Forced Symmetric Formation Control

Daniel Zelazo, *Senior Member, IEEE*, Shin-ichi Tanigawa, and Bernd Schulze

***Abstract*—This work considers the distance constrained formation control problem with an additional constraint requiring that the formation exhibits a specified spatial symmetry. We employ recent results from the theory of symmetry-forced rigidity to construct an appropriate potential function that leads to a gradient dynamical system driving the agents to the desired formation. We show that only $(1 + 1/|\Gamma|)n$ edges are sufficient to implement the control strategy when there are n agents and the underlying symmetry group is Γ. This number is considerably smaller than what is typically required from classic rigidity-theory-based strategies ($2n - 3$ edges). We also provide an augmented control strategy that ensures that the agents can converge to a formation with respect to an arbitrary centroid. Numerous numerical examples are provided to illustrate the main results.**

***Index Terms*—Rigidity theory, symmetry, formation control.**

I. Introduction

MANY applications for cooperative multiagent networks require the agents to arrange themselves into some spatial pattern. This can include alignment of orientations and velocities for flocking behaviors [1], or specific formations, spacecraft constellations for sensing [2] or vehicle platoons for autonomous driving [3]. One of the main challenges for the implementation of these applications is to resolve the tradeoff between sparsity of information exchange with guarantees on the system performance. In the study of formation control problems, this tradeoff is well understood through the lens of *rigidity theory*.

Rigidity theory studies the solution of a set of geometric constraints on a discrete configuration of points in an Euclidean space. These constraints can include distance or bearing constraints between pairs of points. Of interest in rigidity theory is to determine whether the set of polynomial equations representing these constraints 1) has a solution (independence), 2) has locally isolated solutions (rigidity), or 3) has exactly one solution in the given space up to isometric motions (global rigidity) [4], [13].

Received 13 August 2024; accepted 17 November 2024. Date of publication 3 January 2025; date of current version 20 June 2025. The work of Daniel Zelazo was supported in part by the Israel Science Foundation under Grant 453/24 and in part by the Bernard M. Gordon Center for Systems Engineering at the Technion. The work of Shin-ichi Tanigawa was supported by JST PRESTO under Grant JPMJPR2126. Recommended by Associate Editor S. Sundaram. *(Corresponding author: Daniel Zelazo.)*

Daniel Zelazo is with the Faculty of Aerospace Engineering, Technion-Israel Institute of Technology, 3200003 Haifa, Israel (e-mail: dzelazo@technion.ac.il).

Shin-ichi Tanigawa is with the Graduate School of Information Science and Technology, University of Tokyo, Tokyo 1138656, Japan.

Bernd Schulze is with the School of Mathematical Sciences, Lancaster University, LA14YF Lancaster, U.K.

Digital Object Identifier 10.1109/TCNS.2025.3525814

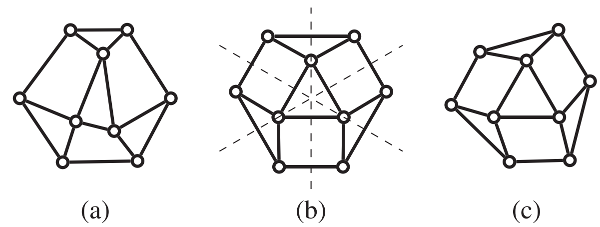


Fig. 1. Framework in (a) is infinitesimally rigid, whereas the framework with dihedral symmetry in (b) is flexible, as shown in (c).

In this article, we will focus on configurations that are constrained by pairwise distances. Such systems are commonly known as (bar-joint) frameworks. Since checking a framework for rigidity is in general very difficult (as it requires solving a system of quadratic equations), a common approach is to check for the linearized (and stronger) notion of rigidity known as "infinitesimal rigidity" (see Section II-A).

The seminal work from [5] provided the first formal result showing that (minimal) infinitesimal rigidity (MIR) of the interaction network in a team of integrator agents is required to ensure that the gradient controller (locally) converges to the correct formation shape. In the Euclidean space $\mathbb{R}^2$, MIR translates to having $2n - 3$ constraints, where n is the number of agents. Since the work in [5], there has been an explosion of research focusing on the formation control problem from a rigidity theory perspective; see the following for an overview [6], [7], [8]. Within the controls community, the main concern of these works, however, focuses on understanding the resulting dynamics of the control strategies, with most efforts on the stability and convergence properties of these systems [9], [10], [11]. In these works, the assumption of infinitesimal rigidity is taken as a kind of architectural requirement for solving the formation control problem. That is, there has not been a concerted effort to exploit results from rigidity theory to relax the infinitesimal rigidity assumption. In this article, we will pursue this using recent advances in the rigidity analysis of symmetric frameworks.

It is well understood in the rigidity community that there are many special configurations that can lead to unexpected flexibility (or rigidity). Symmetry often leads to such special configurations. An example in the plane is shown in Fig. 1. The graph is infinitesimally rigid (in fact, MIR) for almost all realizations as a framework in the plane. See the framework in Fig. 1(a), for example. However, if we place the vertices in the positions shown in Fig. 1(b), then the framework has three reflection symmetries (each of the reflections in the dashed mirror lines maps the framework onto itself) and it is continuously flexible, as indicated in Fig. 1(c).

Note that the motion of the framework in Fig. 1(b) destroys all the reflection symmetries. So while the framework is flexible, it is still "forced-symmetric rigid," in the sense that it does not have a nontrivial motion that preserves the original symmetry of the framework. We will make this precise in Section III-B.

In our previous work [12], we proposed a "symmetry-attaining" formation controller by augmenting the classic gradient-based formation control with another control term that forces the agents into a symmetric position. Under some mild assumptions of the underlying graph, we showed that this approach does not require more communication or sensing than the normal formation controller and can guarantee local convergence to the desired symmetry-constrained formation. Moreover, by means of simulation, we demonstrated that the problem can be solved even for flexible frameworks requiring less than $2n-3$ edges as is typically assumed for the formation control problem in MIR frameworks.

This initial work, however, did not leverage the full potential of recent results characterizing symmetry-forced rigidity. In this article, we extend the results of [12] in the following ways. We develop a concrete mathematical foundation for solving the multiagent formation control problem under symmetry constraints. The theory provides the symmetric counterpart to the ordinary rigidity-based formation control using the modern language of graph rigidity, and in particular that of forced-symmetric rigidity theory. Notably, we show that $(1+1/|\Gamma|)n$ edges are sufficient when the underlying symmetry group is Γ. This is significantly smaller than the bound for infinitesimal rigidity. This improved bound has direct implications for practical implementation as it leads to a reduction of energy consumption and communication bandwidth. Moreover, this approach can be used to augment any multiagent coordination problem with explicit symmetry constraints, providing a new conceptual solution to these problems. In this direction, we first present a detailed overview of results from the rigidity theory of symmetric frameworks. The central objects in this study are the *quotient graphs*, providing a graph-theoretic characterization of the vertex and edge orbits induced by a given symmetry group, and the *orbit rigidity matrix*, which can be thought of as the rigidity matrix associated with the quotient graph of a framework. The orbit rigidity matrix is then used to define an *orbit rigidity formation potential* for solving the forced-symmetric formation control problem. We then provide a stability and convergence analysis of this control law. Finally, we propose a consensus-augmented version of the forced-symmetric formation control problem that ensures the formation converges to a point other than a globally defined origin. Numerous examples are provided throughout to illustrate the main concepts and results.

The rest of this article is organized as follows. Section II presents an overview of geometric rigidity theory and formation control. Section III provides a detailed background on notions of symmetry for graphs and frameworks. Section IV presents the main results of this article with different formation control strategies that exploit symmetry properties of the framework. Finally, Section V concludes this article.

II. Preliminaries From Geometric Rigidity and Formation Control

This section provides an overview of basic concepts from geometric rigidity theory and the formation control literature.

A. Rigidity Theory

A *framework* in $\mathbb{R}^d$ is defined to be a pair $(\mathcal{G},p)$ consisting of a finite simple graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ and a map $p:\mathcal{V}\to\mathbb{R}^d$. It is natural to also consider p as a point in $\mathbb{R}^{d|\mathcal{V}|}$, and we refer to p as a *configuration* of $|\mathcal{V}|$ points in $\mathbb{R}^d$. Frameworks are the fundamental objects of study in geometric rigidity theory, where they are considered as mathematical models of physical structures consisting fixed-length bars (corresponding to the edges of $\mathcal{G}$) that are connected by joints (corresponding to the vertices of $\mathcal{G}$) that allow rotation in any direction. A framework $(\mathcal{G},p)$ is *rigid* if the only edge-length-preserving continuous motions of the vertices arise from isometries of $\mathbb{R}^d$, and *flexible*, otherwise. The rigidity and flexibility analysis of frameworks is a well-developed theory with a rich history and many practical applications (see, e.g., [13, ch. 61], [14], and [15]). In particular, it has recently found important applications in the formation control of multirobot systems [6], [7].

Since for $d\geq 2$, it is NP-hard to determine if a given framework is rigid, a common approach to study the rigidity of frameworks is to linearize the problem by differentiating the length constraints on the edges. This leads to the notion of infinitesimal (or equivalently, static) rigidity. An *infinitesimal motion* of a framework $(\mathcal{G},p)$ in $\mathbb{R}^d$ is an assignment of velocity vectors, one to each vertex, $u:\mathcal{V}\to\mathbb{R}^d$, such that $\langle p_i-p_j, u_i-u_j\rangle = 0$ for all $ij\in\mathcal{E}$, where $p_i=p(i)$ and $u_i=u(i)$ for each $i\in\mathcal{V}$. An infinitesimal motion u of $(\mathcal{G},p)$ is *trivial* if there exists a skew-symmetric matrix S and a vector c such that $u_i=Sp_i+c$ for all $i\in\mathcal{V}$, and $(\mathcal{G},p)$ is *infinitesimally rigid* if every infinitesimal motion of $(\mathcal{G},p)$ is trivial, and *infinitesimally flexible*, otherwise.

The $|\mathcal{E}|\times d|\mathcal{V}|$ matrix corresponding to the linear system above is called the *rigidity matrix* of $(\mathcal{G},p)$, denoted as $R(\mathcal{G},p)$. The row of $R(\mathcal{G},p)$ corresponding to the edge $ij\in\mathcal{E}$ is of the form $\begin{pmatrix} 0\cdots 0 & (p_i-p_j)^T & 0\cdots 0 & (p_j-p_i)^T & 0\cdots 0\end{pmatrix}$. We also employ the useful algebraic representation of the rigidity matrix

$$R(\mathcal{G},p)=\text{diag}\left\{(p_i-p_j)^T\right\}_{ij\in\mathcal{E}}\left(E^T\otimes I_d\right) \qquad (1)$$

where E is the $|\mathcal{V}|\times|\mathcal{E}|$ incidence matrix (using an arbitary orientation of the edges) with $E_{ij}=1$ if the edge e_j leaves vertex i, $E_{ij}=-1$ if the edge e_j enters the vertex i, and $E_{ij}=0$, otherwise. So the kernel of $R(\mathcal{G},p)$ is the space of all infinitesimal motions of $(\mathcal{G},p)$, and it is well known that $(\mathcal{G},p)$ is infinitesimally rigid if and only if the rank of $R(\mathcal{G},p)$ is $d|\mathcal{V}|-\binom{d+1}{2}$, provided that the points p_i affinely span all of $\mathbb{R}^d$ [15].

A *self-stress* of a framework $(\mathcal{G},p)$ is a function $\omega:\mathcal{E}\to\mathbb{R}$ so that the following equation is satisfied at every vertex i:

$$\sum_{\{j:ij\in E\}}\omega_{ij}(p_i-p_j)=0.$$

Equivalently, $\omega \in \mathbb{R}^{|\mathcal{E}|}$ is a self-stress if and only if ω is an element of the cokernel of $R(\mathcal{G}, p)$, i.e., $\omega^T R(\mathcal{G}, p) = 0$. If $(\mathcal{G}, p)$ has no nonzero self-stress, then it is called *independent*. Moreover, an infinitesimally rigid and independent framework is called *isostatic*. Isostatic frameworks are also called *minimally infinitesimally rigid*, as the removal of any edge yields an infinitesimally flexible framework.

While an infinitesimally rigid framework is always rigid, the converse does not hold in general. Asimov and Roth [4], however, showed that for "generic" configurations p (i.e., which form an open dense subset of $\mathbb{R}^{d|\mathcal{V}|}$), infinitesimal rigidity is equivalent to rigidity. The graphs that yield rigid frameworks for generic configurations in the plane have been characterized by Pollaczek–Geiringer [16] and Laman [17], and the condition can be checked in polynomial time. It remains a key open problem to find a characterization of generically rigid graphs in higher dimensions that can be checked in polynomial time deterministically.

B. Formation Control

We now review the well-studied distance-constrained formation control problem [7]. Consider a network of $n = |\mathcal{V}|$ agents described by integrator dynamics

$$\dot{p}_i(t) = u_i(t) \tag{2}$$

where $p_i(t) \in \mathbb{R}^d$ is the position of agent i, and $u_i(t) \in \mathbb{R}^d$ is the control. As in the development of rigidity theory in Section II-A, the configuration of the network is the stack of the agent positions, $p(t) = \begin{bmatrix} p_1^T(t) & \cdots & p_n^T(t) \end{bmatrix}^T$ (similarly defined for $u(t)$). The agents are tasked with attaining a spatial formation using only measurements and/or communication with neighboring agents, as defined by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The formation is specified by a set of desired interagent distances $\mathbf{d}_{ij}$ for each edge $ij \in \mathcal{E}$, and we denote $\mathbf{d}$ as the stack of all desired distances.

It is well-known that a gradient-based control strategy can (locally) solve the formation control problem. In this direction, we define the *formation potential* function

$$F_f(p(t)) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - \mathbf{d}_{ij}^2 \right)^2. \tag{3}$$

Then, the gradient controller $u(t) = -\nabla F_f(p(t))$ solves the formation control problem. That is, the closed-loop system

$$\dot{p}(t) = -R(\mathcal{G}, p(t))^T \left(R(\mathcal{G}, p(t))p(t) - \mathbf{d}^2 \right) \tag{4}$$

satisfies $\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = \mathbf{d}_{ij}$ for all $ij \in \mathcal{E}$. Here, $R(\mathcal{G}, p(t))$ is the rigidity matrix defined in Section II-A. The interested reader may refer to [6] and [9] for more details.

III. Symmetry in Graphs and Frameworks

The main focus of this work is to exploit notions from the rigidity theory of symmetric frameworks to solve the formation control problem. In this section, we provide an overview of symmetric frameworks and their (infinitesimal) rigidity.

A. Symmetry in Graphs

Symmetry in objects can be described mathematically via the fundamental algebraic notion of a group (see, e.g., [18]).

Definition 1: A *group* is defined to be a set Γ together with an operation $\circ$ such that for any two elements $a, b \in \Gamma$, the composition $a \circ b$ is also in Γ. The operation $\circ$ satisfies the associativity law. Moreover, each group has a special element id, called the *identity element*, such that for any element $a \in \Gamma$, $a \circ \mathrm{id} = \mathrm{id} \circ a = a$. Each element a of Γ also has an *inverse* a^{-1} in Γ such that $a \circ a^{-1} = a^{-1} \circ a = \mathrm{id}$. The number of elements in a group is called the *order* of the group. A subset B of Γ that also forms a group under $\circ$ is called a *subgroup* of Γ.

The combinatorial symmetries of a finite simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ are described by its group of automorphisms. An automorphism of $\mathcal{G}$ can be loosely understood as a permutation of $\mathcal{V}$ that maps adjacent vertices to adjacent vertices and nonadjacent vertices to nonadjacent vertices, and hence preserves all structural properties of $\mathcal{G}$.

Definition 2: An *automorphism* of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation $\psi : \mathcal{V} \to \mathcal{V}$ of its vertex set such that $\psi(v)\psi(u) \in \mathcal{E} \Leftrightarrow vu \in \mathcal{E}$.

It is clear, then, that the identity permutation, denoted id, is an automorphism of any graph, and for an automorphism ψ, ψ^{-1} is also an automorphism. Therefore, the set of all automorphisms of $\mathcal{G}$ forms a group under composition of maps. This group is called the *automorphism group* of $\mathcal{G}$ and is denoted by $\mathrm{Aut}(\mathcal{G})$.

A common way to represent a permutation for an automorphism is by a two-row array. For a graph $\mathcal{G}$ with $|\mathcal{V}| = n$ vertices, one can write the automorphism ψ as

$$\psi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \psi(1) & \psi(2) & \cdots & \psi(n) \end{pmatrix}.$$

Equivalently, one can express every permutation more compactly as a composition of disjoint cycles of the permutation. A cycle is a successive action of the permutation that sends a vertex back to itself, i.e., $i \to \psi(i) \to \psi(\psi(i)) \to \cdots \to \psi^k(i) = i$, where $\psi^k = \underbrace{\psi \circ \cdots \circ \psi}_{k \text{ times}}$. Such a cycle is compactly written using the *cycle notation*, denoted by $(i\, \psi(i) \cdots \psi^{k-1}(i))$. The integer k is the *length* of the cycle.

Definition 3: A graph $\mathcal{G}$ is Γ-*symmetric* for any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$. The symmetry is *free* if $\gamma(i) \neq i$ for all $i \in \mathcal{V}$ and nonidentity $\gamma \in \Gamma$.

A key structural property of a Γ-symmetric graph $\mathcal{G}$ is its sets of vertex and edge orbits under Γ. Loosely speaking, the orbit of a vertex i (or edge e) of $\mathcal{G}$ under Γ is the set of vertices (edges, respectively) of $\mathcal{G}$ that i (e, respectively) can be mapped to by elements in Γ.

Definition 4: For a Γ-symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and vertex $i \in \mathcal{V}$, the set $\Gamma_i = \{\gamma(i) \,|\, \gamma \in \Gamma\}$ is called the *vertex orbit* of i. Similarly, for an edge $e = ij \in \mathcal{E}$, the set $\Gamma_e = \{\gamma(i)\gamma(j) \,|\, \gamma \in \Gamma\}$ is termed the *edge orbit* of e.

The size of the vertex orbits depends, of course, on the group Γ. Since all nodes in a given orbit are somehow equivalent under a group action, we often consider *representative vertices* from each vertex orbit. We denote by $\mathcal{V}_0$ the set of representative vertices for each orbit, such that $|\mathcal{V}_0|$ are the number of vertex

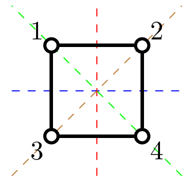


Fig. 2. Cycle graph C_4 has eight automorphisms in $\mathrm{Aut}(\mathcal{G})$.

orbits, and $i \in \mathcal{V}_0$ means that vertex i is only in one vertex orbit. Similarly, we denote by $\mathcal{E}_0$ the set of representative edges from each edge orbit.

For a Γ-symmetric graph $\mathcal{G}$ and a vertex orbit Γ_i of $\mathcal{G}$ under Γ, it follows immediately from the definition of Γ_i that for every vertex j in Γ_i, there is a $\gamma_j \in \Gamma$ such that $\gamma_j(j) = i$. Similarly, for any two vertices $u, v \in \Gamma_i$, there is an automorphism $\gamma_{uv} \in \Gamma$ such that $\gamma_{uv}(u) = v$. With this notation, we then have $\gamma_j = \gamma_{ji}$ when i is the representative vertex of Γ_i.

Example 1: Fig. 2 shows the cycle graph on 4 nodes, C_4. We will identify all the automorphisms of $\mathrm{Aut}(C_4)$. First, consider a clockwise rotation by 90° of C_4 as drawn in the figure. This gives the automorphism

$$\psi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

The cycle notation is $\psi_1 = (1\,2\,4\,3)$. With ψ_1, we also have $\psi_2 = \psi_1^2 = \psi_1 \circ \psi_1, \psi_3 = \psi_1^3$, and $\psi_1^4 = \mathrm{id}$ in $\mathrm{Aut}(\mathcal{G})$, where $\psi_2 = (1\,4)(2\,3)$ and $\psi_3 = (1\,3\,4\,2)$ may be interpreted geometrically as rotations by 180° and 270°. Additional permutations can be found by considering reflections.

Fig. 2 shows four reflection symmetries. Consider first the reflection about the vertical red line, giving the permutation (in cycle notation) $\psi_4 = (1\,2)(3\,4)$. Similarly, the horizontal reflection (blue line) yields $\psi_5 = (1\,3)(2\,4)$, the diagonal reflection (green line) gives $\psi_6 = (1)(2,\,3)(4)$, and the brown line reflection gives $\psi_7 = (1\,4)(2)(3)$. Thus, we have that $\mathrm{Aut}(C_4) = \{\mathrm{id}, \psi_1, \ldots, \psi_7\}$ has eight automorphisms. As an abstract group, it is the dihedral group D_8 of order 8 [18]. Note that any vertex can be mapped to any other under the automorphisms in $\mathrm{Aut}(C_4)$, and hence, C_4 has only one vertex orbit (and only one edge orbit) under $\mathrm{Aut}(C_4)$. If, however, we considered C_4 as a Γ-symmetric graph, where $\Gamma = \{\mathrm{id}, \psi_4\} \subset \mathrm{Aut}(C_4)$, then C_4 has two vertex orbits, namely, $\{1, 2\}$ and $\{3, 4\}$, and three edge orbits, namely, $\{12\}$, $\{34\}$, and $\{13, 24\}$.

B. Symmetry in Frameworks

Having defined notions of symmetries for graphs, we now consider symmetry of frameworks [13, ch. 62].

Definition 5: Let $\mathcal{G}$ be a Γ-symmetric graph, and let Γ be represented as a *point group*, i.e., a subgroup of the orthogonal group $O(\mathbb{R}^d)$, via a homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$. In other words, τ assigns an orthogonal matrix (describing an isometry of $\mathbb{R}^d$ such as a rotation or reflection) to each element of Γ. Then, a framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called $\tau(\Gamma)$-*symmetric* if

$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}. \tag{5}$$

Given a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ and a vertex orbit Γ_i, for every $j \in \Gamma_i$, there is a $\gamma_j \in \Gamma$ such that $\tau(\gamma_j)p_j = p_i$. More generally, for any two vertices $u, v \in \Gamma_i$, there exists $\gamma_{uv} \in \Gamma$ such that $\tau(\gamma_{uv})p_u = p_v$.

We use the standard Schoenflies notation for point groups in this article [19], [20]. The only possible point groups in dimension 2 are the reflection group $\mathcal{C}_s$ (consisting of the identity and a single reflection about an axis σ), the rotational groups $\mathcal{C}_n$ of order n, where $n \geq 1$ (generated by a rotation c_n about the origin in counter-clockwise direction by an angle of $2\pi/n$), and the dihedral groups $\mathcal{C}_{nv}$ of order $2n$, where $n \geq 2$ (generated by a reflection σ and a rotation c_n). If we think of the graph drawing in Fig. 2 as a framework in the plane, for example, then this framework is $\mathcal{C}_{4v}$-symmetric.

If the framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-symmetric, then the configuration p is in a special geometric position that may no longer be "generic." Thus, symmetry can lead to unexpected flexibility (as well as unexpected rigidity), as in Fig. 1. Since symmetry is very common in both natural and man-made structures, the rigidity and flexibility analysis of symmetric frameworks has grown into a major research area over the last two decades; see [13, ch. 62] for a summary of this work.

A fundamental result—based on group representation theory—is that for a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$, there are suitable symmetry-adapted bases of $\mathbb{R}^{|\mathcal{E}|}$ and $\mathbb{R}^{|d\mathcal{V}|}$ that transform the rigidity matrix of $(\mathcal{G}, p)$ into a block-decomposed form [21], [22], [13, ch. 62]. This block-decomposition of $R(\mathcal{G}, p)$ can be used to break up the infinitesimal rigidity analysis of $(\mathcal{G}, p)$ into a number of independent subproblems, one for each block matrix. A number of results concerning the infinitesimal rigidity of symmetric frameworks have been obtained via this approach (see, e.g., [13] and [23]).

Another major research area is to study the rigidity of symmetric frameworks under the additional constraint that any motion must preserve the original symmetry of the framework. If all symmetry-preserving motions of a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ are trivial, then $(\mathcal{G}, p)$ is called *forced* $\tau(\Gamma)$-*symmetric rigid*. This is a weaker notion than rigidity, because a forced $\tau(\Gamma)$-symmetric framework may still have nontrivial motions that destroy the original symmetry of the framework. By differentiating a symmetry-preserving continuous motion of $(\mathcal{G}, p)$, we obtain a $\tau(\Gamma)$-symmetric infinitesimal motion, that is, an infinitesimal motion whose velocity assignments to the points of $(\mathcal{G}, p)$ exhibit exactly the same symmetry as the configuration p.

Definition 6: An infinitesimal motion u of a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric* if

$$\tau(\gamma)(u_i) = u_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}. \tag{6}$$

We say that $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric infinitesimally rigid* if every $\tau(\Gamma)$-symmetric infinitesimal motion is trivial.

A self-stress ω of $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric* if

$$\omega_{\gamma(e)} = \omega_e \quad \text{for all } \gamma \in \Gamma \text{ and all } e \in \mathcal{E}. \tag{7}$$

The framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric independent* if it has no non-zero $\tau(\Gamma)$-symmetric self-stress. Further, $(\mathcal{G}, p)$ is $\tau(\Gamma)$-*symmetric isostatic* if it is $\tau(\Gamma)$-symmetric infinitesimally rigid and $\tau(\Gamma)$-symmetric independent.

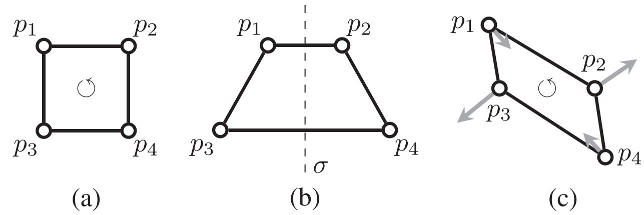


Fig. 3. Symmetric frameworks with C_4 as underlying graph. (a) $\mathcal{C}_{4v}$-symmetric (and hence $\tau(\Gamma)$-symmetric for any subgroup $\tau(\Gamma)$ of $\mathcal{C}_{4v}$). (b) and (c) $\mathcal{C}_s$-symmetric (with respect to the reflection σ) and $\mathcal{C}_2$-symmetric. The framework in (c) has a nontrivial $\mathcal{C}_2$-symmetric infinitesimal motion, which extends to a continuous symmetry-preserving motion.

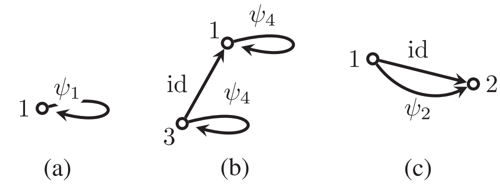


Fig. 4. Each gain graph (a), (b), and (c) corresponds to the frameworks (a), (b), and (c) in Fig. 3.

Note that $(\mathcal{G}, p)$ is $\tau(\Gamma)$-symmetric isostatic if it is minimally $\tau(\Gamma)$-symmetric infinitesimally rigid, in the sense that the removal of any edge orbit Γ_e from $\mathcal{G}$ yields a $\tau(\Gamma)$-symmetric infinitesimally flexible framework [13]. See Fig. 3(a) and (b) for some examples of minimally $\tau(\Gamma)$-symmetric infinitesimally rigid frameworks in the plane.

Example 2: Consider the frameworks in Fig. 3. They are all infinitesimally (in fact, continuously) flexible. The framework in Fig. 3(a) is (minimally) $\mathcal{C}_4$-symmetric infinitesimally rigid, where $\mathcal{C}_4$ is the rotational point group of order 4, as the only $\mathcal{C}_4$-symmetric infinitesimal motions are trivial rotations. [Note that $\mathcal{C}_4$ symmetry implies the larger $\mathcal{C}_{4v}$ symmetry in this example and the framework is also (minimally) $\mathcal{C}_{4v}$-symmetric infinitesimally rigid.]

The frameworks in Fig. 3(b) and (c) are $\mathcal{C}_s$- and $\mathcal{C}_2$-symmetric, respectively. The one in Fig. 3(b) is (minimally) $\mathcal{C}_s$-symmetric infinitesimally rigid, whereas the one in Fig. 3(c) is $\mathcal{C}_2$-symmetric infinitesimally flexible. In fact, it has a continuous motion that preserves the half-turn symmetry.

A key motivation for studying $\tau(\Gamma)$-symmetric infinitesimal rigidity is that a $\tau(\Gamma)$-symmetric infinitesimal motion extends to a continuous, symmetry-preserving motion of the framework, provided that the configuration is sufficiently generic with the given symmetry constraints [13, ch. 62]. In this article, we will exploit the notion of $\tau(\Gamma)$-symmetric infinitesimal rigidity to reduce the communication/sensing requirements in multiagent formations.

One of the block matrices of the block-decomposed rigidity matrix (the block matrix $R_0(\mathcal{G}, p)$ corresponding to the trivial irreducible representation of the group $\tau(\Gamma)$, which assigns 1 to each group element) has the property that its kernel is isomorphic to the space of $\tau(\Gamma)$-symmetric infinitesimal motions and its cokernel is isomorphic to the space of $\tau(\Gamma)$-symmetric self-stresses. Thus, in the study of forced-symmetric (infinitesimal) rigidity, this block matrix plays the same role as the rigidity matrix in the standard nonsymmetric theory. In the next section, we will introduce the *orbit rigidity matrix*, which is equivalent to the block matrix $R_0(\mathcal{G}, p)$, and whose entries have a simple form similar to the entries of the standard rigidity matrix.

C. Orbit Rigidity Matrix

Given a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$, it requires a nontrivial computation to obtain the block matrices of the block-decomposed rigidity matrix of $(\mathcal{G}, p)$. However, it was shown in [24] that the block matrix $R_0(\mathcal{G}, p)$ describing the $\tau(\Gamma)$-symmetric infinitesimal rigidity of $(\mathcal{G}, p)$ is equivalent to another matrix, called the *orbit rigidity matrix*, which can be written down in a simple and direct way, without using any group representation theory.

For simplicity, we will assume from now on that the symmetry is always free (recall Definition 3). While all of our results are expected to extend to the nonfree case, this assumption simplifies the definition of the orbit rigidity matrix, and hence the entire forced-symmetric formation control theory, significantly. So we will leave the nonfree case for future work.

Assumption 1: The $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ is free.

To describe the orbit rigidity matrix, we first need to introduce the notion of a quotient gain graph of a Γ-symmetric graph, which in turn relies on the notions of vertex and edge orbits introduced in Definition 4.

Definition 7: Let $\mathcal{G}$ be Γ-symmetric graph with representative vertex set $\mathcal{V}_0$ and representative edge set $\mathcal{E}_0$. The *quotient* Γ*-gain graph* $\mathcal{G}_0$ of $\mathcal{G}$ is the directed multigraph with vertex set $\mathcal{V}_0$ whose edge set $\mathcal{E}_0$ has the directed edge (i, j) with group label (or *gain*) $\gamma \in \Gamma$, denoted by $((i, j); \gamma)$, for each edge orbit representative $i\gamma(j)$.

Note that different choices of vertex representatives give rise to different, but equivalent quotient Γ-gain graphs. Moreover, for a fixed choice of $\mathcal{V}_0$, the edge $((i, j); \gamma)$ in $\mathcal{E}_0$, where $i \neq j$, is equivalent to the edge $((j, i); \gamma^{-1})$, so the orientation of nonloop edges in $\mathcal{E}_0$ may be reversed by changing the gain to its inverse.

Example 3: The framework (C_4, p) in Fig. 3(a) has $\mathcal{C}_{4v}$ symmetry, but we may consider it as a framework with the smaller rotational $\mathcal{C}_4$ symmetry (where $\mathcal{C}_4$ is generated by the 90° rotation about the origin). Let $\Gamma = \{\text{id}, \psi_1, \psi_1^2, \psi_1^3\}$ be the corresponding subgroup of Aut(C_4), where ψ_1 is the automorphism defined in Example 1. For simplicity, we will identify the groups Aut(C_4) and $\mathcal{C}_4$ in the following discussion. Note that the symmetry $\mathcal{C}_4$ is free and there is only one vertex orbit under $\mathcal{C}_4$. We may pick vertex 1 as its representative. Then, there is only one edge orbit, represented by edge 13, for example, which in the quotient $\mathcal{C}_4$-gain graph of C_4 corresponds to the loop at 1 with gain ψ_1. The quotient $\mathcal{C}_4$-gain graph of C_4 is shown in Fig. 4(a).

For the $\mathcal{C}_s$-symmetric framework in Fig. 3(b), the corresponding subgroup Γ of Aut(C_4) is the group consisting of the identity and $\psi_4 = (12)(34)$, as defined in Example 1. Again we identify Γ and $\mathcal{C}_s$. Since ψ_4 has two cycles (each of length 2), there are two vertex orbits. We may pick the representatives 1 and 3 for these vertex orbits. Then, there are three edge orbits: one of size 2 represented by edge 13, and two of size 1, consisting of edges 12 and 34, respectively. This leads to the quotient $\mathcal{C}_s$-gain graph of C_4 shown in Fig. 4(b). Note that the directed edge (3,1) has the identity gain since it joins two vertex orbit representatives, whereas the loops have the nontrivial gain ψ_4.

Finally, the corresponding subgroup Γ of $\mathrm{Aut}(C_4)$ for the $\mathcal{C}_2$-symmetric framework in Fig. 3(c) consists of the identity and the automorphism $\psi_2 = (14)(23)$. In this case, there are again two vertex orbits, which we may represent by the vertices 1 and 2, and two edge orbits, represented by 12 and 13. The corresponding quotient $\mathcal{C}_2$-gain graph of C_4 is shown in Fig. 4(c). It has two parallel edges from 1 to 2, one with identity gain and one with gain ψ_2.

We are now ready to define the orbit rigidity matrix, which describes the $\tau(\Gamma)$-symmetric infinitesimal rigidity properties of a $\tau(\Gamma)$-symmetric framework. In particular, as we will see in Theorem 1, its kernel and cokernel are isomorphic to the space of $\tau(\Gamma)$-symmetric infinitesimal motions and self-stresses of $(\mathcal{G}, p)$, respectively.

Definition 8: Let $\mathcal{G}$ be a Γ-symmetric graph, where the symmetry is free, and let $(\mathcal{G}, p)$ be a $\tau(\Gamma)$-symmetric framework. Further, let $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$ be the quotient Γ-gain graph of $\mathcal{G}$ and denote $\bar{p} = p|_{\mathcal{V}_0}$, the restriction of the configuration to only the representative vertices in $\mathcal{V}_0$. Then, the *orbit rigidity matrix* $\mathcal{O}(\mathcal{G}_0, \bar{p})$ of $(\mathcal{G}, p)$ is the $|\mathcal{E}_0| \times d|\mathcal{V}_0|$ matrix defined as follows. The row corresponding to an edge $((i,j);\gamma)$, where $i \neq j$, has the form

$$\left(0 \cdots 0 \;\; (\bar{p}_i - \tau(\gamma)\bar{p}_j)^T \;\; 0 \cdots 0 \;\; (\bar{p}_j - \tau(\gamma)^{-1}\bar{p}_i)^T \;\; 0 \cdots 0\right)$$

with the d-dimensional entries $(\bar{p}_i - \tau(\gamma)\bar{p}_j)^T$ and $(\bar{p}_j - \tau(\gamma)^{-1}\bar{p}_i)^T$ being in the columns corresponding to vertices i and j, respectively. The row corresponding to a loop $((i,i);\gamma)$ has the following form:

$$\left(\; 0 \cdots 0 \quad (2\bar{p}_i - \tau(\gamma)\bar{p}_i - \tau(\gamma)^{-1}\bar{p}_i)^T \quad 0 \cdots 0 \;\right)$$

with the d-dimensional entry $(2\bar{p}_i - \tau(\gamma)\bar{p}_i - \tau(\gamma)^{-1}\bar{p}_i)^T$ being in the columns corresponding to vertex i.

Example 4: The framework in Fig. 3(a) has the quotient $\mathcal{C}_4$-gain graph shown in Fig. 4(a). The matrix representing the rotation c_4 by 90° is given by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and its inverse is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. So for $p_1 = \begin{bmatrix} x_1 & y_1 \end{bmatrix}^T$, the orbit rigidity matrix is the 1×2 matrix

$$\mathcal{O}(\mathcal{G}, \bar{p}) = \begin{bmatrix} 2x_1 & 2y_1 \end{bmatrix}.$$

Note that the kernel of this matrix is the 1-D space spanned by the rotational vector $\begin{bmatrix} -y_1 & x_1 \end{bmatrix}^T$ at p_1, which corresponds to the $\mathcal{C}_4$-symmetric infinitesimal rotation of the framework via (6). So this confirms that the framework is $\mathcal{C}_4$-symmetric infinitesimally rigid. Since the cokernel is trivial, the framework has no nontrivial $\mathcal{C}_4$-symmetric self-stress, and hence, the framework is $\mathcal{C}_4$-symmetric isostatic.

The framework in Fig. 3(b) has the quotient $\mathcal{C}_s$-gain graph shown in Fig. 4(b). The matrix representing the reflection σ is given by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and this matrix is equal to its inverse. So for $p_i = \begin{bmatrix} x_i & y_i \end{bmatrix}^T$, $i = 1, 3$, the orbit rigidity matrix is the 3×4 matrix

$$\mathcal{O}(\mathcal{G}, \bar{p}) = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 & x_3 - x_1 & y_3 - y_1 \\ 4x_1 & 0 & 0 & 0 \\ 0 & 0 & 4x_3 & 0 \end{bmatrix}.$$

The kernel of this matrix is the 1-D space spanned by the vector $\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T$, which corresponds to the $\mathcal{C}_s$-symmetric vertical infinitesimal translation of the framework via (6). Thus, the framework is $\mathcal{C}_s$-symmetric infinitesimally rigid. In fact, it is $\mathcal{C}_s$-symmetric isostatic, since the cokernel is again trivial.

Note that since both frameworks are $\tau(\Gamma)$-symmetric isostatic, the removal of any edge orbit, or equivalently the removal of any edge in the quotient Γ-gain graph, yields a $\tau(\Gamma)$-symmetric infinitesimally flexible framework.

Finally, for the framework in Fig. 3(c), the quotient $\mathcal{C}_s$-gain graph is shown in Fig. 4(c). The matrix representing the half-turn is given by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and this matrix is equal to its inverse. For $p_i = \begin{bmatrix} x_i & y_i \end{bmatrix}^T$, $i = 1, 2$, the orbit rigidity matrix is the 2×4 matrix

$$\mathcal{O}(\mathcal{G}, \bar{p}) = \begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 \\ x_1 + x_2 & y_1 + y_2 & x_2 + x_1 & y_2 + y_1 \end{bmatrix}.$$

The 2-D kernel of this matrix consists of a 1-D space of vectors that correspond to infinitesimal rotations, and a 1-D space of vectors that correspond to nontrivial $\mathcal{C}_2$-symmetric infinitesimal motions of the framework [see Fig. 3(c)].

The key properties of the orbit rigidity matrix, established in [24], are summarized in the following theorem.

Theorem 1: Let $(\mathcal{G}, p)$ be a $\tau(\Gamma)$-symmetric framework with orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, \bar{p})$. Then, the following holds.

i) The kernel of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ is isomorphic to the space of $\tau(\Gamma)$-symmetric infinitesimal motions of $(\mathcal{G}, p)$.

ii) The cokernel of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ is isomorphic to the space of $\tau(\Gamma)$-symmetric self-stresses of $(\mathcal{G}, p)$.

More precisely, an element $\bar{u}$ in the kernel of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ assigns a velocity vector to each vertex orbit representative in $\mathcal{V}_0$. From this we can uniquely construct the corresponding $\tau(\Gamma)$-symmetric infinitesimal motion of $(\mathcal{G}, p)$ via (6). Conversely, if we restrict any $\tau(\Gamma)$-symmetric infinitesimal motion of $(\mathcal{G}, p)$ to $\mathcal{V}_0$, then we obtain a vector in the kernel of $\mathcal{O}(\mathcal{G}_0, \bar{p})$. Similarly, each element $\bar{\omega}$ in the cokernel of $\mathcal{O}(\mathcal{G}_0, \bar{p})$ assigns a scalar to each edge orbit representative in $\mathcal{E}_0$. From this we can uniquely construct the corresponding $\tau(\Gamma)$-symmetric self-stress of $(\mathcal{G}, p)$ via (7), and vice versa. See [24] for details.

It is a simple consequence of Theorem 1 that if a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ is $\tau(\Gamma)$-symmetric infinitesimally rigid, then the same is true for all $\tau(\Gamma)$-symmetric frameworks $(\mathcal{G}, q)$ in an open neighborhood of p (and in fact for an open dense subset of the set of $\tau(\Gamma)$-symmetric realizations of $\mathcal{G}$ as a bar-joint framework).

Using the orbit rigidity matrix, efficient Laman-type combinatorial characterizations of the graphs that can be realized as $\tau(\Gamma)$-symmetric infinitesimally rigid frameworks in the plane have been established for $\mathcal{C}_s$, $\mathcal{C}_n$, and $\mathcal{C}_{(2n+1)v}$, $n \geq 1$ (in the case where the symmetry is free) in [25]. The problem remains open for the dihedral groups $\mathcal{C}_{2nv}$. Moreover, as in the nonsymmetric situation, there are no analogs of these results in higher dimensions.

Of importance to this work is a natural corollary of Theorem 1 describing the rank of the orbit rigidity matrix.

Corollary 1: Let $(\mathcal{G}, p)$ be a $\tau(\Gamma)$-symmetric independent framework with orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, \bar{p})$. Then, $\mathcal{O}(\mathcal{G}_0, \bar{p})$ has full row-rank.

In particular, by Corollary 1, the orbit rigidity matrix $\mathcal{O}(\mathcal{G}_0, \bar{p})$ of a $\tau(\Gamma)$-symmetric isostatic framework $(\mathcal{G}, p)$ has full row-rank. In addition, it has the desired rank to guarantee $\tau(\Gamma)$-symmetric infinitesimal rigidity. This rank depends on the point group of $(\mathcal{G}, p)$ [25].

It is useful to keep in mind the parallels between the standard rigidity matrix $R(\mathcal{G}, p)$ and the orbit rigidity matrix. Corollary 1 can thus be thought of as the symmetric counterpart to the known result that the rigidity matrix of an independent framework has full row-rank. If the framework is isostatic, then it also has the desired rank to guarantee infinitesimal rigidity (namely, $2n-3$ in the plane).

IV. Forced Symmetric Formation Control

We now would like to study a variation of the formation control problem where the goal of each agent in the network is to obtain a formation corresponding to a *symmetric configuration*. In other words, starting with a Γ-symmetric graph $\mathcal{G}$ with a homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$, we would like to drive the agents to a special position $p^\star$ such that the framework $(\mathcal{G}, p^\star)$ is $\tau(\Gamma)$-symmetric.

Problem 1: Consider a group of n integrator agents (2) that interact over the Γ-symmetric sensing graph $\mathcal{G}$. Let $p^\star \in \mathbb{R}^{dn}$ be a configuration such that $(\mathcal{G}, p^\star)$ is $\tau(\Gamma)$-symmetric for some desired point group $\tau(\Gamma)$, and let $\mathcal{V}_0$ be a set of representatives of the vertex orbits of $\mathcal{G}$ under Γ. Design a control $u_i(t)$ for each agent i such that the following holds:

i) $\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = \|p_i^\star - p_j^\star\|$ for all $ij \in \mathcal{E}$;

ii) for each $i \in \mathcal{V}_0$, $\lim_{t\to\infty} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\| = 0$ for all $u, v \in \Gamma_i$.

We would like to solve Problem 1 in a distributed fashion, ideally allowing agent i to only obtain information from neighboring agents as defined by $\mathcal{G}$. Before we proceed, we comment on the information needed to solve this problem.

Requirement (i) in Problem 1 is the standard formation control constraint introduced in Section II-B. That is, $\|p_i^\star(t) - p_j^\star(t)\| = \mathbf{d}_{ij}$ are the desired distances between neighboring agents. Requirement (ii) aims to enforce the symmetric position between agents that are in the same vertex orbit. The statement of Problem 1 implicitly assumes that the requirements (i) and (ii) are consistent with each other.

We also point out that it may not be the case that $u, v \in \Gamma_i$ implies that $uv \in \mathcal{E}$. We will show that the following assumption on the subgraph induced by the vertex orbits is sufficient to realize this constraint.

Assumption 2: The subgraph induced by each vertex orbit Γ_i, denoted $\mathcal{G}(\Gamma_i)$, is connected.

Remark 1: Assumption 2 is required to ensure that all agents within the same vertex orbit eventually exchange information with each other. This is analogous to standard connectivity assumptions in consensus algorithms, of which this is a generalization [26]. Relaxation of this assumption would necessitate a more complicated control architecture, including distributed observers, to estimate the missing relative state information.

The vertex orbits also introduce a natural labeling for the agents in the network. Without loss of generality, the vertex orbit Γ_i will contain the vertices $\{i, i+1, \ldots, i+|\Gamma_i|-1\}$. Furthermore, we denote by $\mathcal{E}(\Gamma_i) \subset \mathcal{E}$ the edges in $\mathcal{G}(\mathcal{E}_i)$. Similarly, the state-vector $p(t)$ can also be partitioned as $p(t) = \begin{bmatrix} p^1(t)^T & \cdots & p^{|\mathcal{V}_0|}(t)^T \end{bmatrix}^T$, where $p^i(t) \in \mathbb{R}^{d|\Gamma_i|}$, and $p_1^i(t)$ is the position associated to vertex $i \in \mathcal{V}_0$.

A. Symmetry Potentials and Formation Stabilization

Our approach to solve Problem 1 follows the same gradient dynamical system approach used in solving the standard formation control problem, as introduced in Section II-B. We provide first a brief summary of our result from [12], together with some new results. In this direction, we define a *symmetry-forcing potential*

$$F_s(p(t)) = \frac{1}{2} \sum_{i\in\mathcal{V}_0} \sum_{\substack{u,v\in\Gamma_i \\ uv\in\mathcal{E}}} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\|^2. \tag{8}$$

Here, recall that $\mathcal{V}_0$ is a set of representatives from each of the vertex orbits of a Γ-symmetric graph. The *symmetric formation potential* can then be defined as

$$F(p(t)) = F_f(p(t)) + F_s(p(t)) \tag{9}$$

where $F_f(p(t))$ is the formation potential defined in (3).

We now propose the control

$$u(t) = -\nabla F(p(t)) \tag{10}$$

to solve Problem 1. The closed-loop dynamics then take the following form:

$$\dot{p}(t) = -R(\mathcal{G}, p(t))^T \left(R(\mathcal{G}, p(t))p(t) - \mathbf{d}^2\right) - Qp(t), \tag{11}$$

where Q is a block diagonal matrix with $|\mathcal{V}_0|$ blocks, where each block is $|\Gamma_i|d \times |\Gamma_i|d$. With the labeling of the nodes defined earlier, Q can always be written as

$$Q = \begin{bmatrix} Q_1 & & \\ & \ddots & \\ & & Q_{|\mathcal{V}_0|} \end{bmatrix}$$

and

$$[Q_i]_{uv} = \begin{cases} d_{\Gamma_i}(u)I, & u = v,\ u \in \Gamma_i \\ -\tau(\gamma_{uv}), & uv \in \mathcal{E}, u, v \in \Gamma_i \\ 0, & \text{o.w.} \end{cases}$$

where $d_{\Gamma_i}(u)$ denotes the degree of node u in the induced subgraph $\mathcal{G}(\Gamma_i)$. Observe that Q_i can be expressed as the matrix product $E(\Gamma_i)E(\Gamma_i)^T$, where $E(\Gamma_i)$ has a similar structure as the incidence matrix. For an edge $k = uv$ with $u, v \in \Gamma_i$, one has $[E(\Gamma_i)]_{uk} = I$ if the edge k leaves vertex u, $[E(\Gamma_i)]_{uk} =$

$-\tau(\gamma_{vu})$ if the edge k enters the vertex u, and $[E(\Gamma_i)]_{uk} = 0$ otherwise. Therefore Q_i (and consequently Q) is a positive semidefinite matrix. To further simplify notation, we define $\bar{E}(\Gamma) = \mathrm{diag}\{E(\Gamma_i)\}_{i=1:|\mathcal{V}_0|}$, and therefore, $Q = \bar{E}(\Gamma)\bar{E}(\Gamma)^T$. It follows that any configuration that is in a symmetric position must lie in the kernel of Q.

It is also useful to examine the expression of the closed-loop dynamics for each agent. For an agent i in the vertex orbit Γ_u, the dynamics are

$$\dot{p}_i(t) = \sum_{ij\in\mathcal{E}} (\|p_i(t) - p_j(t)\|^2 - (\mathbf{d}_{ij})^2)(p_j(t) - p_i(t)) + \sum_{\substack{ij\in\mathcal{E}\\ i,j\in\Gamma_u}} (\tau(\gamma_{ij})p_j(t) - p_i(t)).$$

We now show that the closed-loop dynamics (11) has an invariant quantity.

Proposition 1: Consider the closed-loop dynamics (11) and let

$$z(t) = \sum_{v\in\mathcal{V}}\sum_{\gamma\in\Gamma} \tau(\gamma)p_v(t). \tag{12}$$

Then, $z(t)$ remains invariant along the trajectories of (11), i.e., $\dot{z}(t) = 0$.

Proof: We examine the derivative of $z(t)$

$$\dot{z}(t) = \sum_{v\in\mathcal{V}}\sum_{\gamma\in\Gamma} \tau(\gamma)\dot{p}_v(t). \tag{13}$$

We will study separately the contribution of the distance constraint term and the symmetry-forcing term from each agent in (13).

To begin, we examine the symmetry-forcing contribution to $\dot{z}(t)$ from an edge $ij \in \mathcal{E}$ with $i, j \in \Gamma_u$ for any $u \in \mathcal{V}_0$. This leads to

$$\begin{aligned}&\sum_{\gamma\in\Gamma} \tau(\gamma)\left(\tau(\gamma_{ij})p_j - p_i\right) + \sum_{\gamma\in\Gamma} \tau(\gamma)\left(\tau(\gamma_{ji})p_i - p_j\right)\\ &= \left(\left(\sum_{\gamma\in\Gamma}\tau(\gamma)\right)\tau(\gamma_{ij}) - \sum_{\gamma\in\Gamma}\tau(\gamma)\right)p_j(t)\\ &\quad + \left(\left(\sum_{\gamma\in\Gamma}\tau(\gamma)\right)\tau(\gamma_{ji}) - \sum_{\gamma\in\Gamma}\tau(\gamma)\right)p_i(t) = 0\end{aligned}$$

where the last equation follows from the fact that $(\sum_{\gamma\in\Gamma}\tau(\gamma))\tau(\gamma') = \sum_{\gamma\in\Gamma}\tau(\gamma)$ for any $\gamma' \in \Gamma$. Note that this holds for each edge connecting agents in the same vertex orbit.

We now look at the contribution from the distance constraint terms in the agent dynamics. Consider again the edge $ij \in \mathcal{E}$ (with no restriction on agent j being in the same vertex orbit of agent i). Let $a_{ij} = a_{ji} = \|p_i - p_j\|^2 - \mathbf{d}_{ij}^2$. Then, it is straightforward to verify that

$$\sum_{\gamma\in\Gamma}\tau(\gamma)a_{ij}(p_j - p_i) + \sum_{\gamma\in\Gamma}\tau(\gamma)a_{ji}(p_i - p_j) = 0.$$

This also holds for each edge in $\mathcal{E}$. Together this shows that $\dot{z}(t) = 0$ as claimed. □

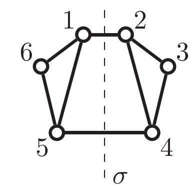


Fig. 5. $\tau(\Gamma)$-symmetric graph with y-axis symmetry not satisfying Assumption 2.

We now present the main result of [12].

Theorem 2 (See [12]): Consider a team of n agents (2) interacting over a Γ-symmetric graph $\mathcal{G}$ (with a homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$) satisfying Assumption 2 in $\mathbb{R}^d$, and let

$$\mathcal{F}_f = \{p \in \mathbb{R}^{dn} \,|\, \|p_i - p_j\| = \mathbf{d}_{ij}\, ij \in \mathcal{E}\}$$

and

$$\mathcal{F}_s = \{p \in \mathbb{R}^{dn} \,|\, \tau(\gamma)(p_i) = p_{\gamma(i)} \quad \forall\gamma \in \Gamma,\, i \in \mathcal{V}\}.$$

Then, for initial conditions $p_i(0)$ satisfying

$$\sum_{ij\in\mathcal{E}} (\|p_i(0) - p_j(0)\| - \mathbf{d}_{ij}^2) \le \epsilon_1,$$

$$\text{and } \|p_i(0) - \tau(\gamma_j)p_j(0)\|^2 \le \epsilon_2$$

for all $i \in \mathcal{V}_0$ and $j \in \Gamma_i$, for a sufficiently small and positive constant ϵ_1 and ϵ_2, the control

$$u = -\nabla F(p(t)) \tag{14}$$

renders the set $\mathcal{F}_f \cap \mathcal{F}_s$ exponentially stable, i.e.,

$$\lim_{t\to\infty}\|p_i(t) - p_j(t)\| = \mathbf{d}_{ij} \text{ and } \lim_{t\to\infty}\tau(\gamma)(p_i(t)) = \lim_{t\to\infty}p_{\gamma(i)}(t)$$

for all $\gamma \in \Gamma, i \in \mathcal{V}$.

The implementation of the control (11) assumes that agents within the same vertex orbit are able to exchange information with each other (i.e., Assumption 2 is satisfied). This may not be the case for different symmetries, as illustrated in the next example.

Example 5: Consider the graph in Fig. 5 . This is a $\tau(\Gamma)$-symmetric framework with the point group symmetry specified by the reflection in the y-axis. The vertex orbits for this symmetry are $\Gamma_1 = \Gamma_2 = \{1, 2\}$, $\Gamma_3 = \Gamma_6 = \{3, 6\}$, and $\Gamma_4 = \Gamma_5 = \{4, 5\}$. Note that the nodes in the orbit Γ_3 are not connected by edges in $\mathcal{G}$. Since Assumption 2 is not satisfied, implementation of (10) requires establishing additional communication between nodes 3 and 6.

We also note that the implementation of (11) requires $|\mathcal{E}|$ communication/sensing links for each distance constraint, and that the subgraph induced by the vertex orbits are connected. In fact, Assumption 2 can be relaxed further to require only a spanning tree for each subgraph induced by the vertex orbits.

Corollary 2: The conditions stated in Theorem 2 hold if the induced subgraph of each vertex orbit, $\mathcal{G}(\Gamma_i)$, is a spanning tree.

The proof remains unchanged from Theorem 2. The corollary shows that symmetry between agents in the same vertex orbit, say Γ_i, can be maintained with only $|\Gamma_i| - 1$ edges. Nevertheless, the control proposed in Theorem 2 still requires to use all the edges encoding the distance constraints. We now show that this too is redundant and only edges from the representative edge set $\mathcal{E}_0$ need to be considered.

B. Orbit Rigidity Potentials and Formation Stabilization

In this direction, we define a new potential function that aims to satisfy the distance constraints over the representative edges in the edge orbits

$$F_e(p(t)) = \frac{1}{4} \sum_{e=ij\in\mathcal{E}_0} \left(\|p_i - \tau(\gamma)p_j\|^2 - \mathbf{d}^2_{i\gamma(j)} \right)^2$$

where $\gamma \in \Gamma$ is the label of the edge in the quotient gain graph. Thus, the new forced-symmetric formation control potential can be expressed as

$$F(p(t)) = F_e(p(t)) + F_s(p(t)) \tag{15}$$

where $F_s(p(t))$ was defined in (8). As before, we now propose the control

$$u(t) = -\nabla F(p(t)). \tag{16}$$

To help simplify notations, we denote by $p_0(t)$ the restriction of the configuration vector $p(t)$ to only the agents in the representative vertex set $\mathcal{V}_0$. The remaining agents are denoted by the vector $p_f(t)$, so with an appropriate labeling of the agents we can write $\tilde{p}(t) = Pp(t) = \begin{bmatrix} p_0^T(t) & p_f^T(t) \end{bmatrix}^T$, for some permutation matrix P. Then, the control for each agent $i \in \mathcal{V}_0$ can be expressed as

$$u_i(t) = u_i^{(a)}(t) + u_i^{(b)}(t) + u_i^{(c)}(t) \tag{17}$$

where

$$u_i^{(a)}(t) = \sum_{\substack{i\gamma(j)\in\mathcal{E}_0 \\ j\in\mathcal{V}_0,\, i\neq j}} \left(\|p_i(t) - \tau(\gamma)p_j(t)\|^2 - \mathbf{d}^2_{ij}\right)\left(\tau(\gamma)p_j(t) - p_i(t)\right)$$

$$u_i^{(b)}(t) = -\sum_{i\gamma(i)\in\mathcal{E}_0} \left(\|(I-\tau(\gamma))p_i\|^2 - \mathbf{d}^2_{i\gamma(i)}\right)\left(2I - \tau(\gamma) - \tau(\gamma)^{-1}\right)p_i$$

$$u_i^{(c)}(t) = \sum_{ij\in\mathcal{E}(\Gamma_i)} \left(\tau(\gamma_{ij})p_j(t) - p_i(t)\right).$$

The control for the agents in $\mathcal{V} \setminus \mathcal{V}_0$ is simply

$$u_i(t) = \sum_{ij\in\mathcal{E}(\Gamma_v)} \left(\tau(\gamma_{ij})p_j(t) - p_i(t)\right) \tag{18}$$

for each $v \in \mathcal{V}_0$.

When expressing the dynamics in a state-space form, the orbit rigidity matrix explicitly appears, and we obtain

$$\begin{bmatrix} \dot{p}_0(t) \\ \dot{p}_f(t) \end{bmatrix} = \begin{bmatrix} -\mathcal{O}^T(\mathcal{G}_0, p_0(t))\left(\mathcal{O}(\mathcal{G}_0, p_0(t))p_0(t) - \mathbf{d}_0^2\right) \\ 0 \end{bmatrix} - PQP^T \begin{bmatrix} p_0(t) \\ p_f(t) \end{bmatrix}. \tag{19}$$

Here, $\mathbf{d}_0$ are the distance constraints for the edges in $\mathcal{E}_0$. It is interesting to observe the structure of this system. The orbit rigidity matrix in (19) plays the role of the rigidity matrix for (4) in preserving the distances.

It is now useful to define an error system to study the stability and convergence properties of the system. In this direction, we let

$$\bar{\sigma}(t) = \mathcal{O}(\mathcal{G}_0, p_0(t))p_0(t) - \mathbf{d}_0^2, \text{ and } \bar{q}(t) = \bar{E}(\Gamma)^T P^T p(t)$$

with $\bar{e}(t) = \begin{bmatrix} \bar{\sigma}(t)^T & \bar{q}(t)^T \end{bmatrix}^T$. Then, the error dynamics can be expressed as

$$\begin{aligned} \begin{bmatrix} \dot{\bar{\sigma}}(t) \\ \dot{\bar{q}}(t) \end{bmatrix} &= -\begin{bmatrix} \mathcal{O}\mathcal{O}^T & \mathcal{O}\bar{E}_0(\Gamma) \\ \bar{E}_0^T(\Gamma)\mathcal{O}^T & \bar{E}^T(\Gamma)\bar{E}(\Gamma) \end{bmatrix} \begin{bmatrix} \bar{\sigma}(t) \\ \bar{q}(t) \end{bmatrix} \\ &= -\begin{bmatrix} \begin{bmatrix} \mathcal{O} & 0 \end{bmatrix} \\ \bar{E}^T(\Gamma)P^T \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathcal{O}^T \\ 0^T \end{bmatrix} & P\bar{E}(\Gamma) \end{bmatrix} \begin{bmatrix} \bar{\sigma}(t) \\ \bar{q}(t) \end{bmatrix}. \end{aligned} \tag{20}$$

Here, $\begin{bmatrix} \bar{E}_0(\Gamma)^T & \bar{E}_f(\Gamma)^T \end{bmatrix}^T = P\bar{E}(\Gamma)$. We refer to (20) as the *orbit error system*. For notational simplicity, we have dropped the explicit dependence of the orbit rigidity matrix on $\mathcal{G}_0$ and $p_0(t)$. We also can characterize its equilibria by the set

$$\mathcal{X}_0 = \left\{(\bar{\sigma}, \bar{q}) \mid \mathcal{O}^T\bar{\sigma} + \bar{E}_0(\Gamma)\bar{q} = 0 \text{ and } \bar{E}_f(\Gamma)\bar{q} = 0\right\}. \tag{21}$$

Our first result shows that the set $\mathcal{X}_0$ is asymptotically stable.

Theorem 3: Consider the error system (20) for a $\tau(\Gamma)$-symmetric framework satisfying Assumption 1. The set of equilibria (20) is asymptotically stable. Furthermore, on the set $\mathcal{X}_0$, the control $u(t) = 0$ and the configuration vector $p(t)$ converge to a fixed point under the dynamics (19).

Proof: We define the Lyapunov function

$$V(\bar{e}(t)) = \frac{1}{2}\begin{bmatrix} \bar{\sigma}(t)^T & \bar{q}(t)^T \end{bmatrix} \begin{bmatrix} \bar{\sigma}(t) \\ \bar{q}(t) \end{bmatrix} = \frac{1}{2}\bar{e}(t)^T\bar{e}(t).$$

The derivative of V along the trajectories of (20) is

$$\begin{aligned} \dot{V}(\bar{e}(t)) &= -\bar{e}(t)^T \underbrace{\begin{bmatrix} \begin{bmatrix} \mathcal{O} & 0 \end{bmatrix} \\ \bar{E}^T(\Gamma)P^T \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathcal{O}^T \\ 0^T \end{bmatrix} & P\bar{E}(\Gamma) \end{bmatrix}}_{\mathcal{M}} \bar{e}(t) \\ &= -\left\| \begin{bmatrix} \begin{bmatrix} \mathcal{O}^T \\ 0^T \end{bmatrix} & P\bar{E}(\Gamma) \end{bmatrix} \bar{e}(t) \right\|^2 \leq 0. \end{aligned}$$

Note that for any $\rho > 0$, the set $\Psi(\rho) = \{\bar{e} : V(\bar{e}) \leq \rho\}$ such that $\bar{e}(0) \in \Psi(\rho)$ is compact and positively invariant with respect to (20). Therefore, by LaSalle's Theorem, every solution starting in $\Psi(\rho)$ must approach the largest invariant set where $\dot{V}(\bar{e}) = 0$, which is precisely the set $\mathcal{X}_0$.

Now, observe from (19) that the control can be expressed as follows:

$$u(t) = -\begin{bmatrix} \mathcal{O}^T \\ 0 \end{bmatrix} \bar{\sigma}(t) - P\bar{E}(\Gamma)\bar{q}.$$

Therefore, on the set $\mathcal{X}_0$, it follows that $u(t) \equiv 0$. Since $u(t) \to 0$, it follows that $p(t)$ must converge to a constant value. □

Theorem 3 guarantees that both the orbit error dynamics and the formation dynamics behave nicely. However, the set $\mathcal{X}_0$ itself may be difficult to characterize, and Theorem 3 does not guarantee, for example, convergence to the correct symmetric formation shape. Analogous to classic results from formation control (see [9]), by imposing additional assumptions on the framework, we can show that, in fact, the error dynamics locally converge exponentially fast to the origin.

Theorem 4: Let $p^\star$ be the target formation satisfying conditions (i) and (ii) of Problem 1, and assume that $(\mathcal{G}, p^\star)$ is a $\tau(\Gamma)$-symmetric isostatic framework satisfying Assumption 1. Furthermore, assume that Assumption 2 holds and that the matrices $E(\Gamma_i)$ are constructed using only a spanning tree subgraph of $\mathcal{G}(\Gamma_i)$. Then, the origin is a locally exponentially stable equilibrium of (20).

Proof: Let $\Psi(\rho) = \{\bar{e} : V(\bar{e}) \leq \rho\}$ for a sufficiently small ρ such that for all points in $\Psi(\rho)$, the formation is $\tau(\Gamma)$-symmetric isostatic. By Corollary 1, the orbit rigidity matrix must have full row-rank on this set. We now consider the set $\mathcal{X}_0$ in (20). Note that by Assumption 2, the matrices $E(\Gamma_i)$ have full column rank, and in particular, so must $\bar{E}_f(\Gamma)$. Therefore, $\bar{q} = 0$. Similarly, since $\mathcal{O}^T$ also has full column rank in $\Psi(\rho)$, it follows that $\bar{\sigma} = 0$, and we conclude that $\mathcal{X}_0 = \{0\}$.

We conclude the proof using the same Lyapunov function from the proof of Theorem 3. Observe that due to the assumptions of the corollary, $\lambda_{\min} = \min_{e \in \Psi(\rho)} \lambda(\mathcal{M}) > 0$. Since $\Psi(\rho)$ is compact, the existence of $\lambda_{\min}$ is guaranteed. Then one has

$$\dot{V}(\bar{e}(t)) \leq -\lambda_{\min} \|\bar{e}(t)\|^2 = -2\lambda_{\min} V(\bar{e}(t))$$

which indicates that $V(e(t))$ is negative definite for $e(t) \in \Psi(\rho) \setminus \{0\}$. Thus, the exponential stability of the equilibrium $e = 0$ in the error system (20) is proved. □

In summary, minimally forced-symmetric rigidity provides an architecture for Problem 1 that ensures (local) exponential stability to the desired symmetric formation shape. As in rigidity-based formation control strategies, this result is local since we must still be concerned with flip ambiguities of the framework. We also note that the controlled system (19) requires fewer edges than the ordinary rigidity-based control.

Theorem 5: In the setting of Theorem 4, the controlled system (19) uses at most $(1 + 1/|\Gamma|)|\mathcal{V}|$ edges.

Proof: Let $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$ be the quotient graph as used in the description of the control. By the assumption that the vertex set of each orbit induces a tree in the communication graph, the control uses $|\mathcal{E}_0| + (|\Gamma| - 1)|\mathcal{V}_0|$ edges. It is known that if $(\mathcal{G}, p^*)$ is a $\tau(\Gamma)$-symmetric isostatic framework, then $|\mathcal{E}_0| \leq 2|\mathcal{V}_0|$; see, e.g., [13, Thm. 62.1.4]. This relation is the symmetric analog of the well-known fact that any isostatic framework has $2|\mathcal{V}| - 3$ edges. Hence, the number of edges in the control is bounded by $(|\Gamma| + 1)|\mathcal{V}_0|$. By the assumption that Γ acts freely on $\mathcal{V}$, every vertex orbit is of size $|\Gamma|$, and hence, $|\mathcal{V}| = |\Gamma||\mathcal{V}_0|$. Thus, $(|\Gamma| + 1)|\mathcal{V}_0| = (1 + 1/|\Gamma|)|\mathcal{V}|$. □

The bound in Theorem 5 is significantly smaller than that required for stabilizing infinitesimally rigid frameworks without any additional symmetry constraints (which requires at least $2|\mathcal{V}| - 3$ edges).

It should also be noted that if a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p^*)$ is infinitesimally rigid in the ordinary sense (that is, the framework has a rigidity matrix with rank $2|\mathcal{V}| - 3$), then one can always extract a subgraph $\mathcal{H}$ such that $(\mathcal{H}, p^*)$ is $\tau(\Gamma)$-symmetric isostatic. Hence, as long as Assumption 2 holds, our control can be applied to a subgraph of $\mathcal{G}$ with $|\mathcal{E}_0| + (|\Gamma| - 1)|\mathcal{V}_0|$ edges.

Example 6: We now consider the graph on $n = 10$ nodes shown in Fig. 6(a). This is a $\tau(\Gamma)$-symmetric framework with Γ corresponding to the rotational group, where $\tau(\gamma)$ is the rotation matrix of $2\pi/5$ radians. Note that this graph has 15 edges. To solve the formation control problem using the standard approach in (4), we would require an additional two edges to ensure MIR of the framework (17 total). On the other hand, the control strategy in (19) requires only nine edges, with the subgraph shown in Fig. 6(b). The quotient graph for this framework is shown in Fig. 6(c). Fig. 6(d) shows the resulting trajectories when running (19).

Remark 2: One well-known challenge in formation controllers of the form (4) is to avoid the invariant subspace of colinear solutions [5]. While the symmetry-forced formation control (19) is not immune to this problem, it depends greatly on the chosen symmetry. For example, rotational symmetries, as in Example 6, do not have colinear solutions as an invariant set, while a reflection symmetry only has colinear invariant solutions that are orthogonal to the mirror axis.

Of note is that the symmetries used to define the $\tau(\Gamma)$-symmetric framework are defined with respect to a common inertial frame [see Fig. 6(d)]. In the sequel, we propose a modification to (20) to relax this point.

C. Symmetry Forced Formations With Centroid Consensus

The $\tau(\Gamma)$-symmetric frameworks by definition have the point-group symmetries defined with respect to some fixed inertial point (the origin). As seen in Fig. 6(d), for agent initial conditions that are far from the origin, the system in (19) will drive the agent to the correct formation, but with respect to the origin. In fact, it would be more desirable for the agents to be able to arrange themselves to the correct symmetric formation with respect to *any* point. This is further a necessary requirement if we want to include formation maneuvering as well.

In this direction, we propose the addition of a consensus term that will allow the agents to distributedly agree on a different origin that is a function of the initial conditions. Each agent will augment its state-vector with the centroid estimate, $r_i(t) \in \mathbb{R}^d$. The classic consensus algorithm is then run on these virtual states

$$\dot{r}_i(t) = \sum_{ij \in \mathcal{E}} (r_j(t) - r_i(t)) \Leftrightarrow \dot{r}(t) = -(L(\mathcal{G}) \otimes I_d) r(t) \quad (22)$$

where $L(\mathcal{G})$ is the combinatorial graph Laplacian matrix of $\mathcal{G}$. It is well-known that under the assumption that $\mathcal{G}$ is connected, we have [26]

$$r(t) \to \mathbf{r}^\star = \mathbb{1}_{|\mathcal{V}|} \otimes \left((1/|\mathcal{V}|)(\mathbb{1}^T \otimes I_d) r(0) \right). \quad (23)$$

We are now able to use the virtual state $r_i(t)$ for each agent to effectively shift the origin in the control (19).

We define the shifted state $\bar{c}(t) = \begin{bmatrix} c_0^T(t) & c_f^T(t) \end{bmatrix}^T = P(p(t) - r(t))$. The modified control, together with (21), then

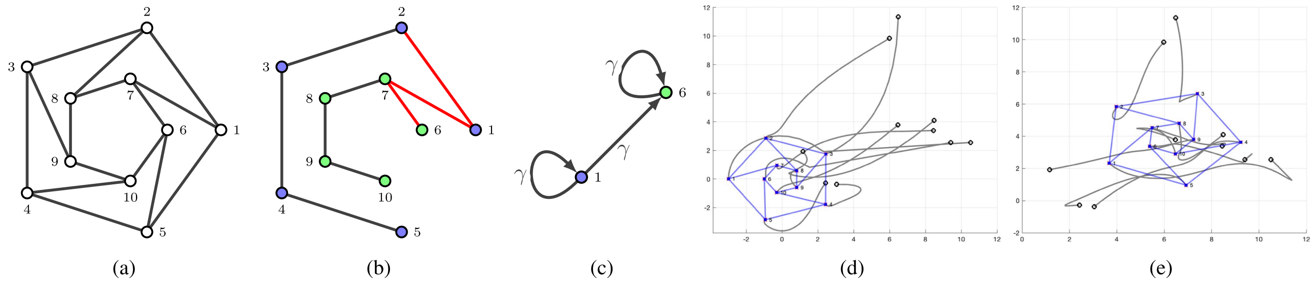


Fig. 6. Results from Example 6. We consider the $\tau(\Gamma)$-symmetric framework in (a) with the $2\pi/5$ rotational symmetry. The framework has two vertex orbits, indicated by the blue and green nodes in (b), along with three edges in the edge orbit [red edges in (b)]. For the implementation of (19), we only require a spanning tree subgraph from each $\mathcal{G}(\Gamma_i)$ and the edges in $\mathcal{E}_0$, shown in (b). (c) Corresponding quotient gain graph. (d) and (e) Resulting trajectories without and with the additional consensus term, respectively. (a) A $\tau(\Gamma)$-symmetric isostatic framework. (b) Subgraph used to implement control strategy. (c) Quotient graph. (d) Trajectories generated from (19). (e) Trajectories with centroid consensus term, (22) and (24).

takes the following form:

$$\begin{bmatrix}\dot{p}_0(t)\\ \dot{p}_f(t)\end{bmatrix} = \begin{bmatrix}-\mathcal{O}^T(\mathcal{G}_0, c_0(t))\left(\mathcal{O}(\mathcal{G}_0, c_0(t))c_0(t) - \mathbf{d}_0^2\right)\\ 0\end{bmatrix} - PQP^T\begin{bmatrix}c_0(t)\\ c_f(t)\end{bmatrix}. \tag{24}$$

The combined dynamics (21) and (23) are in a cascade form. Following the same approach as before, we define the error signals:

$$\bar{\sigma}(t) = \mathcal{O}(\mathcal{G}_0, c_0(t))c_0(t) - \mathbf{d}_0^2, \text{ and } \bar{q}(t) = \bar{E}(\Gamma)^T P^T \bar{c}(t).$$

The error dynamics can now be expressed as

$$\begin{bmatrix}\dot{\bar{\sigma}}(t)\\ \dot{\bar{q}}(t)\end{bmatrix} = -\begin{bmatrix}\mathcal{O}(\mathcal{G}_0, c_0(t))\mathcal{O}^T(\mathcal{G}_0, c_0(t)) & \mathcal{O}(\mathcal{G}_0, c_0(t))\bar{E}_0(\Gamma)\\ \bar{E}_0^T(\Gamma)\mathcal{O}^T(\mathcal{G}_0, c_0(t)) & \bar{E}^T(\Gamma)\bar{E}(\Gamma)\end{bmatrix}\begin{bmatrix}\bar{\sigma}(t)\\ \bar{q}(t)\end{bmatrix} + \begin{bmatrix}\begin{bmatrix}\mathcal{O}(\mathcal{G}_0, c_0(t)) & 0\end{bmatrix} P(L(\mathcal{G})\otimes I_d)\\ \bar{E}^T(\Gamma)(L(\mathcal{G})\otimes I_d)\end{bmatrix} r(t). \tag{25}$$

To assist in our analysis of the cascade system (22) and (25), we perform a simple change of coordinates. Let $\hat{p}(t) = p(t) - \mathbf{r}^\star$, $\hat{r}(t) = r(t) - \mathbf{r}^\star$. With this definition, note that $\hat{c}(t) = \hat{p}(t) - \hat{r}(t) = c(t)$, similarly for the error signals $\bar{\sigma}(t)$ and $\bar{q}(t)$ (i.e., $\hat{\sigma}(t) = \bar{\sigma}(t)$ and $\hat{q}(t) = \bar{q}(t)$). With this definition, the shifted cascade system can be represented as

$$\begin{bmatrix}\dot{\hat{\sigma}}(t)\\ \dot{\hat{q}}(t)\end{bmatrix} = -\begin{bmatrix}\mathcal{O}(\mathcal{G}_0, \hat{c}_0(t))\mathcal{O}^T(\mathcal{G}_0, \hat{c}_0(t)) & \mathcal{O}(\mathcal{G}_0, \hat{c}_0(t))\bar{E}_0(\Gamma)\\ \bar{E}_0^T(\Gamma)\mathcal{O}^T(\mathcal{G}_0, \hat{c}_0(t)) & \bar{E}^T(\Gamma)\bar{E}(\Gamma)\end{bmatrix}\begin{bmatrix}\hat{\sigma}(t)\\ \hat{q}(t)\end{bmatrix} + \begin{bmatrix}\begin{bmatrix}\mathcal{O}(\mathcal{G}_0, \hat{c}_0(t)) & 0\end{bmatrix} P(L(\mathcal{G})\otimes I_d)\\ \bar{E}^T(\Gamma)(L(\mathcal{G})\otimes I_d)\end{bmatrix} \hat{r}(t) \tag{26}$$

$$\dot{\hat{r}}(t) = -(L(\mathcal{G})\otimes I_d)\hat{r}(t). \tag{27}$$

We will now show that the cascade system (26) and (24) is locally exponentially stable. To do so, we must first show that these dynamics are locally input-to-state stable [27].

Theorem 6: Let $p^\star$ be the target formation such that $p^\star$ satisfies conditions (i) and (ii) of Problem 1, and assume that $(\mathcal{G}, p^\star)$ is a $\tau(\Gamma)$-symmetric isostatic framework satisfying Assumption 1. Assume that Assumption 2 holds and that the matrices $E(\Gamma_i)$ are constructed using only a spanning tree subgraph of $\mathcal{G}(\Gamma_i)$. Then, the cascade system (26) and (24) is locally exponentially stable.

Proof: First, we establish that (26) is locally input-to-state stable while interpreting the signal $\hat{r}(t)$ as the input. When $\hat{r}(t) = 0$, we have that $\hat{c}_0(t) = \hat{p}_0(t)$ and the dynamics (26) reduce to (20). Using the same arguments as in Theorem 4, this system is locally exponentially stable, which shows the system must be locally input-to-state stable [27]. To establish that the cascade system is exponentially stable, all we need is to recall that (24) is exponentially stable and converges from any initial condition to the origin [26]. It can then be concluded that the cascade system is also locally exponentially stable [28]. □

Finally, we can establish the agent trajectories that will then converge to a symmetric configuration with respect to the centroid of the virtual state $r(t)$, defined in (22).

Theorem 7: Consider the cascade dynamics (22) and (23) and assume that $(\mathcal{G}, p^\star - \mathbf{r}^\star)$ is a $\tau(\Gamma)$-symmetric isostatic framework. Then, for all initial conditions sufficiently close to $p^\star$, and $\mathbf{r}^\star$ given in (22), $p(t)$ exponentially converges to $p^\star$.

Proof: From Theorem 6, we have that $\hat{\sigma}(t)$ and $\hat{q}(t)$ converge to the origin. The remainder of the proof follows the same argument as Theorem 3 but on the shifted state $\hat{p}(t)$. Therefore, since $\hat{p}(t)$ locally and exponentially converges to a $\tau(\Gamma)$-symmetric isostatic framework, $\hat{p}(t) \to p^\star - \mathbf{r}^\star$, it follows that $p(t) \to p^\star + \mathbf{r}^\star$. □

This result assumes that $p(0) - \mathbf{r}^\star$ is sufficiently close to a $\tau(\Gamma)$-symmetric configuration. This may be problematic since in general the vector $\mathbf{r}^\star$ may not be globally known to all agents. On the other hand, it is only as restrictive as Theorem 3 and so does not introduce any new conservativeness.

Example 6 (Continued): We return to the same setup as Example 6 and now implement the centroid consensus version of the dynamics (23). The resulting trajectories can be seen in

Fig. 6(e). Here, we initialize the virtual state $r(t)$ to be the initial condition of the agents, $r(0) = p(0)$. The formation converges to the correct symmetry-forced configuration, but now with respect to the centroid of the initial conditions.

V. Conclusion

This work presented a formation control strategy for controlling a network of agents to a formation characterized by its symmetry properties. Leveraging recent results from symmetry-forced rigidity theory, a gradient control strategy was derived with two main components based on a chosen symmetry group of the formation: one maintains distances between edges in the edge orbits, while the other forces agents within the same vertex orbit to a symmetric position. The stability results turn out to be related to properties of the so-called orbit rigidity matrix, which is a central object in symmetry-force rigidity theory. While the results in this work focused only on symmetries that are free, extending these ideas to the nonfree case should be straightforward and is the topic of future research.

References

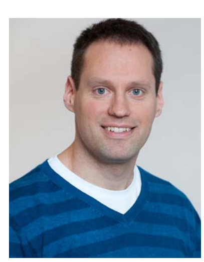
[1] Y. Igarashi, T. Hatanaka, M. Fujita, and M. W. Spong, "Passivity-based attitude synchronization in $SE(3)$," *IEEE Trans. Control Syst. Technol.*, vol. 17, no. 5, pp. 1119–1134, Sep. 2009.
[2] P. A. Rosen et al., "Synthetic aperture radar interferometry," *Proc. IEEE*, vol. 88, no. 3, pp. 333–382, Mar. 2000.
[3] S.-L. Dai, S. He, H. Lin, and C. Wang, "Platoon formation control with prescribed performance guarantees for USVs," *IEEE Trans. Ind. Electron.*, vol. 65, no. 5, pp. 4237–4246, May 2018.
[4] L. Asimow and B. Roth, "The rigidity of graphs," *Trans. Amer. Math. Soc.*, vol. 245, pp. 279–289, 1978.
[5] L. Krick, M. E. Broucke, and B. A. Francis, "Stabilisation of infinitesimally rigid formations of multi-robot networks," *Int. J. Control*, vol. 82, no. 3, pp. 423–439, 2009.
[6] M. De Queiroz, X. Cai, and M. Feemster, *Formation Control of Multi-Agent Systems: A Graph Rigidity Approach*. Hoboken, NJ, USA: Wiley, 2019.
[7] H.-S. Ahn, *Formation Control: Approaches for Distributed Agents*. Berlin, Germany: Springer, 2020.
[8] S. Zhao and D. Zelazo, "Bearing rigidity theory and its applications for control and estimation of network systems: Life beyond distance rigidity," *IEEE Control Syst. Mag.*, vol. 39, no. 2, pp. 66–83, Apr. 2019.
[9] Z. Sun, S. Mou, B. D. Anderson, and M. Cao, "Exponential stability for formation control systems with generalized controllers: A unified approach," *Syst. Control Lett.*, vol. 93, pp. 50–57, Jul. 2016.
[10] D. V. Dimarogonas and K. H. Johansson, "On the stability of distance-based formation control," in *Proc. 47th IEEE Conf. Decis. Control*, 2008, pp. 1200–1205.
[11] F. Dörfler and B. Francis, "Geometric analysis of the formation problem for autonomous robots," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2379–2384, Oct. 2010.
[12] D. Zelazo, B. Schulze, and S. Tanigawa, "Stabilization of symmetric formations," in *Proc. IFAC World Congr.*, 2023, pp. 11316–11321.
[13] *Handbook of Discrete and Computational Geometry*, J. E. Goodman, C. D. Tóth, and J. O'Rourke, Eds., 3rd ed., CRC Press, Boca Raton, FL, USA, 2018.
[14] R. Connelly and S. D. Guest, *Frameworks, Tensegrities, and Symmetry*. Cambridge, U.K.: Cambridge Univ. Press, 2022.
[15] W. Whiteley, "Some matroids from discrete applied geometry," in *Matroid Theory.*Providence, RI, USA: Amer. Math. Soc., 1996, pp. 171–311.
[16] H. Pollaczek-Geiringer, "Über die Gliederung ebener Fachwerke," *ZAMM-J. Appl. Math. Mech./Angewandte Mathematik Mechanik*, vol. 7, pp. 58–72, 1927.
[17] G. Laman, "On graphs and rigidity of plane skeletal structures," *J. Eng. Math.*, vol. 4, pp. 331–340, 1970.
[18] D. Dummit and R. Foote, *Abstract Algebra*. Hoboken, NJ, USA: Wiley, 2004.
[19] S. L. Altmann and P. Herzig, *Point-Group Theory Tables*. Oxford, U.K.: Clarendon Press, 1994.
[20] P. W. Atkins, M. S. Child, and C. S. G. Phillips, *Tables for Group Theory*. London, U.K.: Oxford Univ. Press, 1970.
[21] R. D. Kangwai and S. D. Guest, "Symmetry-adapted equilibrium matrices," *Int. J. Solids Structures*, vol. 37, pp. 1525–1548, 2000.
[22] J. C. Owen and S. C. Power, "Frameworks symmetry and rigidity," *Int. J. Comput. Geometry Appl.*, vol. 20, pp. 723–750, 2010.
[23] R. Ikeshita, "Infinitesimal rigidity of symmetric frameworks," master's thesis, Dept. Math. Inform., Graduate Sch. Inf. Sci. Technol., Univ. Tokyo, 2015.
[24] B. Schulze and W. J. Whiteley, "The orbit rigidity matrix of a symmetric framework," *Discrete Comput. Geometry*, vol. 46, pp. 561–598, 2011.
[25] T. Jordán, V. E. Kaszanitzky, and S. Tanigawa, "Gain-sparsity and symmetry-forced rigidity in the plane," *Discrete Comput. Geometry Int. J. Math. Comput. Sci.*, vol. 55, no. 2, pp. 314–372, 2016.
[26] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks* (Ser. Princeton Series in Applied Mathematics). Princeton, NJ, USA: Princeton Univ. Press, 2010.
[27] E. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 1283–1294, Sep. 1996.
[28] V. Sundarapandian, "Local and global asymptotic stability of nonlinear cascade interconnected systems," *Math. Comput. Modelling*, vol. 40, no. 1, pp. 227–232, 2004.

Daniel Zelazo (Senior Member, IEEE) received the B.Sc. and M.Eng. degrees in electrical engineering and computer science from the Massachusetts Institute of Technology, Cambridge, MA, USA, in 1999 and 2001, respectively, and the Ph.D. degree in aeronautics and astronautics from the University of Washington, Seattle, WA, USA, in 2009.

From 2010 to 2012, he was a Postdoctoral Research Associate and Lecturer with the Institute for Systems Theory and Automatic Control, University of Stuttgart, Stuttgart, Germany. He is an Associate Professor of aerospace engineering with the Technion-Israel Institute of Technology, Haifa, Israel. His research focuses on multiagent systems.

Shin-ichi Tanigawa received the Master of Engineering and Doctor of Engineering degrees from Kyoto University, Kyoto, Japan, in 2007 and 2010, respectively.

He was a JSPS Postdoctoral Fellow with Kyoto University, from 2010 to 2011, and with CWI, Amsterdam, The Netherlands, from 2015 to 2017. He is currently an Associated Professor of Mathematical Informatics with University of Tokyo. His research interests include discrete and computational geometry, graph theory, and combinatorial optimization.



Bernd Schulze received the Ph.D. degree in mathematics from York University, North York, ON, Canada, in 2009.

Before arriving at Lancaster in 2012, he was Postdoc with the TU Berlin, Berlin, Germany, from 2009 to 2011, and with the Fields Institute, Toronto, ON, Canada, in 2011. He is currently a Reader in mathematics with Lancaster University, Lancaster, U.K. He studied mathematics and computer science at the FU Berlin, Berlin, and at Western Michigan University, Kalamazoo, MI, USA. His research interests include applied discrete geometry, graph theory, combinatorial optimization, symmetry, and algebraic methods in discrete mathematics.