_u - \delta_w\| > \varrho + d + \eta$, *and suppose that the positions in the formation are assigned to the agents so as to minimize* $\sum_u \|p_u(0) - \delta_u\|^2$ *over all assignments. If*

$$\|p_u(0) - p_w(0)\| > \frac{\varrho}{\sqrt{1 - \big((\varrho + d + \eta)/\underline{\delta}\big)^2}} \quad \forall u \neq w, \quad (30)$$

then Assumption 2 holds.

Proof. Fix $u \neq w$ and let $\alpha := p_u(0) - p_w(0)$, $\beta := \delta_u - \delta_w$. Exchanging the positions assigned to u and w does not decrease the cost, that is, $\|p_u(0) - \delta_u\|^2 + \|p_w(0) - \delta_w\|^2 \le \|p_u(0) - \delta_w\|^2 + \|p_w(0) - \delta_u\|^2$, which is equivalent to $\alpha^\top \beta \ge 0$: the angle between α and β is at most $\pi/2$. In the plane spanned by α and β, the unit vectors a with $a^\top \alpha > \varrho$ are those at an angle less than $\varphi_\alpha := \arccos(\varrho/\|\alpha\|)$ from α, and those with $a^\top \beta \ge \varrho + d + \eta$ are at an angle of at most $\varphi_\beta := \arccos((\varrho + d + \eta)/\|\beta\|)$ from β. A common element exists if $\varphi_\alpha + \varphi_\beta > \pi/2$, that is, if $\cos \varphi_\alpha < \sin \varphi_\beta$, which is implied by (30) since $\|\beta\| \ge \underline{\delta}$. □

We now design a protocol that keeps each pair separated along its axis. For each pair let $b_{uw} = b_{wu} \colon (0, \infty) \to [0, \infty)$ be a continuously differentiable, convex and non-increasing *barrier function* such that $b_{uw}(h) \to \infty$ as $h \to 0$ and $b_{uw}(h) = 0$ for $h \ge \bar{h}_{uw}$, where the activation width satisfies

$0 < \bar{h}_{uw} \le c_{uw}$. An example is $b_{uw}(h) = k(\bar{h}_{uw} - h)^2/h$ for $h < \bar{h}_{uw}$, with $k > 0$. Let $\beta_{uw}(x) := -b'_{uw}(h_{uw}(x)) \ge 0$. The flow (8) of agent u is modified to

$$\dot{x}_u = -\big[(L_{\mathcal{G}(e)} \otimes I_\nu)x\big]_u + \frac{1}{n_u(m)} \sum_{\substack{u' \in c_u(m) \\ w \notin c_u(m)}} \beta_{u'w}(x)\, a_{u'w}. \quad (31)$$

In other words, two agents of different clusters repel each other along their separating axis, with equal and opposite terms, when their separation along the axis is smaller than $\bar{h}_{uw}$. Moreover, in the same way as the neighborhood merge shares the links of a cluster among its members, all agents of a cluster apply the average of the repulsion terms of its members, so that a cluster still translates rigidly. Since $\bar{h}_{uw} \le c_{uw}$, a repulsion term is active only if $a_{uw}^\top (x_u - x_w) < 0$, that is, only if the pair is closer along its axis than it will be in the formation.

A jump can itself cause a collision, since it displaces the agents of two clusters. A jump with $g^{(ik)}$ is therefore enabled only if the state after the jump leaves a margin η to all constraints. With $\mu := \mu^{(ik)}(\xi)$ and $\chi := \chi^{(ik)}(\xi)$ in (10), this jump is enabled in

$$D^{\mathrm{s}}_{ik} := D_{ik} \cap \big\{\xi \colon h_{uw}(\chi) \ge \eta \ \ \forall u, w \text{ with } \mu_{uw} = 0\big\}. \quad (32)$$

Let $\mathcal{K}^{\mathrm{s}} := \{\xi \in \mathcal{K} \colon x \in \Omega(m)\}$. The jump set is $D^{\mathrm{s}} := \mathcal{K}^{\mathrm{s}} \cap \bigcup_{ik \in \mathcal{E}_{\mathrm{d}}} D^{\mathrm{s}}_{ik}$, the flow set C^{s} consists of the $\xi \in \mathcal{K}^{\mathrm{s}}$ such that, for every $ik \in \mathcal{E}_{\mathrm{d}}$ with $m_{ik} = 0$, either $\|x_i - x_k\| \ge d$ or $h_{uw}(\chi^{(ik)}(\xi)) \le \eta$ for some u, w with $\mu^{(ik)}_{uw}(\xi) = 0$, and the jump map is (13) with D^{s}_{ik} in place of D_{ik}. We denote by $\mathscr{H}^{\mathrm{s}}$ the hybrid system with these data and the flow (31).

Theorem 3 (Collision-free aggregating consensus)**.** *Let Assumptions 1 and 2 hold, and let $\phi = (x, e, m)$ be a maximal solution of $\mathscr{H}^{\mathrm{s}}$ with $\phi(0,0) = (x_0, e_0, m_0) \in \mathcal{K}^{\mathrm{s}}$. Then ϕ is complete, has $J \le N-1$ jumps, and:*

(i) No collisions: $h_{uw}(x(t,j)) > 0$ *for all* $(t,j) \in \operatorname{dom}\phi$ *and all* u, w *with* $m_{uw}(t,j) = 0$. *Consequently,* $\|p_u(t,j) - p_w(t,j)\| > \varrho$ *for all* $u \ne w$ *and all* $(t,j) \in \operatorname{dom}\phi$.

(ii) *Statements (i)–(iii) of Theorem 2 hold, with the same time* $T^\star$ *in* (21).

Proof. Let $B(\xi) := \sum b_{uw}(h_{uw}(x))$, the sum being over the unordered pairs with $m_{uw} = 0$, and let $\Pi := P(P^\top P)^{-1}P^\top$ be the orthogonal projector onto the vectors that are constant on the clusters, with P the characteristic matrix of $\pi(m)$. Then (31) reads $\dot{x} = -(L \otimes I_\nu)x - (\Pi \otimes I_\nu)\nabla_x B$, with $L = L_{\mathcal{G}(e)}$. On $\mathcal{K}$, $(\Pi \otimes I_\nu)x = x$ and, by Lemma 2, $(\Pi \otimes I_\nu)(L \otimes I_\nu)x = (L \otimes I_\nu)x$. Hence merged agents keep equal velocities, and flows remain in $\mathcal{K}$.

We first prove (i). Along flows, $F := \frac{1}{2}x^\top(L \otimes I_\nu)x + B$ satisfies $\dot{F} = -\|(L \otimes I_\nu)x + (\Pi \otimes I_\nu)\nabla_x B\|^2 \le 0$. Hence B remains bounded on each interval of flow, and every h_{uw} with $m_{uw} = 0$ stays bounded away from zero. After a jump, $h_{uw} \ge \eta$ for all pairs that are not merged, by (32). This proves the first claim in (i). The second follows from $\|p_u - p_w\| \ge a_{uw}^\top(p_u - p_w) = h_{uw}(x) + \varrho$ for pairs that are not merged. For merged pairs, $p_u - p_w = \delta_u - \delta_w$, whose norm is at least $c_{uw} + \varrho \ge \varrho + d + \eta$ if the pair merged during the solution, and exceeds ϱ by Assumption 2 if it was merged initially.

We next prove (ii). The repulsion terms of a pair are equal and opposite and $\mathbf{1}^\top\Pi = \mathbf{1}^\top$, so $\bar{x}$ is constant during flows, and jumps preserve it as in Theorem 2(i). With $\tilde{x} := x - \mathbf{1} \otimes \bar{x}$, which satisfies $(\Pi \otimes I_\nu)\tilde{x} = \tilde{x}$,

$$\dot{W} = -\tilde{x}^\top(L \otimes I_\nu)\tilde{x} + \sum \beta_{uw}\, a_{uw}^\top(x_u - x_w) \le -2\lambda_2(\mathcal{G}(e))W,$$

because $\beta_{uw} > 0$ only if $a_{uw}^\top(x_u - x_w) < 0$, as noted after (31). Thus (19) and (20) hold unchanged. For Theorem 2(iii), it suffices to show that (24) holds with C^{s} in place of C. Let $\xi \in \mathcal{K}^{\mathrm{s}}$ with $\|x\|_{\mathcal{A}_x} < d/\sqrt{2}$ and $m \ne \mathbf{1}$, and let $ik \in \mathcal{E}_{\mathrm{d}}$ be the pair found in the proof of Theorem 2(iii). By (23), all pairwise distances between the states are smaller than d, and so are those of $\chi = \chi^{(ik)}(\xi)$, whose components are convex combinations of those of x. Hence $h_{uw}(\chi) > c_{uw} - d \ge \eta$ for all u, w, the jump with $g^{(ik)}$ is enabled by (32), and $\xi \notin C^{\mathrm{s}}$.

Finally, $\mathcal{K}^{\mathrm{s}}$ is not closed and the flow map is singular on its boundary, so the hybrid basic conditions of Lemma 5(i) do not carry over to $\mathscr{H}^{\mathrm{s}}$, and we prove existence, completeness and $J \le N-1$ directly. Every point of $\mathcal{K}^{\mathrm{s}}$ lies in $C^{\mathrm{s}} \cup D^{\mathrm{s}}$, because a point outside C^{s} has a pair within d whose guard holds. At a point of $C^{\mathrm{s}} \setminus D^{\mathrm{s}}$, since $\xi \notin D^{\mathrm{s}}_{ik}$, every $ik \in \mathcal{E}_{\mathrm{d}}$ with $m_{ik} = 0$ satisfies $\|x_i - x_k\| > d$ or $h_{uw}(\chi^{(ik)}(\xi)) < \eta$ for some u, w with $\mu^{(ik)}_{uw}(\xi) = 0$. A pair at distance exactly d whose jump violates the guard thus satisfies the second, strict, condition. These conditions are open and the flow map is continuous on $\Omega(m)$, so a solution of (31) exists and remains in C^{s} for a short time. Along flows, $B \le F$ is bounded by its value at the start of the interval, so each h_{uw} with $m_{uw} = 0$ is bounded below by a positive constant, and x is bounded because $\bar{x}$ is constant and W is non-increasing. Flows thus remain in a compact subset of $\Omega(m)$ and cannot escape in finite time. Jumps map D^{s} into $\mathcal{K}^{\mathrm{s}}$ by Lemma 5(ii) and (32), and each jump merges two clusters. □

Theorem 3 shows that, for untangled configurations, collision avoidance comes at no cost in the guarantees. The reason is that the repulsion acts along a fixed axis and only on one side of the position that the pair has in the formation, so that it never opposes the decrease of the disagreement and cannot create a deadlock. Combined with Proposition 2, the theorem provides a collision-free assembly protocol from every initial configuration that is sufficiently spread out.

Remark 4. A pair of agents that must exchange sides to reach its positions in the formation admits no separating axis satisfying Assumption 2. For such a pair the axis has to rotate while the agents go around each other. Re-selecting the axis does not affect the safety argument, as long as $h_{uw} > 0$ for the new axis, but the sign condition in the expression of $\dot{W}$, and with it the guarantee of progress, is lost while the axis is not compatible with the formation. The construction applies also in the setting of Section VI-C, with W_0 in place of W. Collisions with the leader, when it is a physical object, are not covered.

C. Leader–follower networks

So far the agents meet, or the formation is completed, wherever their centroid happens to be. Many tasks prescribe the location instead, and only a few agents know it. This is captured by a *leader*: an additional node 0 at a fixed position $q \in \mathbb{R}^\nu$, of infinite mass. All graphs are now on the vertex set $\{0\} \cup \mathbb{N}_N$, e and m have one component for each pair of this set, and $\mathcal{G}_\mathrm{d}$ is a graph on it. The agents with a link to the leader are *informed*: they measure $q - x_i$. The leader and the agents merged with it do not move, and the other agents run the consensus protocol over $\mathcal{G}(e)$, in which the leader takes part as a neighbor of the informed agents. With $x_0 := q$ and $\dot{q} = 0$,

$$\dot{x}_i = (1 - m_{0i}) \sum_{k \in \mathcal{N}_{\mathcal{G}(e)}(i)} (x_k - x_i), \qquad i \in \mathbb{N}_N. \tag{33}$$

Jumps are those of Section IV, except that a merge whose set S contains the leader resets all agents of S to q instead of the center of mass. We denote the resulting hybrid system by $\hat{\mathscr{H}}$ and its admissible set by $\hat{\mathcal{K}}$. In the coordinates of Section VI-A, $x_i = q$ means that agent i is at its prescribed position $q + \delta_i$. The leader may be a goal location for the formation or, as we will see in Section VII, a stationary object with attachment points $q + \delta_i$.

The merge of Definition 1 now also spreads information: a merged cluster is adjacent to every neighbor of any of its members, including the leader, so that an agent that merges with an informed agent becomes informed. Knowledge of the position of the leader thus propagates through the network by docking.

The analysis is that of Theorem 2, with the leader in place of the centroid. For $\xi \in \hat{\mathcal{K}}$, let $\mathcal{F}(\xi) \subseteq \mathbb{N}_N$ be the set of agents that have not merged with the leader, and let $\Gamma(\xi) := [L_{\mathcal{G}(e)}]_{\mathcal{F}\mathcal{F}}$ be the principal submatrix of $L_{\mathcal{G}(e)}$ indexed by $\mathcal{F}(\xi)$, that is, the Laplacian of $\mathcal{G}(e)$ grounded at the leader and at the agents merged with it. Let $\gamma(\xi) := \lambda_{\min}(\Gamma(\xi))$, which is positive because $\mathcal{G}(e)$ is connected [2], with $\gamma := +\infty$ when all agents have merged with the leader. Let $y := (x_1 - q, \dots, x_N - q)$ and $W_0 := \frac{1}{2}\|y\|^2$.

Corollary 2 (Leader–follower aggregating consensus)**.** *Let $\mathcal{G}_\mathrm{d}$ be connected, let $\phi = (x, e, m)$ be a maximal solution of $\hat{\mathscr{H}}$ with $\phi(0,0) \in \hat{\mathcal{K}}$, and let $\gamma_0 := \gamma(\phi(0,0))$. Then ϕ is complete, has at most N jumps, and satisfies the following.*

(i) Monotone decay: *during flows, $\dot{W}_0 \le -2\gamma W_0$, W_0 does not increase at jumps, and γ is non-decreasing along ϕ. Consequently,*

$$\|y(t,j)\| \le e^{-\gamma_0 t}\|y(0,0)\| \qquad \forall (t,j) \in \operatorname{dom}\phi.$$

(ii) Finite-time aggregation at the leader: *let*

$$T_0^\star := \max\Big\{0,\ \frac{1}{\gamma_0} \ln \frac{\sqrt{2}\,\|y(0,0)\|}{d}\Big\}.$$

Then $t_J \le T_0^\star$, and $x_i(t,J) = q$ for all $i \in \mathbb{N}_N$ and all $t \ge t_J$.

Proof. Lemma 5 holds with the same proof on $N+1$ nodes, which gives at most N jumps. Since the agents merged with the leader are at q, (33) gives $\dot{y}_\mathcal{F} = -(\Gamma \otimes I_\nu) y_\mathcal{F}$ for the errors $y_\mathcal{F}$ of the agents in $\mathcal{F}$, so $\dot{W}_0 \le -2\gamma W_0$. A jump within $\mathcal{F}$ decreases W_0 as in (19), and a jump with the leader sets the errors of S to zero. A jump adds edges to $\mathcal{G}(e)$ and removes agents from $\mathcal{F}$, which replaces Γ by a principal submatrix of a matrix dominating it, so γ does not decrease by Cauchy's interlacing theorem. Finally, $\|x_i - x_k\| \le \sqrt{2}\|y\|$ and $\|x_i - q\| \le \|y\|$, so if $\|y\| < d/\sqrt{2}$ and $m \ne \mathbf{1}$ the connectedness of $\mathcal{G}_\mathrm{d}$ provides an enabled merge, and the argument of Theorem 2(iii) applies. □

Leader–follower consensus over a fixed graph reaches the leader only asymptotically, at the rate γ_0. Here the agents reach it exactly, in finite time, and γ increases with each merge. A single informed agent suffices, and merging among agents is what lets the knowledge of the informed agents spread before the leader is reached.

VII. Numerical Examples

We illustrate the results with two planar examples: a team of robots, only a few of which know where a table is, that gathers around its legs and carries it (Section VI-C), and the self-assembly of a hexagonal structure, with and without collision avoidance (Sections VI-A and VI-B). Linear flows are integrated exactly, docking events are located by bisection, and simultaneous dockings are taken in random order, which the results allow.

A. Gathering around a table and carrying it

Twelve robots of diameter 0.3 have to carry a table of size 2×1.2, which rests at a location known to four of them, to a goal where it has to be set down turned by $90°$. Three robots are assigned to each leg, with attachment points on a circle of radius 0.3 around the leg, and one robot of each team is informed. The robots start at random positions in $[-6, 6]^2$, attachment points are assigned by a minimum-cost matching [24], and the robots initially communicate over the symmetrized 3-nearest-neighbor graph. Robots assigned to the same leg can dock with each other and every robot can attach to the table, and $d = 0.1$. For this instance $\gamma_0 = 0.280$, and Corollary 2 guarantees that all robots hold the table by $T_0^\star = 20.6$.

Fig. 5 shows the solution. The eight dockings between teammates take place between $t = 0.25$ and $t = 5.66$. With each of them, the knowledge of the location of the table passes to the robots that did not have it, all twelve robots are informed from $t = 5.66$ on, and γ grows from 0.28 to 1.00. The four teams attach to their legs at $T = 7.30$, after which the error is identically zero. The robots and the table then follow a cubic Bézier reference $(q_\mathrm{r}, \theta_\mathrm{r})$ of length 11.96 for the pose of the table, along which the orientation goes from 0 to $90°$, and the table is set down in the goal pose at $t = 13.30$. Each robot applies the velocity of its attachment point on a rigid body that follows the reference, plus the correction $q_\mathrm{r} - x_i$ with $x_i := p_i - R(\theta_\mathrm{r})\delta_i$, where R is the rotation matrix. Since $x_i = q_\mathrm{r}$ at T, the correction vanishes and the robots and the table move as one rigid body.

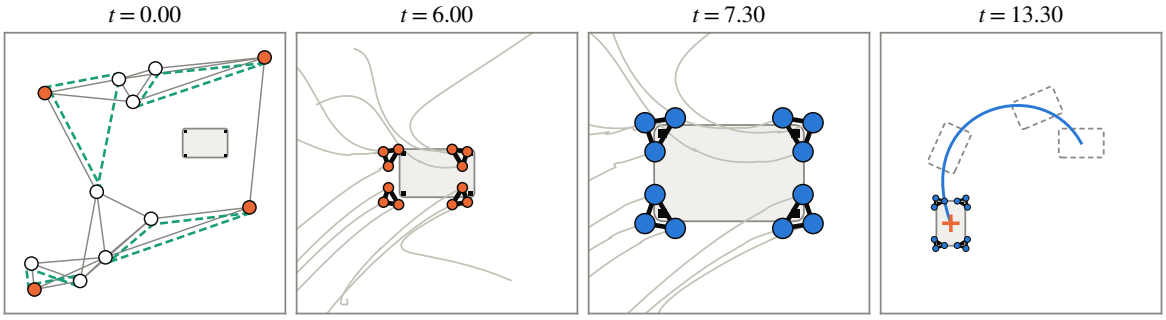


Fig. 5. Twelve robots gather around the legs of a table, attach to it, and carry it to a goal. First panel: initial positions (robots enlarged), interaction graph among the robots (solid, informed robots shaded), and docking graph among the robots (dashed), which joins each team of three. Every robot can also dock with the table. Next panels: teams have docked (thick links) and all robots are informed, then all robots hold the table at $T = 7.30$. Last panel: the table is carried to the goal (cross) while turning by $90°$.

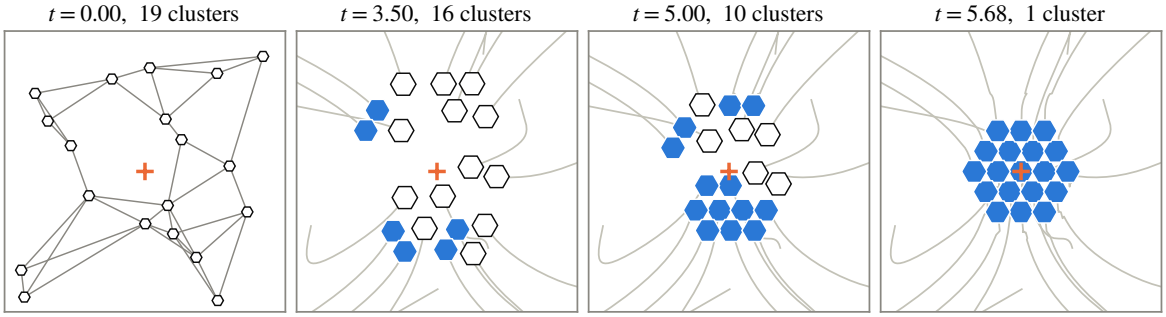


Fig. 6. Self-assembly of $N = 19$ hexagonal modules with the protocol (25)–(27). Left: initial positions and graph $\mathcal{G}(e_0)$. Other panels: trajectories up to the indicated time (magnified by 2.25), docked modules filled, initial centroid marked by a cross.

B. Self-assembly and collision avoidance

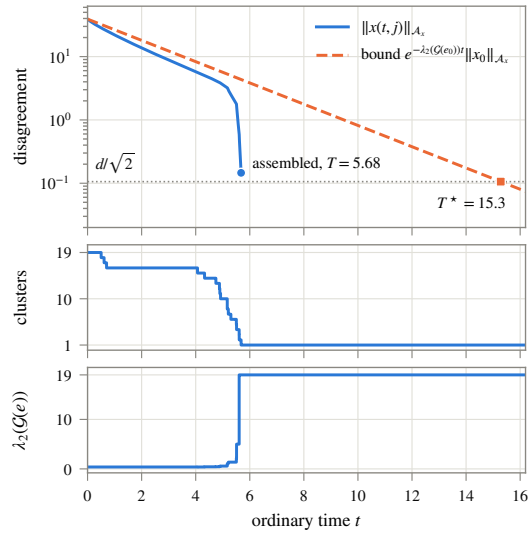


Fig. 7. Assembly of Fig. 6. Top: disagreement $\|x\|_{\mathcal{A}_x}$, zero after the last docking at $T = 5.68$, and the bound (20), which reaches $d/\sqrt{2}$ at $T^\star = 15.3$. Middle: number of clusters. Bottom: algebraic connectivity of $\mathcal{G}(e)$.

We next consider the planar assembly of a hexagonal structure from $N = 19$ modules. The positions δ_i are the centers of a hexagonal patch of two rings with unit spacing, and $\mathcal{G}_{\mathrm{d}}$ joins modules that are adjacent in the structure. The modules start from random positions in $[-12, 12]^2$, at least 2.5 apart, positions in the structure are assigned by a minimum-cost matching, and $\mathcal{G}(e_0)$ is the symmetrized 3-nearest-neighbor graph. For a typical instance with $d = 0.15$ and $\lambda_0 = 0.387$, shown in Fig. 6, the structure is completed after $N - 1 = 18$ dockings at $T = 5.68$, at the centroid of the initial positions, while (21) gives $T^\star = 15.28$. The difference is explained by $\lambda_2(\mathcal{G}(e))$, which grows from 0.39 to 19 along the solution (Fig. 7).

Over 50 random initial conditions and tolerances d between 0.4 and 0.01, the assembly time is below $T^\star$ in every instance, and its mean grows linearly in $\ln(1/d)$, as predicted by (21), from 4.7 to 20.3. The mean of $T^\star$ grows from 22.8 to 40.1, so the bound is conservative by a factor between 4.8 and 2, and is tighter for small tolerances.

Finally, we illustrate Theorem 3 for modules whose bodies are discs of diameter $\varrho = 0.75$, with $d = 0.15$ and $\eta = 0.05$. For these values (30) requires the modules to be initially at least 2.40 apart, so by Proposition 2 all 50 initial conditions are untangled. The barrier functions are $b_{uw}(h) = k(\bar{h}_{uw} - h)^2/h$ with $k = 0.02$ and $\bar{h}_{uw} = \min\{c_{uw}, 0.4\}$. With the nominal protocol, the bodies of two modules overlap at some time in 20 of the 50 runs, by up to 0.66. With the protocol of Section VI-B no overlap occurs in any run, all runs complete the structure before $T^\star$, and the assembly time increases by 3.7% on average. Fig. 8 shows the run in which the nominal protocol produces the deepest overlap.

VIII. Concluding Remarks

We modeled consensus among physically merging agents as a hybrid system in which a jump combines a state merge

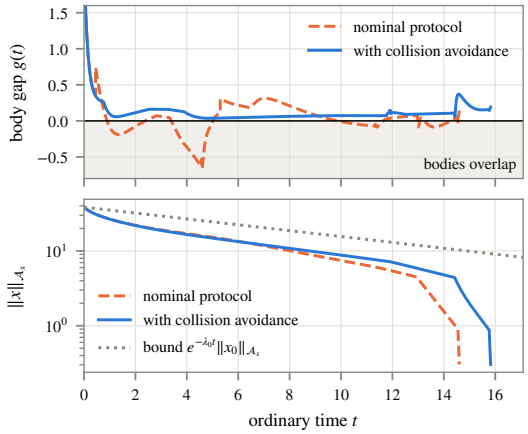


Fig. 8. Hexagonal assembly with disc-shaped modules of diameter $\varrho = 0.75$, with the nominal protocol and with that of Section VI-B, from the same initial condition. Top: body gap $g(t) := \min\{\|p_u(t) - p_w(t)\| - \varrho \colon m_{uw}(t) = 0\}$ between undocked modules, negative when two bodies overlap. It jumps up when the closest pair docks. Bottom: disagreement and the bound (20).

with a neighborhood merge. The network generated by the protocol is a blow-up of a contraction of the initial graph, and all agents merge at the initial centroid in finite time, with a bound that does not depend on the order of the merges. The framework extends to formations, collision avoidance, and leader–follower networks. The model is idealized, since the agents share a common orientation and docking is instantaneous. Future work includes tangled configurations, attitude dynamics, agents with inertia, for which docking is an inelastic impact, directed graphs, and clusters that split.

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References

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[2] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ: Princeton University Press, 2010.

[3] J. Cortés, S. Martínez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Transactions on Automatic Control*, vol. 51, no. 8, pp. 1289–1298, 2006.

[4] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, 2015.

[5] D. Saldaña, B. Gabrich, G. Li, M. Yim, and V. Kumar, "ModQuad: The flying modular structure that self-assembles in midair," in *Proc. IEEE International Conference on Robotics and Automation*, 2018, pp. 691–698.

[6] I. O'Hara, J. Paulos, J. Davey, N. Eckenstein, N. Doshi, T. Tosun, J. Greco, J. Seo, M. Turpin, V. Kumar, and M. Yim, "Self-assembly of a swarm of autonomous boats into floating structures," in *Proc. IEEE International Conference on Robotics and Automation*, 2014, pp. 1234–1240.

[7] E. Bray and R. Groß, "Recent developments in self-assembling multi-robot systems," *Current Robotics Reports*, vol. 4, no. 4, pp. 101–116, 2023.

[8] D. Rus, B. Donald, and J. Jennings, "Moving furniture with teams of autonomous robots," in *Proc. IEEE/RSJ International Conference on Intelligent Robots and Systems*, 1995, pp. 235–242.

[9] E. Tuci, M. H. M. Alkilabi, and O. Akanyeti, "Cooperative object transport in multi-robot systems: A review of the state-of-the-art," *Frontiers in Robotics and AI*, vol. 5, p. 59, 2018.

[10] N. Mathews, A. L. Christensen, R. O'Grady, F. Mondada, and M. Dorigo, "Mergeable nervous systems for robots," *Nature Communications*, vol. 8, p. 439, 2017.

[11] N. Leiter and D. Zelazo, "The aggregating consensus protocol—a case study of behavioral multi-agent systems," in *Proc. 58th Israel Annual Conference on Aerospace Sciences*, Haifa, Israel, 2018, pp. 1037–1051.

[12] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton, NJ: Princeton University Press, 2012.

[13] S. Phillips, Y. Li, and R. G. Sanfelice, "A hybrid consensus protocol for pointwise exponential stability with intermittent information," *IFAC-PapersOnLine*, vol. 49, no. 18, pp. 146–151, 2016.

[14] C. De Persis and R. Postoyan, "A Lyapunov redesign of coordination algorithms for cyber-physical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 2, pp. 808–823, 2017.

[15] P. Frasca, S. Tarbouriech, and L. Zaccarian, "Hybrid models of opinion dynamics with opinion-dependent connectivity," *Automatica*, vol. 100, pp. 153–161, 2019.

[16] J. Cortés, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, no. 11, pp. 1993–2000, 2006.

[17] N. Monshizadeh, H. L. Trentelman, and M. K. Camlibel, "Projection-based model reduction of multi-agent systems using graph partitions," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 2, pp. 145–154, 2014.

[18] N. Leiter and D. Zelazo, "Graph-based model reduction of the controlled consensus protocol," in *Proc. 20th IFAC World Congress*, Toulouse, France, 2017, pp. 9456–9461.

[19] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph Classes: A Survey*, ser. SIAM Monographs on Discrete Mathematics and Applications. Philadelphia, PA: Society for Industrial and Applied Mathematics, 1999.

[20] D. M. Cardoso, M. A. A. de Freitas, E. A. Martins, and M. Robbiano, "Spectra of graphs obtained by a generalization of the join graph operation," *Discrete Mathematics*, vol. 313, no. 5, pp. 733–741, 2013.

[21] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 2, pp. 298–305, 1973.

[22] L. Wang, A. D. Ames, and M. Egerstedt, "Safety barrier certificates for collisions-free multirobot systems," *IEEE Transactions on Robotics*, vol. 33, no. 3, pp. 661–674, 2017.

[23] M. Jankovic, M. Santillo, and Y. Wang, "Multiagent systems with CBF-based controllers: Collision avoidance and liveness from instability," *IEEE Transactions on Control Systems Technology*, vol. 32, no. 2, pp. 705–712, 2024.

[24] M. Turpin, N. Michael, and V. Kumar, "CAPT: Concurrent assignment and planning of trajectories for multiple robots," *The International Journal of Robotics Research*, vol. 33, no. 1, pp. 98–112, 2014.