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Stabilization of Symmetric Formations ⋆

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Abstract: This work proposes a solution to the distance-constrained formation control problem to attain symmetric formations. Utilizing recent results from rigidity theory for symmetric frameworks, we design a gradient-based control strategy that simultaneously drives the agents to the desired inter-agent distances and also a special position characterizing additional symmetry constraints on the graph. We show that for graphs where there exist edges between agents in the same vertex orbit induced by the automorphism group, no additional information exchange links are required. Furthermore, leveraging the symmetry constraints of the system it is possible to solve the formation control problem with fewer edges than standard approaches. In particular, the framework is not required to be minimally infinitesimally rigid in this case. Our results are demonstrated with a numerical example.

Keywords: Multi-agent systems; distributed control and estimation; nonlinear cooperative control; rigidity theory; formation control; symmetry

1. INTRODUCTION

In the study of multi-agent coordination problems, formation control takes a leading role. It is central to many application domains, including vehicle platooning (Dai et al. (2018)), synthetic aperture interferometry (Rosen et al. (2000)), and surveillance networks (Drake et al. (2005)), to mention just a few. The design of distributed control protocols to solve the formation control problem has also been an active research topic in the systems and controls community (Ahn (2020); De Queiroz and Feemster (2019); Oh et al. (2015); Zhao and Zelazo (2019)).

Gradient-based control strategies for solving the formation control problem are appealing because they lead to distributed protocols. More interesting is the connection between the potential functions used to define the control laws, and the combinatorial theory of rigidity. Rigidity theory considers questions of the following form: given a discrete configuration of geometric objects (points, lines, etc.) in a given Euclidean space (called a framework), and a set of geometric constraints (distances, bearings, etc.), determine whether the set of polynomial equations representing these constraints (a) has a solution (independence); (b) has locally isolated solutions (rigidity); or (c) has exactly one solution in the given space up to isometric motions (global rigidity) (Asimow and Roth (1978); Jackson (2007)). The seminal work from Krick et al. (2009) provided the first formal result showing that (minimal) infinitesimal rigidity of the interaction network in a team of integrator agents is required to ensure that a the gradient controller (locally) converges to the correct formation shape.

Since the work of Krick et al. (2009), there has been an increased interest in understanding the interplay between the system dynamics induced by the gradient controller and the combinatorial properties of rigid graphs. Fundamental to these works is the conclusion that infinitesimal rigidity of a framework is an *architectural requirement* of a formation-seeking multi-agent system. For planar formations with n agents (frameworks embedded in $\mathbb{R}^2$), this amounts to having at least $2n-3$ edges in the interaction graph, providing a lower-bound on the sparsity of information exchange in the system.

Motivated by this discussion, we first ask if it is possible to solve the formation control problem in $\mathbb{R}^2$ with *less* than $2n-3$ edges, i.e., over frameworks that are not minimally infinitesimally rigid. It turns out by exploiting properties found in *symmetric* frameworks this is indeed possible. The impact of symmetry on the rigidity and flexibility of frameworks has received a lot of interest in recent years; see (Schulze and Whiteley (2017b)), for example, for a summary of results. There are two different settings for studying the rigidity of symmetric frameworks: 1) the symmetry is *incidental* and might be broken by any motions; 2) the symmetry is *forced* by something in the construction, and will be preserved by any motions. For our purposes, we can make use of results from the theory of forced-symmetric rigidity. In particular, for a framework to be forced-symmetric infinitesimally rigid we don't necessarily need it to have $2n-3$ edges. Instead

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there are adapted counts involving the number of edge and vertex orbits under the symmetry group; see (Jordán et al. (2016); Schulze and Whiteley (2011)), for example.

The contributions of this work are as follows. We provide a gradient-based control strategy that aims to drive a team of integrator agents to a geometric shape characterized by a given forced symmetry. We show that under certain conditions, the communication and sensing requirements of the network, in terms of the number of edges, are less than what is needed in the standard gradient-based formation control strategies, such as in Krick et al. (2009).

The paper outline is as follows. Section 2 presents preliminary background on rigidity theory and formation control. In Section 3, symmetry notions for frameworks are introduced. The main results are presented in Section 4 with a numerical example given in Section 5. Finally, some concluding remarks are offered in Section 6.

2. PRELIMINARIES FROM GEOMETRIC RIGIDITY AND FORMATION CONTROL

This section provides an overview of basic concepts from geometric rigidity theory and the formation control literature. It will serve as the foundation for the main results in this work.

2.1 Rigidity Theory

A *framework* in $\mathbb{R}^d$ is defined to be a pair $(\mathcal{G}, p)$ consisting of a finite simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a map $p : \mathcal{V} \to \mathbb{R}^d$. We may think of p as a point in $\mathbb{R}^{d|\mathcal{V}|}$ in which case we refer to p as a *configuration* of $|\mathcal{V}|$ points in $\mathbb{R}^d$. A framework $(\mathcal{G}, p)$ is *rigid* if the only edge-length-preserving continuous motions of the vertices arise from isometries of $\mathbb{R}^d$, and *flexible* otherwise. The rigidity and flexibility analysis of frameworks is a well-developed theory with a rich history and many practical applications (see, e.g. Schulze and Whiteley (2017a); Connelly and Guest (2022); Whiteley (1996)).

A common approach to study the rigidity of frameworks is to linearise the problem by differentiating the length constraints on the edges. This leads to the notion of infinitesimal (or equivalently, static) rigidity. An *infinitesimal motion* of a framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is an assignment of velocity vectors, one to each vertex, $u : V \to \mathbb{R}^d$, such that

$$\langle p_i - p_j, u_i - u_j\rangle = 0 \quad \text{for all } ij \in \mathcal{E}, \tag{1}$$

where $p_i = p(i)$ and $u_i = u(i)$ for each i. An infinitesimal motion u of $(\mathcal{G}, p)$ is *trivial* if there exists a skew-symmetric matrix S and a vector c such that $u_i = Sp_i + c$ for all $i \in \mathcal{V}$, and $(\mathcal{G}, p)$ is *infinitesimally rigid* if every infinitesimal motion of $(\mathcal{G}, p)$ is trivial, and *infinitesimally flexible* otherwise.

The $|\mathcal{E}| \times d|\mathcal{V}|$ matrix corresponding to the linear system in (1) is called the *rigidity matrix* of $(\mathcal{G}, p)$, denoted as $R(\mathcal{G}, p)$. The row of $R(\mathcal{G}, p)$ corresponding to the edge $ij \in \mathcal{E}$ is of the form

$$\begin{pmatrix} 0 \cdots 0 & (p_i - p_j)^T & 0 \cdots 0 & (p_j - p_i)^T & 0 \cdots 0 \end{pmatrix}.$$

We also employ the useful algebraic representation of the rigidity matrix,

$$R(\mathcal{G}, p) = \text{diag}\{(p_i - p_j)^T\}_{ij\in\mathcal{E}}(E^T \otimes I_d), \tag{2}$$

where E is the directed $|\mathcal{V}| \times |\mathcal{E}|$ incidence matrix with $E_{ij} = 1$ if the edge e_j leaves vertex i, $E_{ij} = -1$ if the edge e_j enters the vertex i, and $E_{ij} = 0$ otherwise. So the kernel of $R(\mathcal{G}, p)$ is the space of all infinitesimal motions of $(\mathcal{G}, p)$, and it is well known that $(\mathcal{G}, p)$ is infinitesimally rigid if and only if the rank of $R(\mathcal{G}, p)$ is $d|\mathcal{V}| - \binom{d+1}{2}$, provided that the points p_i affinely span all of $\mathbb{R}^d$ (Whiteley (1996)). While an infinitesimally rigid framework is always rigid, the converse does not hold in general. Asimov and Roth, however, showed that for 'generic' configurations p (i.e. an open dense subset of configurations), infinitesimal rigidity is equivalent to rigidity (Asimow and Roth (1978)).

2.2 Formation Control

We review the now well-studied distance-constrained formation control problem (Oh et al. (2015)). Consider a network of $n = |\mathcal{V}|$ agents described by integrator dynamics,

$$\dot{p}_i(t) = u_i(t), \tag{3}$$

where $p_i(t) \in \mathbb{R}^d$ is the position of agent i, and $u_i \in \mathbb{R}^d$ is the control. As in the development of rigidity theory in §2.1, the configuration of the network is the stack of the agent positions, $p(t) = \begin{bmatrix} p_1^T(t) & \cdots & p_n^T(t) \end{bmatrix}^T$ (similary defined for $u(t)$). The agents are tasked with attaining a spatial formation using only measurements and/or communication with neighboring agents, as defined by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The formation is specified by a set of desired inter-agent distances, $d_{ij}^\star$, for each edge $ij \in \mathcal{E}$, and we denote $d^\star$ as the stack of all desired distances.

It is well-known that a gradient-based control strategy can solve the formation control problem. In this direction, we define the *formation potential* function,

$$F_f(p(t)) = \frac{1}{4}\sum_{ij\in\mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2\right)^2. \tag{4}$$

Then the gradient controller $u(t) = -\nabla F_f(p(t))$ solves the formation control problem. That is, the closed-loop system

$$\begin{aligned} \dot{p}(t) &= -\nabla F_f(p(t)) \\ &= -R(\mathcal{G}, p(t))^T \left(R(\mathcal{G}, p(t))p(t) - (d^\star)^2\right) \end{aligned} \tag{5}$$

satisfies $\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = d_{ij}^\star$ for all $ij \in \mathcal{E}$. Here, $R(\mathcal{G}, p(t))$ is the rigidity matrix defined in §2.1.

3. SYMMETRY IN GRAPHS AND FRAMEWORKS

3.1 Symmetry in graphs

Symmetry in objects can be described mathematically via the fundamental algebraic notion of a group (see e.g. Dummit and Foote (1991)).

Definition 1. *A group is defined to be a set, Γ, together with an operation $\circ$, such that for any two elements $a, b \in \Gamma$, $a \circ b$ is also in Γ. The operation $\circ$ satisfies the associativity law. Moreover, each group has a special element* id, *called the* identity element, *such that for any element $a \in \Gamma$, $a \circ \text{id} = \text{id} \circ a = a$. Each element a of Γ also has an* inverse a^{-1} *in Γ such that $a \circ a^{-1} = a^{-1} \circ a = \text{id}$. The number of elements in a group is called the* order *of the group. A subset B of Γ that also form a group under $\circ$ is called a* subgroup *of Γ.*

The combinatorial symmetries of a finite simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ are described by its group of automorphisms. An automorphism of $\mathcal{G}$ can be loosely understood as a permutation of $\mathcal{V}$ that maps adjacent vertices to adjacent vertices, and non-adjacent vertices to nonadjacent vertices, and hence preserves all structural properties of $\mathcal{G}$. The formal definition is as follows.

Definition 2. *An* automorphism *of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation $\psi : \mathcal{V} \to \mathcal{V}$ of its vertex set such that*

$$\psi(v)\psi(u) \in \mathcal{E} \Leftrightarrow vu \in \mathcal{E}.$$

It is clear, then, that the identity permutation, denoted id, is an automorphism of any graph, and for an automorphism ψ, ψ^{-1} is also an automorphism. So it is easy to see that the set of all automorphisms of $\mathcal{G}$ forms a group under composition of maps. This group is called the *automorphism group* of $\mathcal{G}$ and is denoted by $\mathrm{Aut}(\mathcal{G})$.

A common way to represent a permutation for an automorphism is by a two-row array. For a graph $\mathcal{G}$ with $|\mathcal{V}| = n$ vertices, one can write the automorphism ψ as

$$\psi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \psi(1) & \psi(2) & \cdots & \psi(n) \end{pmatrix}.$$

Equivalently, one can express every permutation more compactly as a composition of disjoint cycles of the permutation. A cycle is a successive action of the permutation that sends a vertex back to itself, i.e., $i \to \psi(i) \to \psi(\psi(i)) \to \cdots \to \psi^k(i) = i$, where $\psi^k = \underbrace{\psi \circ \cdots \circ \psi}_{k \text{ times}}$. Such a cycle is compactly written using the *cycle notation*, denoted by $(i\,\psi(i) \cdots \psi^{k-1}(i))$. The integer k is the *length* of the cycle.

Definition 3. *A graph $\mathcal{G}$ is* Γ-symmetric *for any subgroup $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$. The symmetry is* free *if $\gamma(i) \neq i$ for all $i \in \mathcal{V}$ and non-identity $\gamma \in \Gamma$.*

A key structural property of a Γ-symmetric graph $\mathcal{G}$ is its sets of vertex and edge orbits under Γ. Loosely speaking, the orbit of a vertex i (or edge e) of $\mathcal{G}$ under Γ is the set of vertices (edges, resp.) of $\mathcal{G}$ that i (e, resp.) can be mapped to by elements in Γ.

Definition 4. *For a Γ-symmetric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and vertex $i \in \mathcal{V}$, the set $\Gamma_i = \{\gamma(i) \,|\, \gamma \in \Gamma\}$ is called the* vertex orbit *of i. Similarly, for an edge $e = ij \in \mathcal{E}$, the set $\Gamma_e = \{\gamma(i)\gamma(j) \,|\, \gamma \in \Gamma\}$ is termed the* edge orbit *of e.*

The size of the vertex orbits depends, of course, on the sub-group Γ. Since all nodes in a given orbit are somehow equivalent under a group action, we often consider *representative vertices* from each vertex orbit. We denote by $\mathcal{V}_0$ the set of representative vertices for each orbit, such that $|\mathcal{V}_0|$ are the number of vertex orbits, and $i \in \mathcal{V}_0$ means that vertex i is only in one vertex orbit.

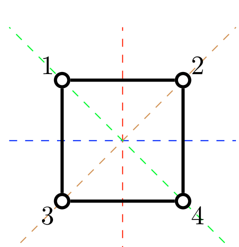


Fig. 1. The cycle graph C_4 has 8 automorphisms in $\mathrm{Aut}(\mathcal{G})$.

Example 1. *Figure 1 shows the cycle graph on 4 nodes, C_4. We will identify all the automorphisms of $\mathrm{Aut}(C_4)$. First, consider a clock-wise rotation by $90°$ of C_4 as drawn in the figure. This gives the automorphism*

$$\psi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

The cycle notation is $\psi_1 = (1\,3\,4\,2)$. With ψ_1, we also have $\psi_1^2 = \psi_1 \circ \psi_1, \psi_1^3$ and $\psi_1^4 = \mathrm{id}$ in $\mathrm{Aut}(\mathcal{G})$, where $\psi_2 = (1\,4)(2\,3)$ and $\psi_3 = (1\,2\,4\,3)$ may be interpreted geometrically as rotations by $180°$ and $270°$. Additional permutations can be found by considering reflections. Figure 1 shows 4 reflection symmetries. Consider first the reflection about the vertical red line, giving the permutation (in cycle notation) $\psi_4 = (1\,2)(3\,4)$. Similarly, the horizontal reflection (blue line) yields $\psi_5 = (1\,3)(2\,4)$, the diagonal reflection (green line) gives $\psi_6 = (1)(2,3)(4)$, and the brown line reflection $\psi_7 = (1\,4)(2)(3)$. Thus, we have that $\mathrm{Aut}(C_4) = \{\mathrm{id}, \psi_1, \ldots, \psi_7\}$ has 8 automorphisms. As an abstract group, it is the dihedral group D_8 of order 8 (Dummit and Foote (2004)). Note that any vertex can be mapped to any other under the automorphisms in $\mathrm{Aut}(C_4)$ and hence C_4 has only one vertex orbit (and only one edge orbit) under $\mathrm{Aut}(C_4)$. If, however, we considered C_4 as a Γ-symmetric graph, where Γ is the subgroup of $\mathrm{Aut}(C_4)$ of order 2 consisting of the identity and ψ_4, for example, then C_4 has two vertex orbits, namely $\{1, 2\}$ and $\{3, 4\}$, and three edge orbits, namely $\{12\}, \{34\}$ and $\{13, 24\}$.

3.2 Symmetry in frameworks

Having defined notions of symmetries for graphs, we now consider symmetry of frameworks.

Definition 5. *Let $\mathcal{G}$ be a Γ-symmetric graph, and let Γ be represented as a* point group*, i.e., a subgroup of the orthogonal group $O(\mathbb{R}^d)$, via a homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$. In other words, τ assigns an orthogonal matrix (describing an isometry of $\mathbb{R}^d$ such as a rotation or reflection) to each element of Γ. Then a framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called* $\tau(\Gamma)$-symmetric *if*

$$\tau(\gamma)(p_i) = p_{\gamma(i)} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in \mathcal{V}. \qquad (6)$$

We use the standard Schoenflies notation for point groups in this paper (Altmann and Herzig (1994); Atkins et al. (1970)). The only possible point groups in dimension 2 are the reflection group $\mathcal{C}_s$ (consisting of the identity and a single reflection about an axis σ), the rotational groups $\mathcal{C}_n$ of order n, where $n \geq 1$ (generated by a rotation c_n about the origin in counter-clockwise direction by an angle of $2\pi/n$), and the dihedral groups $\mathcal{C}_{nv}$ of order $2n$, where $n \geq 2$ (generated by a reflection σ and a rotation c_n). If we think of the graph drawing in Figure 1 as a framework in the plane, for example, then this framework is $\mathcal{C}_{4v}$-symmetric.

If a framework (G, p) is $\tau(\Gamma)$-symmetric, then the configuration p is in a special geometric position that may no longer be 'generic'. Thus, symmetry can lead to unexpected infinitesimal flexibility (as well as unexpected rigidity). Since symmetry is very common in both natural and man-made structures, the rigidity and flexibility analysis of symmetric frameworks has grown into a major research area over the last two decades; see (Schulze and Whiteley (2017b)) for a summary of this work.

4. STABILIZATION OF SYMMETRIC CONFIGURATIONS

We now would like to study a variation of the formation control problem where the goal of each agent in the network is to obtain a formation corresponding to a *symmetric configuration*. In other words, starting with a Γ-symmetric graph $\mathcal{G}$, which can be drawn with maximum point group symmetry $\mathcal{S}$ in $\mathbb{R}^d$, we would like to drive the agents to a special position $p^\star$ such that the framework $(\mathcal{G}, p^\star)$ is $\tau(\Gamma)$-symmetric for a desired subgroup of $\mathcal{S}$.

For a Γ-symmetric graph $\mathcal{G}$ and a vertex orbit Γ_i of $\mathcal{G}$ under Γ, it follows immediately from the definition of Γ_i (Definition 4) that for every vertex j in Γ_i there is a $\gamma_j \in \Gamma$ such that $\gamma_j(j) = i$. So for a $\tau(\Gamma)$-symmetric framework $(\mathcal{G}, p)$ and for every $j \in \Gamma_i$, there is a $\gamma_j \in \Gamma$ such that $\tau(\gamma_j)p_j = p_i$ for all $j \in \Gamma_i$. With this notion in place we now formally state the control problem.

Problem 1. *Consider a group of n integrator agents* (3) *that interact over the Γ-symmetric sensing graph $\mathcal{G}$. Let $p^\star \in \mathbb{R}^{dn}$ be a configuration such that $(\mathcal{G}, p^\star)$ is $\tau(\Gamma)$-symmetric for some desired point group $\tau(\Gamma)$, and let $\mathcal{V}_0$ be a set of representatives of the vertex orbits of $\mathcal{G}$ under Γ. Design a control $u_i(t)$ for each agent i such that*

(i) $\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = \|p_i^\star - p_j^\star\|$ *for all* $ij \in \mathcal{E}$*;*
(ii) for each $i \in \mathcal{V}_0$, $\lim_{t\to\infty} \|p_i(t) - \tau(\gamma_j)p_j(t)\| = 0$ *for all* $j \in \Gamma_i$.

We would like to solve Problem 1 in a distributed fashion, ideally allowing agent i to only obtain information from neighboring agents as defined by $\mathcal{G}$. Before we proceed, we comment on the information needed to solve this problem.

Requirement (i) in Problem 1 is the standard formation control constraint introduced in §2.2. That is, $\|p_i^\star - p_j^\star\| = d_{ij}^\star$ are the desired distances between neighboring agents. Requirement (ii) aims to enforce the symmetric position between agents that are in the same vertex orbit. Here we point out that it may not be the case that $j \in \Gamma_i$ implies that $ij \in \mathcal{E}$. We will comment on this point later.

Our approach to solve Problem 1 follows the same gradient dynamical system approach used in solving the standard formation control problem. In this direction, we define a *symmetric potential*

$$F_s(p(t)) = \frac{1}{2}\sum_{i\in\mathcal{V}_0}\sum_{j\in\Gamma_i} \|p_i(t) - \tau(\gamma_j)p_j(t)\|^2. \tag{7}$$

The *symmetric formation potential* can then be defined as

$$F(p(t)) = F_f(p(t)) + F_s(p(t)), \tag{8}$$

where $F_f(p(t))$ is the formation control potential defined in (4). Here recall that $\mathcal{V}_0$ is a set of representatives from each of the vertex orbits of a Γ-symmetric graph.

We now propose the control

$$u(t) = -\nabla F(p(t)), \tag{9}$$

to solve Problem 1. The closed-loop dynamics then take the form

$$\dot{p}(t) = -R(\mathcal{G}, p(t))^T \left(R(\mathcal{G}, p(t))p(t) - (d^\star)^2\right) - Qp(t),$$

where Q is a block matrix and

$$Q_{ij} = \begin{cases} (|\Gamma_i| - 1)I, & i = j,\, i \in \mathcal{V}_0 \\ -\tau(\gamma_j), & i \in \mathcal{V}_0, j \in \Gamma_i \\ I, & i = j,\, j \notin \mathcal{V}_0,\, j \in \Gamma_i \\ -\tau(\gamma_j)^{-1}, & j \in \mathcal{V}_0,\, i \in \Gamma_j \\ 0, & \text{o.w.} \end{cases}.$$

Note that without loss of generality, we can label the nodes such that the matrix Q thus has a block-diagonal structure with $|\mathcal{V}_0|$ blocks, where each block is $|\Gamma_i|d \times |\Gamma_i|d$,

$$Q = \begin{bmatrix} \bar{Q}_1 & & \\ & \ddots & \\ & & \bar{Q}_{|\mathcal{V}_0|} \end{bmatrix}.$$

Furthermore, since the off-diagonal blocks Q_{ij} is an element of the orthogonal group $O(\mathbb{R}^d)$ (see Definition 5), it follows that $\tau(\gamma_j)^{-1} = \tau(\gamma_j)^T$ and therefore Q is a symmetric matrix. Finally, each block matrix of $\bar{Q}_i$ can be shown to be positive-semidefinte, since $\det(\bar{Q}_i) = 0$ and all the principle minors are strictly positive.

Before presenting the main result, we introduce an additional assumption on the graph structure.

Assumption 1. *For each $i \in \mathcal{V}_0$ and $j \in \Gamma_i \setminus \{i\}$, the edge ij is in $\mathcal{E}$.*

Assumption 1 ensures that the resulting controller (9) does not require additional communication/sensing links beyond the given graph $\mathcal{G}$. Relaxing this assumption is the subject of ongoing research.

Theorem 1. *Consider a team of n agents* (3) *interacting over a Γ-symmetric graph $\mathcal{G}$ satisfying Assumption 1, that can be drawn with maximum point group symmetry $\mathcal{S}$ in $\mathbb{R}^d$, and let*

$$\mathcal{F}_f = \{p \in \mathbb{R}^{dn} \,|\, \|p_i - p_j\| = d_{ij}^\star\, ij \in \mathcal{E}\},$$

and

$$\mathcal{F}_s = \{p \in \mathbb{R}^{dn} \,|\, \tau(\gamma)(p_i) = p_{\gamma(i)}\, \forall\gamma \in \Gamma,\, i \in \mathcal{V}\}.$$

Then for initial conditions $p_i(0)$ satisfying

$$\sum_{ij\in\mathcal{E}} (\|p_i(0) - p_j(0)\| - d_{ij}^\star)^2 \leq \epsilon_1,$$

and

$$\|p_i(0) - \tau(\gamma_j)p_j(0)\|^2 \leq \epsilon_2$$

for all $i \in \mathcal{V}_0$ and $j \in \Gamma_i$, for a sufficiently small and positive constant ϵ_1 and ϵ_2, the control

$$u = -\nabla F(p(t)), \tag{10}$$

renders the set $\mathcal{F}_f \cap \mathcal{F}_s$ exponentially stable, i.e.

$$\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = d_{ij}^\star$$

and

$$\lim_{t\to\infty} \tau(\gamma)(p_i(t)) = \lim_{t\to\infty} p_{\gamma(i)}(t) \quad \text{for all } \gamma \in \Gamma, i \in \mathcal{V}.$$

Proof. Let $\sigma_k(t) = \|p_i(t) - p_j(t)\|^2 - (d_{ij}^\star)^2$ represent the distance error for the agents incident to the edge $k = ij \in \mathcal{E}$. Let $q_\ell(t) = p_i(t) - \tau(\gamma_j)p_j(t)$ for $i \in \mathcal{V}_0$ and $j \in \Gamma_i$ be the position error relative to the special position in the group Symmetry S. We can now define the Lyapunov function

$$V(\sigma(t), q(t)) = \frac{1}{4}\sigma(t)^T\sigma(t) + \frac{1}{2}q(t)^T q(t),$$

where $\sigma(t)$ is the stack of all the distance errors, and $q(t)$ the stack of the position errors. Differentiating along the system trajectories, we obtain $\dot{V}(\sigma(t), q(t)) = \frac{1}{2}\sigma(t)^T R(\mathcal{G}, p)u(t) + q(t)^T T u(t)$, where T is the block matrix of the form

$$T = \begin{bmatrix} \bar{T}_1 & & \\ & \ddots & \\ & & \bar{T}_{|\mathcal{V}_0|} \end{bmatrix}$$

with $\bar{T}_i \in \mathbb{R}^{(|\Gamma_i|-1)d \times d|\Gamma_i|}$. Each block row of $\bar{T}_i$ has the identity I for the entry corresponding to the representative vertex, and $-\tau(\gamma_j)$ in the jth block entry for each node in the vertex orbit. With this structure we have that $T^T T = Q$. It also follows that $q(t) = Tp(t)$.

Plugging in (10) now gives

$$\begin{aligned} \dot{V}(\sigma(t), q(t)) &= -\sigma^T R(\mathcal{G}, p) \left(R(\mathcal{G}, p)^T \sigma + Qp \right) \\ &\quad - q(t)^T T \left(R(\mathcal{G}, p)^T \sigma + Qp \right) \\ &= -\sigma^T R(\mathcal{G}, p) R(\mathcal{G}, p)^T \sigma - 2\sigma^T R(\mathcal{G}, p) Qp - p^T Q^2 p \\ &= -(R(\mathcal{G}, p)^T \sigma + Qp)^T (R(\mathcal{G}, p)^T \sigma + Qp) \\ &= -\|R(\mathcal{G}, p)^T \sigma + Qp\|^2 \leq 0. \end{aligned}$$

The requirement that the initial condition is sufficiently close to the target formation is standard in the formation control literature, and ensures the gradient dynamical system converges to a critical point corresponding to the correct formation.

□

Remark 1. *The existence of flip ambiguities common in distance constrained formation control problems exist also here. That is, if the initial conditions are not sufficiently close to the target formation, a symmetric and equivalent formation may be attained that is not the desired shape.*

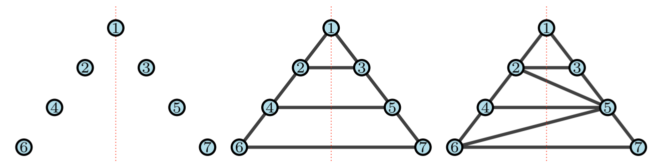
5. AN EXAMPLE

The use of formation flight for aircraft originated in the first world war to offer fighter pilots improved visual communication with squadrons and defensive advantages (Wikipedia (2022)). The original formation, referred to as the *Vic* formation, is meant to emulate the V-formation seen by migrating birds. Since WWI, many works have studied the advantages of these formations in terms of fuel efficiency and drag reduction, as well as control strategies for maintaining the formation Seanor et al. (2006); Pachter et al. (2001); Stipanović et al. (2004).

Our interest in the Vic formation is the natural symmetry it possesses, as can be seen in Figure 2(a). In this example we consider $n = 7$ agents and note that the vertex orbits are $\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}$, where the symmetry "mirror" passes through vertex 1 (marked by the red dashed line). The symmetry we are concerned with is the reflection along the y-axis, and this framework is $\tau(\Gamma)$-symmetric, with $\tau(\gamma) : (x, y) \mapsto (-x, y)$. This can be represented by the linear map

$$\tau(\gamma) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We will demonstrate our control strategy on the flexible graph shown in Figure 2(b). We will also employ the standard formation controller (5) to highlight the advantages of our method.



(a) Vic formation. (b) Flexible framework. (c) Rigid framework.

Fig. 2. Example of the symmetric Vic formation on 7 nodes. Figure (b) includes edges between nodes in the same vertex orbit but is still flexible, while (c) is a minimally rigid framework.

Before presenting the simulation results, we first make a comment on the number of edges (representing information exchange) required for our strategy compared to the classic formation controller. Here we observe that Figure 2(b) has 9 edges, while the minimally rigid graph in Figure 2(c) has 13 edges. More generally, for a Vic formation with n (odd) nodes, the symmetry forced graph requires only $3(n-1)/2$ edges, compared to $2n - 3$ edges for minimal rigidity. For large n this requires significantly less number of edges.

For the numerical simulation, we aim to attain a Vic formation that is congruent to that shown in Figure 2(a), with $p_1^\star = (0, 0)$, and the remaining agents on each "wing" of the formation spaced $\sqrt{2}$ units from each other. Figure 3(a) shows the agent trajectories and Figure 3(b) the distance measurements between each agent when the symmetry-forcing control (10) is applied. It can be seen that the agents converge to the correct formation with correct inter-agent distances. On the other-hand, if only the formation controller, as in (5) is applied on the graph in Figure 2(b), the agents do not converge to the correct shape, shown in Figure 4.

6. CONCLUDING REMARKS

In this work we demonstrated how to exploit properties of symmetric frameworks to drive a team of integrator agents into a symmetric formation with specified distances. Of note is that this approach for certain graphs does not require additional communication between agents, and also can be implemented with graphs that are not minimally infinitesimally rigid. In this way formation control problems can be solved for sparser networks than typically required in formation control. In future work we will characterize more formally the class of graphs that can take advantage of this approach, and how to modify the control strategy for graphs that do not (i.e., graphs for which there are no edges between nodes in the same vertex orbit).

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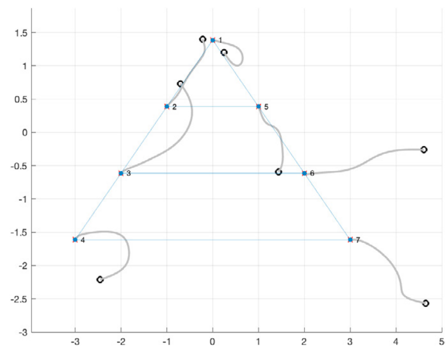
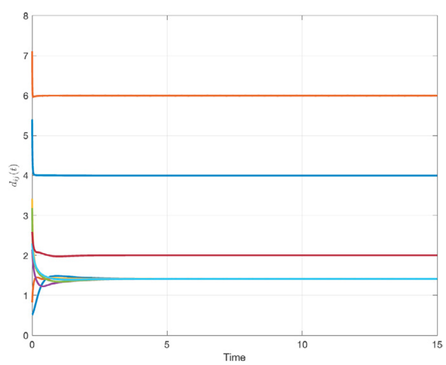


(a) Agent trajectories.



(b) Distances along trajectories.

Fig. 3. The trajectories of the forced symmetry formation controller (9). The agents converge to the desired distances with the correct symmetries. The black circles represent the initial conditions and the red crosses the final position. The inter-agent distances also converge to the correct values.

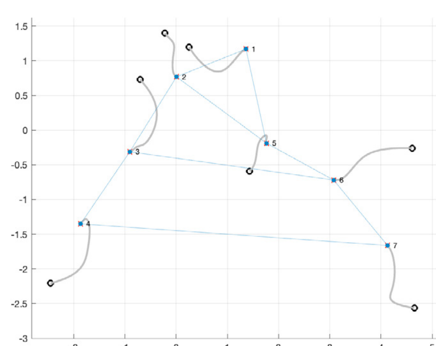


Fig. 4. Trajectories of the formation controller (5) for the flexible framework in Figure 2(b), without the symmetry forcing term. Since the framework is not rigid, the formation does not converge to the correct symmetric position. The distances, however, do converge to the correct values (not shown as identical to Figure 3(b)).

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