Combinatorial Admissibility in Control-Affine Networks

Daniel Zelazo, *Senior Member, IEEE* and Louis Theran.

***Abstract*— We study synchronization of heterogeneous control-affine nonlinear agents interconnected through diffusive relative-output measurements. Motivated by recent geometric edge-space formulations for formation control, we separate the design into an *edge-space* step, which specifies a stabilizing model evolution for relative outputs, and a *lift* step, which realizes the prescribed edge motion using the agents' allowable input directions under the control-affine geometry. We introduce an admissibility notion that characterizes when an edge-driven diffusive design is feasible. We relate generic admissibility to structured rank and maximum matchings in an associated bipartite graph, yielding checkable combinatorial certificates that show how graph topology and actuation structure jointly limit achievable edge dynamics. The results are illustrated on synchronization of nonlinear oscillators.**

I. Introduction

One of the canonical approaches for the coordination of multi-agent systems is through diffusive coupling. In this architecture, agents interact through relative information with their neighbors, and a collective behavior, like synchronization or formation keeping, emerges from local rules. For integrator and linear agent models this viewpoint leads to a mature theory that connects convergence to graph connectivity and Laplacian structure [1]. For nonlinear agents, however, there is an additional layer that is not apparent in the linear setting. The coupling law specifies how relative quantities should evolve, while the plant dynamics may constrain which relative motions can actually be produced through the available input directions. Understanding this gap between "desired" and "realizable" relative behavior is central to synchronization beyond the linear regime.

The study of synchronization of nonlinear agents has a rich and mathematically diverse history. Passivity theory has proven to be an effective framework for studying diffusively coupled networks. Leveraging powerful interconnection results between passive systems, local dissipativity properties can be combined with the graph structure to build Lyapunov functions and prove convergence [2], [3]. When the objective is output agreement for heterogeneous agents or in the presence of exogenous signals, static diffusive couplings may be insufficient and dynamic edge couplings become natural [4]–[6]. Another powerful approach is incremental stability via contraction theory, which yields global synchronization conditions and exponential convergence rates for coupled nonlinear oscillators and related networks [7].

Geometric approaches have also been used to study synchronization problems. When the state evolves on a nonlinear space such as a Lie group or a Riemannian manifold, intrinsic notions of distance and gradient flows provide natural coordination laws and clarify global topological obstructions. This has been studied as an optimization problem over manifolds [8] and Riemannian consensus analyses under curvature assumptions [9]. Related geometric control viewpoints interpret coordination laws as gradient-like flows on quotient spaces induced by symmetries in the problem formulation [10].

The geometric approach adopted here is inspired by the edge-space viewpoint introduced in our previous work [11] to study formation control problems. In [11], a geometric template that separates coordination into an *edge-space design* step and a *node-space realization* step was introduced. The starting point is a non-linear measurement map from configurations to edge measurements, whose image forms a feasible set in measurement space; an artificial edge dynamical system constrained to evolve on this feasible set (e.g., via a Riemannian gradient flow) is proposed, and then the node dynamics are obtained by lifting the edge dynamics to node space. The strength of this approach lies in the fact that the edge dynamics are easier to analyze and yield a more flexible design space. In [11], the node dynamics are integrators, so any edge dynamics admit *some* lifting: once a feasible edge velocity is specified, it can always be implemented by a suitable choice of node velocities. The present note studies a complementary situation of a synchronization problem in which the measurement map factors through a linear graph operator, but the node dynamics are constrained. The main question is then how to design edge dynamics that are realizable. The answer turns out to depend not just on the edge dynamics but on a structural compatibility between the graph topology and the agents' actuation directions—a condition that is automatically satisfied for integrator node dynamics but becomes the central obstruction in the control-affine case we study here.

Our contributions are as follows. First, we formulate diffusive synchronization for heterogeneous non-linear control-affine agents in an explicit edge-space geometry: relative outputs evolve on a feasible set induced by the measurement map, and we design stabilizing edge dynamics directly in that space. Second, we show how to realize the desired edge evolution by a minimum-norm lifting law for control-affine node dynamics, identifying a feasibility condition that we term *admissibility*, that characterizes when the edge-driven design can be implemented by the available input directions.

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D. Zelazo is with the Stephen B. Klein Faculty of Aerospace Engineering, Technion-Israel Institute of Technology, Haifa 320003, Israel dzelazo@technion.ac.il

L. Theran is with the School of Mathematics and Statistics, University of St Andrews, Scotland lst6@st-andrews.ac.uk

Third, we provide tractable, structure-based admissibility tests by relating generic feasibility to the structured rank of the lifted edge map and to maximum matchings in an associated bipartite graph, thereby making the interaction between graph topology and actuation constraints explicit. Finally, we illustrate the theory on a nonlinear oscillator synchronization example, where the matching certificate accurately predicts success and failure under two actuation patterns.

Notation: Let $\mathbb{R}$ denote the real numbers. For an integer n, $\mathbf{1}_n \in \mathbb{R}^n$ is the all-ones vector and I_n is the $n \times n$ identity. For matrices, $\operatorname{Im}(\cdot)$, $\operatorname{Ker}(\cdot)$, and $\operatorname{rk}(\cdot)$ denotes the image, kernel, and rank, while $(\cdot)^\top$ transpose, and $(\cdot)^\dagger$ the Moore–Penrose pseudoinverse. For vectors x_i, $\operatorname{col}(x_1, \ldots, x_n)$ denotes vertical stacking, and $\operatorname{blkdiag}(\cdot)$ the block-diagonal operator. The Kronecker product is $\otimes$.

II. Problem Setup

We consider a network of N agents indexed by $\mathcal{V} := \{1, \ldots, N\}$. Agent i has state $x_i \in \mathbb{R}^{n_i}$, input $u_i \in \mathbb{R}^{p_i}$, and measured/regulated output $y_i \in \mathbb{R}^d$. The dynamics are assumed to be *control-affine*,

$$\dot{x}_i = f_i(x_i) + G_i(x_i)\, u_i, \qquad y_i = h_i(x_i), \tag{1}$$

where $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $G_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times p_i}$, and $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^d$ are analytic. We stack the states, inputs, and outputs as $x := \operatorname{col}(x_1, \ldots, x_N)$, $u := \operatorname{col}(u_1, \ldots, u_N)$, and $y := \operatorname{col}(y_1, \ldots, y_N) = h(x) \in \mathbb{R}^{Nd}$. We similarly stack the drift and input matrices as $f(x) := \operatorname{col}(f_1(x_1), \ldots, f_N(x_N))$ and $G(x) := \operatorname{blkdiag}(G_1(x_1), \ldots, G_N(x_N))$.

Agents exchange relative output information over a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{E}| = m$. For clarity, we begin with an undirected graph and assign an arbitrary orientation to each edge. Let $B \in \mathbb{R}^{N \times m}$ denote the incidence matrix. For an oriented edge $e = (i, j)$, the corresponding column b_e has -1 in row i, $+1$ in row j, and 0 elsewhere.

Define the *edge disagreement* (relative output) signal as

$$z \;:=\; (B^\top \otimes I_d)\, y \;\in\; \mathbb{R}^{md}.$$

Then the control goal is *output agreement* (consensus) across the network,

$$\lim_{t \to \infty} \|z(t)\| = 0, \tag{2}$$

i.e., $y_i(t) - y_j(t) \to 0$ for every edge $(i, j) \in \mathcal{E}$. If $\mathcal{G}$ is connected, (2) implies each agent $y_i(t) \to \bar{y}(t)$ for a common trajectory $\bar{y}(t) \in \mathbb{R}^d$.

Equivalently, agreement corresponds to convergence to the *agreement manifold*,

$$\mathcal{A} \;:=\; \{x :\; (B^\top \otimes I_d)\, h(x) = 0\} \;=\; F^{-1}(0), \tag{3}$$

where $F : \mathbb{R}^{\sum n_i} \to \mathbb{R}^{md}$ is the *edge map* defined as

$$F(x) := (B^\top \otimes I_d)\, h(x). \tag{4}$$

We focus on distributed controllers that depend on the local state and relative outputs,

$$u_i = \kappa_i\Big(x_i,\; \{\, y_j - y_i :\; j \in \mathcal{N}_i \,\}\Big),$$

where $\mathcal{N}_i$ denotes the neighbor set of node i in $\mathcal{G}$. A canonical static diffusive coupling architecture takes the form

$$u_i = \phi_i(x_i) \;-\; \sum_{j \in \mathcal{N}_i} a_{ij}\, \psi\big(y_i - y_j\big),$$

with weights $a_{ij} > 0$ and typically $\psi(\cdot)$ odd and monotone (e.g., $\psi(\eta) = \eta$). When ϕ_i is zero and ψ is the identity, the coupling is a graph Laplacian.

III. Edge-Space Geometry and a Model System

We begin with the edge map introduced in (4), which relates the state of each agent to the edge state as $z = F(x)$. Let

$$Q := \operatorname{Im} F \subseteq \mathbb{R}^{md}$$

denote the feasible set of edge disagreements. Define the Jacobian of F by

$$J(x) := DF(x) = (B^\top \otimes I_d)\, Dh(x), \tag{5}$$

where $Dh(x) = \operatorname{blkdiag}(Dh_1(x_1), \ldots, Dh_N(x_N))$.

Assumption 1 (Constant-rank neighborhood): There exists an open neighborhood Ω of the agreement set $\mathcal{A} = F^{-1}(0)$ such that $\operatorname{rk}(J(x))$ is constant for all $x \in \Omega$.

Under Assumption 1, the image $Q_\Omega := F(\Omega)$ is an immersed (and locally embedded) submanifold of $\mathbb{R}^{md}$, and for $z = F(x)$ with $x \in \Omega$ the tangent space satisfies

$$T_z Q_\Omega = \operatorname{Im} J(x).$$

In the sequel we restrict attention to Ω and write Q in place of Q_Ω.

Following the same ideas as from [11], we consider an artificial system of edges with state $z \in \mathbb{R}^{md}$ with integrator dynamics,

$$\dot{z} = v. \tag{6}$$

We aim to design a stabilizing edge feedback $v^\star(z)$ on Q that drives $z \to 0$. In this direction, we consider the Riemannian gradient flow for $V_e(z) = \frac{1}{2}\|z\|^2$, leading to

$$v^\star(F(x)) = -\Pi(x)\, F(x), \tag{7}$$

where $\Pi(x) : \mathbb{R}^{md} \to T_{F(x)}Q$ is the orthogonal projector onto $T_{F(x)}Q$. To avoid ambiguity when multiple x map to the same z, we write the projector as a function of x,

$$\Pi(x) := J(x) J(x)^\dagger,$$

so that $\Pi(x)$ is the orthogonal projector onto $\operatorname{Im}(J(x)) = T_{F(x)}Q$.

We now must map the edge flow dynamics in (7) to the node dynamics of the system. Differentiating $z = F(x)$ along (1) yields

$$\dot{z} = J(x) f(x) + J(x) G(x)\, u.$$

To realize the desired edge velocity $v^\star(z)$ in (7), we require

$$J(x) G(x)\, u = v^\star(F(x)) - J(x) f(x). \tag{8}$$

Among all feasible inputs, we select the minimum-norm lift,

$$\begin{aligned} u^\star(x) &:= \arg\min_{u} \ \|u\|^2 \\ &\text{s.t.} \quad J(x)G(x)\,u = v^\star(F(x)) - J(x)f(x). \end{aligned} \tag{9}$$

When the constraint is feasible, this yields the explicit *model controller*,

$$\begin{aligned} u^\star(x) &= \big(J(x)G(x)\big)^\dagger \Big(v^\star(F(x)) - J(x)f(x)\Big) \\ &= \big(J(x)G(x)\big)^\dagger \Big(-\Pi(x)\,F(x) - J(x)f(x)\Big). \end{aligned} \tag{10}$$

The remainder of the paper analyzes (i) conditions under which (9) is feasible and yields closed-loop agreement, and (ii) how to obtain distributed approximations and directed variants, culminating in a combinatorial admissibility criterion.

IV. Edge-Driven Controller Families and Distributed Implementations

This section develops an *edge-driven controller family* that extends the model controller (10). The key observation is that the lift constraint (8) is typically *underdetermined*: many inputs can realize the same desired edge velocity. We exploit this freedom to (i) parameterize a family of controllers consistent with a prescribed edge flow, and (ii) identify distributed specializations that recover classical diffusive coupling architectures.

A. A family of lifts for a prescribed edge flow

Let

$$A(x) := J(x)G(x) \in \mathbb{R}^{md\times p} \tag{11}$$

$$b(x) := v^\star(F(x)) - J(x)f(x) \in \mathbb{R}^{md}, \tag{12}$$

where $p := \sum_i p_i$. Then the lift constraint (8) is the linear equation

$$A(x)u = b(x). \tag{13}$$

Whenever (13) is feasible, the set of all solutions is an affine space. In particular, the Moore–Penrose solution (10) is the unique solution with minimum Euclidean norm, and the general solution is

$$u(x) = u^\star(x) + \big(I - A(x)^\dagger A(x)\big)\,w(x),$$

where $w(x) \in \mathbb{R}^p$ is an arbitrary (possibly state-dependent) signal. The matrix $\mathcal{M}(x) := I - A(x)^\dagger A(x)$ is the orthogonal projector onto $\mathrm{Ker}(A(x))$. Thus, the term $\mathcal{M}(x)w(x)$ does not affect the induced edge velocity $\dot z$ and can be used to enforce secondary objectives (e.g., input saturation handling, additional regulation tasks, or distributed implementability).

Remark 1 (Edge-driven family in "gain" form): In the special case $f \equiv 0$ (or after exact drift compensation), the controller can be written in the simpler gain form

$$u(x) = -\nu(x)\,F(x),$$

where $\nu(x) \in \mathbb{R}^{p\times md}$ is chosen so that the induced edge dynamics satisfy $\dot z = -\eta(x)z$ with $\eta(x) = A(x)\nu(x)$. The model controller corresponds to $\nu(x) = A(x)^\dagger\Pi(x)$. Distributed realizations (below) correspond to sparse choices of $\nu(x)$.

The edge-space design $v^\star$ is realizable by the physical agents when the required right-hand side lies in the range of $A(x)$. In particular, we require that inputs can generate *all feasible* edge directions, captured by the map $J(x)$. This will be formalized as *admissibility* and linked to combinatorial certificates in subsequent sections.

B. Distributed implementation for relative degree-one dynamics

We now identify a broad and practically important class of distributed controllers obtained by combining *local right inversion* of the agent output dynamics with a *diffusive outer loop*. In this direction, we further assume each agent admits relative-degree-one output dynamics, expressed as

$$\dot y_i = a_i(x_i) + H_i(x_i)u_i, \tag{14}$$

where $H_i(x_i) \in \mathbb{R}^{d\times p_i}$ is right invertible on the region of interest. Define a virtual input $v_i \in \mathbb{R}^d$ and apply the minimum-norm local right inverse

$$u_i = H_i(x_i)^\dagger\big(v_i - a_i(x_i)\big), \tag{15}$$

which enforces $\dot y_i = v_i$. In other words, we can feedback linearize the output dynamics to generate integrator dynamics. This now leads to our first result.

Theorem 1: Suppose each agent admits relative-degree-one output dynamics (14) on a forward-invariant set $\mathcal{X}_i$, and $H_i(x_i)$ is right invertible for all $x_i \in \mathcal{X}_i$. Apply the local right-inversion controller (15) and let the virtual input be chosen by the diffusive outer loop

$$v_i \;=\; -\sum_{j=1}^{N} w_{ij}\big(y_i - y_j\big), \tag{16}$$

with weights $w_{ij} \ge 0$ of a (directed) graph $\mathcal{G}$. If $\mathcal{G}$ contains a (directed) spanning tree, then the outputs reach consensus as $t \to \infty$.

The proof is standard and is therefore omitted; see [1].

Remark 2 (Relation to the edge-driven family): The distributed controller (15)–(16) can be interpreted as a sparse realization of an edge-driven law: the outer loop prescribes a feasible edge evolution (through v), while the local inversions realize it agentwise without forming global pseudoinverses such as $(J(x)G(x))^\dagger$. This provides a bridge between classical diffusive coupling and the edge-space design-and-lift template. $\diamondsuit$

C. Admissibility of edge-driven designs

The model-based construction developed above designs a desired edge velocity $v^\star(F(x))$ on the feasible edge set and then selects an input u by enforcing the instantaneous constraint (8). This subsection formalizes when that constraint is feasible and clarifies the closed-loop implications.

Definition 1 (Admissibility): Let Ω be the constant-rank neighborhood from Assumption 1. Define $J(x) := DF(x)$ and $A(x) := J(x)G(x)$ as in (5) and (12). The agent–network pair is

(a) *exactly admissible on* Ω if

$$\operatorname{Im}\big(A(x)\big) = \operatorname{Im}\big(J(x)\big), \tag{17}$$

for all $x \in \Omega$;

(b) *locally generically admissible* if, (17) holds for almost all $x \in \Omega$;

(c) *generically admissible* if (17) holds for almost all x.

Condition (17) states that the inputs can generate *every feasible infinitesimal edge motion*: since feasible edge velocities satisfy $\dot{z} \in T_{F(x)}Q = \operatorname{Im}(J(x))$, admissibility ensures that any such $\dot{z}$ can be realized through the control channel $G(x)$.

Remark 3 (Relationship between admissibility concepts): Because $A(x)$ is the restriction of $J(x)$ to the image of $G(x)$, (17) is equivalent to the transversality statement $\operatorname{Im} G(x) + \operatorname{Ker} J(x) = \mathbb{R}^{\sum n_i}$ and therefore to the equality of the ranks of $A(x)$ and $J(x)$. The failure of transversality is expressible as a polynomial in x. Thus, if $G(x)$ and $J(x)$ are analytic and transversality holds at one point x, then there is an $\Omega \ni x$ such that exact admissibility holds on Ω and, moreover, the controller is generically admissible. In particular, in these cases, local generic admissibility and admissibility are equivalent concepts. For more general classes of functions, such as smooth functions, the two may be different.

Remark 4 (Exact vs. least-squares lifting): When $b(x) \in \operatorname{Im}(A(x))$, the constraint $A(x)u = b(x)$ is feasible and the minimum-norm lift (10) enforces the desired edge velocity exactly. When $b(x) \notin \operatorname{Im}(A(x))$, the same pseudoinverse expression yields the minimum-norm least-squares solution, and the realized edge velocity is the orthogonal projection of $b(x)$ onto $\operatorname{Im}(A(x))$. $\Diamond$

Lemma 1 (Exact realization under admissibility): Suppose Assumption 1 holds on Ω and let $\Pi(x) := J(x)J(x)^\dagger$. Consider the projected-gradient model choice $v^\star(F(x)) = -\Pi(x)F(x)$ and the lifted controller (10) written as

$$u^\star(x) = A(x)^\dagger\big(v^\star(F(x)) - J(x)f(x)\big). \tag{18}$$

If the admissibility condition (17) holds at $x \in \Omega$, then the lift constraint (8) is feasible at x and the closed-loop edge dynamics satisfy

$$\dot{z} = v^\star(F(x)) = -\Pi(x)\, z, \qquad z = F(x), \tag{19}$$

at that point.

Proof: By construction, $v^\star(F(x)) = -\Pi(x)F(x) \in \operatorname{Im}(J(x))$ because $\Pi(x)$ projects onto $\operatorname{Im}(J(x))$. Under admissibility, $\operatorname{Im}(J(x)) = \operatorname{Im}(A(x))$, hence $b(x) = v^\star(F(x)) - J(x)f(x) \in \operatorname{Im}(A(x))$ and the constraint is feasible. For the feasible linear equation $A(x)u = b(x)$, the pseudoinverse yields a solution satisfying $A(x)u^\star = b(x)$, which together with $\dot{z} = J(x)f(x) + A(x)u$ gives (19). ∎

Lemma 1 shows that admissibility is the precise condition ensuring the physical closed-loop reproduces the designed (projected) edge gradient flow. In particular, along trajectories that remain in Ω and satisfy admissibility pointwise, the edge Lyapunov function $V_e(z) = \frac{1}{2}\|z\|^2$ satisfies (because z is in the image of Π and Π is a projection),

$$\dot{V}_e(z) = -z^\top \Pi(x) z = -\|z\|^2 = -2V_e(z) < 0,$$

so $z \to 0$ exponentially quickly by Grönwall's inequality [12]. Since $\mathcal{A} = F^{-1}(0)$ is a smooth embedded submanifold, fix a compact neighborhood $K \subset \Omega$ of a point $x^\star \in \mathcal{A}$. By continuity, $J(x)$ is bounded on K. Therefore, under admissibility and exponential stability of the edge model, the standard ISS argument of [11] yields local exponential convergence of the lifted node dynamics to $\mathcal{A}$.

Remark 5 (Why Theorem 1 is always admissible): In the relative-degree-one specialization (14)–(15), the controller satisfies the stronger agent-by-agent transversality condition $\operatorname{Im} G_i(x_i) + \operatorname{Ker} \mathrm{d}_{x_i} h_i = \mathbb{R}^{n_i}$ because $\operatorname{Im} H_i(x_i) = \mathbb{R}^d$. Since the graph operator is linear, this implies admissibility at x_i.

For underactuated agents or restricted actuation directions, admissibility can fail even on connected graphs. In the next subsection we derive checkable sufficient conditions for (generic) admissibility in terms of structured rank and maximum matchings of an associated bipartite graph.

D. Generic admissibility and combinatorial certificates

The admissibility condition in Definition 1 is analytic and state-dependent through $J(x)$ and $G(x)$. In this subsection we derive *structure-based* sufficient conditions that can be checked combinatorially, starting with the undirected setting.

Fix a region Ω and suppose that the zero/non-zero pattern of $Dh_i(x_i)G_i(x_i)$ is constant on Ω. We assume the controller is analytic, so that $A(x)$ achieves its maximal rank for almost all $x \in \Omega$. We now identify a structural property of the graph $\mathcal{G}$ that implies the controller is generically admissible.

For a connected, undirected graph, $\mathcal{G}$ the edge disagreement vector $z = F(x)$ must lie in an $(N-1)d$-dimensional linear subspace cut out by cycle relations (i.e., the sum of its components around any cycle is zero). Hence, the generic rank of $A(x)$ is at most $(N-1)d$. We describe a combinatorial condition that implies equality, from which generic admissibility follows.

Let $\mathcal{T} \subseteq \mathcal{E}$ be a spanning tree with $|\mathcal{T}| = N-1$, and let $B_\mathcal{T} \in \mathbb{R}^{N\times(N-1)}$ denote the incidence matrix restricted to the edges of $\mathcal{T}$ (with arbitrary orientation). Define the tree-edge disagreement map,

$$F_\mathcal{T}(x) := (B_\mathcal{T}^\top \otimes I_d) h(x),$$

and its Jacobian

$$J_\mathcal{T}(x) := DF_\mathcal{T}(x) = (B_\mathcal{T}^\top \otimes I_d)\, Dh(x).$$

Since $B_\mathcal{T}$ has rank $N-1$, the tree-edge coordinates span the disagreement subspace; in particular, $\operatorname{rk}(J_\mathcal{T}(x)) = \operatorname{rk}(J(x))$ generically, and it suffices to certify that $A_\mathcal{T}(x) := J_\mathcal{T}(x)G(x)$ has rank $(N-1)d$ generically.

To this end, we introduce the following assumption.

Assumption 2 (Generic independence of nonzero entries): For any spanning tree $\mathcal{T}$ of $\mathcal{G}$, the nonzero entries in $A_\mathcal{T}(x)$ can be varied independently by changing x.

Construct a bipartite graph $\mathcal{H}_\mathcal{T} = (\mathcal{L}, \mathcal{R}, \mathcal{E})$ using the zero/non-zero pattern of $A_\mathcal{T}(x)$ as follows:

- Left vertices $\mathcal{L}$ index the $(N-1)d$ tree-edge coordinates: $\mathcal{L} := \{(e,k) : e \in \mathcal{T},\ k \in \{1,\dots,d\}\}$.
- Right vertices $\mathcal{R}$ index the input channels: $\mathcal{R} := \{(i,\ell) : i \in \mathcal{V},\ \ell \in \{1,\dots,p_i\}\}$.
- An edge $((e,k),(i,\ell)) \in \mathcal{E}$ is present iff the entry of $A_\mathcal{T}(x)$ corresponding to row (e,k) and column (i,ℓ) is structurally nonzero.

Definition 2 (Matching): A *matching* in a bipartite graph $\mathcal{H} = (\mathcal{L}, \mathcal{R}, \mathcal{E})$ is a set of edges no two of which share a common endpoint. A *maximum matching* is a matching of maximum cardinality with size denoted by $\nu(\mathcal{H})$.

Theorem 2 (Matching certificate for generic admissibility): Let $\mathcal{G}$ be connected. Under Assumption 2, if there is a spanning tree $\mathcal{T}$ of $\mathcal{G}$ such that

$$\nu(\mathcal{H}_\mathcal{T}) = (N-1)d, \tag{20}$$

then $\operatorname{rk}(A_\mathcal{T}(x)) = (N-1)d$ for almost all $x \in \Omega$. Consequently,

$$\operatorname{rk}(A(x)) = \operatorname{rk}(J(x)) = (N-1)d$$

for almost all $x \in \Omega$, i.e., the system is generically admissible on Ω.

Proof: The upper bound on the rank of $J(x)$ was discussed above. By [13, Prop. 2.4], which applies because of Assumption 2, for any fixed $\mathcal{T}$, the rank of $A_\mathcal{T}(x)$ is equal to $\nu(\mathcal{H}_\mathcal{T})$ for almost all x. Hence, we get a matching lower bound for the rank of $A_\mathcal{T}(x)$, if $\nu(\mathcal{H}_\mathcal{T}) = (N-1)d$ for some $\mathcal{T}$. In this case, we have

$$(N-1)d \le \operatorname{rk} A_\mathcal{T}(x) \le \operatorname{rk} A(x) \le \operatorname{rk} J(x) \le (N-1)d,$$

so equality holds throughout and we are done. ■

Remark 6 (Hall condition and design interpretation): A matching of size $(N-1)d$ exists iff Hall's condition holds [14]: $|\Gamma(\mathcal{S})| \ge |\mathcal{S}|$ for every $\mathcal{S} \subseteq \mathcal{L}$, where $\Gamma(\mathcal{S})$ is the neighbor set in $\mathcal{H}_\mathcal{T}$. In design terms, every subset of disagreement coordinates must be influenced by at least as many independent input channels.

Theorem 2 gives a purely combinatorial sufficient condition for (generic) feasibility of the edge lift. In Section V we illustrate how the matching fails when a row of a local actuation map is structurally zero, leading to persistent disagreement in the corresponding output component.

V. Simulation: Admissible vs. Non-Admissible Edge Actuation on a Limit Cycle

We illustrate the admissibility results on a synchronization problem for nonlinear oscillators. We compare two actuation patterns on the same network: one that is admissible and achieves synchronization, and one that is non-admissible.

In this direction, we consider three identical nonlinear oscillators coupled over the undirected path graph 1–2–3. Each agent has state $x_i \in \mathbb{R}^2$ with $x_i := \begin{bmatrix} r_i & \theta_i \end{bmatrix}^T$, $r_i > 0$, $\theta_i \in \mathbb{S}^1$, and evolves according to the control-affine dynamics (1). The drift f is chosen so that $r_i = 1$ is a stable limit cycle and θ_i rotates at constant speed $\omega > 0$:

$$f(x_i) = \begin{bmatrix} (1-r_i^2)r_i \\ \omega \end{bmatrix}.$$

The control input is the *same signal type* in both cases, $u_i := \begin{bmatrix} u_{r,i} & u_{\theta,i} \end{bmatrix}^T \in \mathbb{R}^2$, and the difference between the two models is encoded entirely in G_i.

Each oscillator is observed through its Cartesian position on the plane,

$$y_i = h(x_i) := \begin{bmatrix} r_i \cos\theta_i & r_i \sin\theta_i \end{bmatrix}^T \in \mathbb{R}^2, \tag{21}$$

and we stack $y = \operatorname{col}(y_1, y_2, y_3) \in \mathbb{R}^6$. Synchronization corresponds to $y_i(t) - y_j(t) \to 0$ for all i, j.

a) Case A (admissible): We take the control input as $u_i := \operatorname{col}(u_{r,i}, u_{\theta,i}) \in \mathbb{R}^2$ and select $G_i^{(A)} = I_2$, $i = 1,2,3$. Thus both radial and angular channels are available, and the input can directly modify both the radius and the phase of each oscillator.

b) Case B (non-admissible): We use the same input signal $u_i := \operatorname{col}(u_{r,i}, u_{\theta,i}) \in \mathbb{R}^2$ but restrict actuation to the radial direction by choosing $G_i^{(B)} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $i = 1,2,3$. Equivalently, the angular channel is unavailable and is discarded by $G_i^{(B)}$. In particular, $\dot\theta_i = \omega$ for all i, so phase offsets cannot be corrected by feedback and planar synchronization fails for generic initial conditions.

Let $F(y) := (B^\top \otimes I_2)y$ denote the edge map introduced earlier (specialized here to the linear relative-measurement setting), and define the edge disagreement as $z := F(y) = (B^\top \otimes I_2)\, y \in \mathbb{R}^4$. Following the edge-space construction of Section III (model integrator (6) and projected-gradient choice (7)), we prescribe the virtual edge dynamics $\dot z = v^\star(z)$. In this example the feasible edge set is the linear subspace $Q = \operatorname{Im}(F)$, so the orthogonal projector onto $T_z Q$ is the identity. Consequently, the projected Riemannian gradient flow for $V_e(z) = \frac{1}{2}\|z\|^2$ reduces to the Euclidean gradient flow $v^\star(z) = -k\, z, \quad k > 0$, which is exactly the edge-space "Laplacian" feedback. The admissibility question is whether the control directions available to the oscillators span the edge-velocity directions required by above, i.e., whether the lift constraint admits an exact solution so that the closed-loop satisfies $\dot z = v^\star(z)$. When this fails, the pseudoinverse lift yields the closest achievable edge velocity (the orthogonal projection of $v^\star(z)$ onto the achievable edge-velocity subspace), which is precisely what the matching test in Section IV-D certifies.

Because the graph is a tree and the output dimension is $d = 2$, there are $(n-1)d = 4$ independent disagreement coordinates. The admissibility question is whether the available control directions can realize these four independent edge-space directions.

For the path graph 1–2–3, the tree edge map is $F_\tau(x) = (B_\tau^\top \otimes I_2)h(x)$ with $h_i(r_i, \theta_i)$ given in (21). Differentiating yields $J_\tau(x) = DF_\tau(x) = (B_\tau^\top \otimes I_2)Dh(x)$, where

$$Dh_i(r_i, \theta_i) = \begin{bmatrix} \cos\theta_i & -r_i \sin\theta_i \\ \sin\theta_i & r_i \cos\theta_i \end{bmatrix} =: J_i. \tag{22}$$

The tree lift matrix is $A_\tau(x) := J_\tau(x)G$, where $G = \mathrm{blkdiag}(G_1, G_2, G_3)$ encodes the available inputs.

Case A (radial and tangential actuation). Now, we compute the lift matrix as

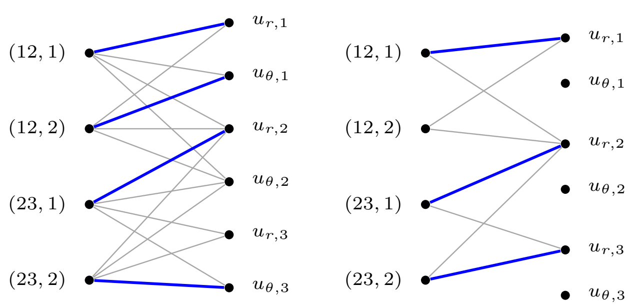
$$A_\tau^{(A)}(x) = \begin{bmatrix} -J_1 & J_2 & 0 \\ 0 & -J_2 & J_3 \end{bmatrix} \in \mathbb{R}^{4\times 6}. \tag{23}$$

Thus the first edge block (rows for 12) depends only on node-1 and node-2 inputs, and the second edge block (rows for 23) depends only on node-2 and node-3 inputs; each 2×2 block is generically dense. The bipartite matching graph is shown in Fig. 1a and corresponding trajectories in Fig. 2a.

Case B (radial-only actuation). We use the same input signal $u_i = \mathrm{col}(u_{r,i}, u_{\theta,i}) \in \mathbb{R}^2$ as in Case A, but restrict actuation by choosing $G_i^{(B)}$, so the angular channel is unavailable. The tree lift matrix $A_\tau(x) = J_\tau(x)G^{(B)}$ is computed as

$$A_\tau^{(B)}(x) = \begin{bmatrix} -a_1 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & a_3 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 6}, \tag{24}$$

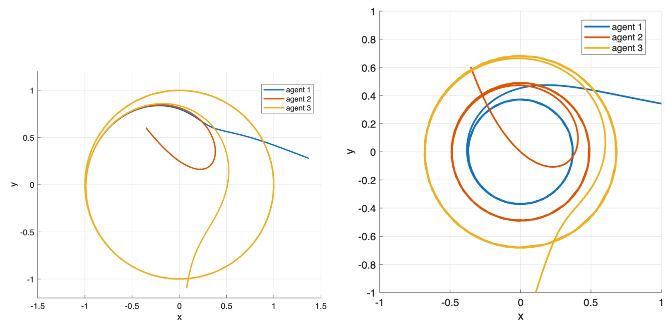
where $a_i := \begin{bmatrix} \cos\theta_i & \sin\theta_i \end{bmatrix}^T \in \mathbb{R}^2$, and each displayed block is a 2×1 column and the zero columns correspond to the unavailable inputs $u_{\theta,i}$. Consequently the bipartite graph associated with (24) contains the isolated vertices $u_{\theta,1}, u_{\theta,2}, u_{\theta,3}$ (Fig. 1b), and the maximum matching satisfies $\nu \le 3 < 4 = (N-1)d$. Thus $\mathrm{rk}(A_\tau^{(B)}(x)) < (N-1)d$ generically and the lift constraint is generically infeasible. Consistent with this prediction, Case B exhibits persistent phase offsets and fails to synchronize in the phase plane (Fig. 2b).



(a) Case A (admissible): $\nu = 4$. Highlighted edges indicate one maximum matching.

(b) Case B (non-admissible): $\nu = 3 < 4$. Highlighted edges indicate one maximum matching.

Fig. 1. Bipartite graphs associated with the sparsity pattern of $A_\tau(x)$. Gray edges show structural nonzeros; colored edges show a maximum matching.



(a) Case A (admissible): phase-plane trajectories synchronize.

(b) Case B (non-admissible): phase-plane trajectories remain phase-shifted.

Fig. 2. Phase-plane trajectories for the oscillator network under the two actuation patterns.

VI. Concluding Remarks

This paper developed an edge-driven geometric framework for diffusive coupling of heterogeneous control-affine networks. The design philosophy is to construct a stabilizing control in the edge space and then generate a feasible node input through an appropriate lift from edge space to node space. We identify *admissibility* as a structural condition linking the interconnection graph with the actuation limits of the agents. We further provided generic/combinatorial certificates, based on structured rank and maximum matchings, that predict when edge-driven diffusive designs can be realized and when agreement is obstructed.

A natural next step is to extend this geometric admissibility viewpoint to settings where *both* layers are nonlinear: i.e., a setting with nonlinear measurement maps (e.g., distances, bearings, or other nonlinear relative outputs) and nonlinear node dynamics.

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