Boundary Filter Design for Biorthogonal Filter Banks

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Abstract

In order for filter banks to be able to process finite length input signals, appropriate modifications must be made to either the input signal (via signal extension techniques) or to the filter bank itself (boundary filters). This paper shows how to choose appropriate boundary filters for the analysis portion of any filter bank. The boundary filters can be chosen to satisfy any number of criteria, such as frequency selectivity or number of vanishing moments, and are only required to be linearly independent of the original filter bank filters. The synthesis filter bank is found by calculating the dual of the analysis bank, which is the same as finding the inverse. This results in a biorthogonal, perfect reconstruction filter bank.

I. Introduction

Many applications in multirate digital signal processing requires the analysis of finite length input signals. Multirate filter banks provide an excellent and efficient means of analyzing infinite length input signals, but stumbles when the input signal is finite length. Filter banks for finite length signals have been extensively studied, and there are many solutions to this problem. These solutions include boundary filter design [1-6] and signal extension techniques [6][7]. Signal extension techniques are computationally simple, but generally result in an expansive filter bank (more sub-band coefficients than the original length of the input signal), and/or exhibit non-ideal behavior at the boundaries. Boundary filter design techniques are length preservative, but can not guarantee certain properties (i.e. frequency selectivity or number of vanishing moments) for a wide range of analysis filters (i.e. minimum phase filters).

This paper describes a method for choosing boundary filters for the analysis bank of any filter bank. The synthesis bank is found by calculating the dual of the analysis bank, thus creating a biorthogonal filter bank. The filter bank is guaranteed to be perfect reconstruction and sub-band length preservative. Furthermore, the boundary filters can be chosen to satisfy various criteria such as frequency selectivity or number of vanishing moments.

The remainder of this paper will provide a brief review of filter bank theory. A short review of previously used techniques for finite length signal analysis will also be presented. Section III will present our design technique followed by a short example in section IV and conclusion in section V. A brief discussion on future work is given in section VI.

II. Filter Banks

It is well known that filter banks can be interpreted from a linear algebra perspective [6]. The linear algebra perspective provides a powerful and notationally compact representation of filter banks. For simplicity, we will study the 2-channel case, although all results can be extended to M channels. Figure 1 shows a block diagram of a 2-channel filter bank. The filters $\mathbf{h}_0$ and $\mathbf{h}_1$ are respectively, the analysis, or decomposition, low and high pass filters. The filters $\mathbf{f}_0$ and $\mathbf{f}_1$ are the synthesis, or reconstruction filters. In a perfect reconstruction (PR) filter bank $\mathbf{x} = \mathbf{x}'$.

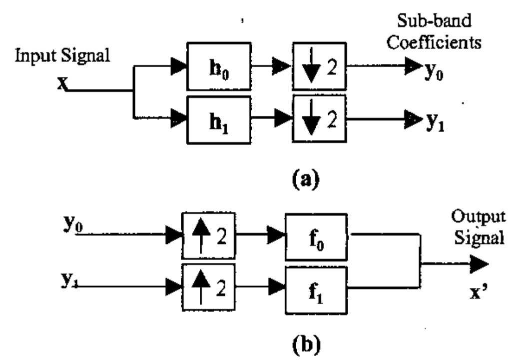

Figure 1. 2-Channel Filter Bank (a) Analysis (b) Synthesis

For an infinite length input vector $\mathbf{x}$, and finite impulse response (FIR) analysis and synthesis filters, the analysis and synthesis portions of the filter bank can be expressed as infinite banded block toeplitz matrices. The block diagram in Figure 1 can now be expressed as a linear transformation of the input vector $\mathbf{x}$.

$$\mathbf{Hx} = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \end{bmatrix} \quad (1)$$

$$\mathbf{F}\begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \end{bmatrix} = \mathbf{x}' \quad (2)$$

In Equation 1, $\mathbf{H}$ is the matrix representation of the analysis filter bank, and $\mathbf{F}$ is the matrix representation of the synthesis filter bank. Note that the analysis matrix can be premultiplied by a permutation matrix $\mathbf{P}$ to achieve an interleaving of the low-pass and high-pass sub-band coefficients ($\mathbf{F}$ must then be multiplied by $\mathbf{P}^T$). Figure 2

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shows the structure of the analysis and synthesis filter bank matrices after a permutation matrix is applied.

$$\mathbf{H}=\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & h_0(N_a-1) & h_0(N_a-2) & h_0(N_a-3) & \cdots & h_0(1) & h_0(0) & 0 & \cdots \\ \cdots & h_1(N_a-1) & h_1(N_a-2) & h_1(N_a-3) & \cdots & h_1(1) & h_1(0) & 0 & \cdots \\ \cdots & 0 & 0 & h_0(N_a-1) & \cdots & h_0(3) & h_0(2) & h_0(1) & \cdots \\ \cdots & 0 & 0 & h_1(N_a-1) & \cdots & h_1(3) & h_1(2) & h_1(1) & \cdots \\ & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \end{bmatrix}$$

(a)

$$\mathbf{F}=\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & f_0(0) & f_1(0) & 0 & 0 & \cdots \\ \cdots & f_0(1) & f_1(1) & 0 & 0 & \cdots \\ \cdots & f_0(2) & f_1(2) & f_0(0) & f_1(0) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & f_0(N_s-2) & f_1(N_s-2) & f_0(N_s-4) & f_1(N_s-4) & \cdots \\ \cdots & f_0(N_s-1) & f_1(N_s-1) & f_0(N_s-3) & f_1(N_s-3) & \cdots \\ \cdots & 0 & 0 & f_0(N_s-2) & f_1(N_s-2) & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(b)

Figure 2. (a) Structure of infinite analysis filter bank matrix. h_0 and h_1 both have length N_a. The analysis matrix contains the time-reversed analysis filters with double shifted rows from the decimation operator. (b) Synthesis filter bank matrix. Synthesis filters are lengths N_s. Columns are double shifted from the interpolation operator.

In a PR filter bank, it is required that

$$\mathbf{FH} = \mathbf{I}. \qquad (3)$$

In this case, **F** is the dual of **H** and the filter bank is considered to be biorthogonal. If in addition $\mathbf{F} = \mathbf{H}^T$, then the filter bank is an orthogonal filter bank. Combining the results of Equations 1, 2, and 3 we have

$$\mathbf{FHx} = \mathbf{x}. \qquad (4)$$

Thus, the entire filter bank operation can be thought of as a projection of the input vector **x** onto the column space of the synthesis matrix **F**.

When the input vector **x** is finite length, modifications to the above description must be made. Signal extension techniques have been extensively studied and include zero-padding, periodic extension, and symmetric extension [6][7]. These techniques are computationally simple but generally introduce unnatural jumps in the signal, and at times can create awkward representations of the original signal depending on the application. These techniques also generally result in redundant sub-band information. Boundary filter designs are attractive because they guarantee sub-band length preservation, and can be designed to closely match the steady-state behavior of the filter bank. Boundary filter design techniques have been proposed in [1-5]. These techniques involve truncating the infinite analysis and synthesis matrices and then adding rows and columns to make the matrices square and invertable. Further manipulation allows the boundary filters to satisfy certain requirements, such as frequency selectivity or coding gain. However, for certain types of filters (i.e. minimum phase filters), these techniques can only guarantee Equation 4, but other criteria like frequency selectivity can not be achieved.

The method proposed in this paper allows the design of boundary filters for the analysis filter bank to satisfy any number of requirements, such as frequency selectivity, or number of vanishing moments, regardless of the prototype filters used for the filter bank. The synthesis filter bank is then found by calculating the dual of the analysis. Conceptual simplicity is emphasized over computational efficiency.

III. Boundary Filter Design

One of the nice properties of biorthogonal filter banks is that the analysis or the synthesis matrices are not required to be orthogonal. For a biorthogonal filter bank, the rows of **H**, or the columns of **F**, are only required to be linearly independent of each other. For finite length input vectors the analysis and synthesis matrices must also become finite. Furthermore, to ensure length preservation of the sub-band coefficients, the matrices must also be square.

Simply truncating the infinite matrices **H** and **F** to square matrices will not guarantee perfect reconstruction. Equation 3 will no longer hold. However, it is desirable, if not a necessity, for the "middle", or steady state, portion of the square matrices to have an identical structure and behavior as its infinite counterpart. Therefore, to begin the boundary filter design process, a rectangular truncation of the analysis matrix is performed, as outlined by [4]. The truncated matrix will be referred to as $\mathbf{H}_t$. The size of $\mathbf{H}_t$ depends on the length of the input vector, the number of channels in the filter bank, and the length of the analysis filters. For simplicity, we will continue working with 2-channel filter banks, and assume that the input vector length is of the form $\mathbf{L} = \mathbf{2p} + \mathbf{N_a}$. **L** is the length of the input vector, **p** is a positive integer value, and $\mathbf{N_a}$ is the length of the analysis filters. The size of $\mathbf{H}_t$ is **2(p+1)xL**. For most applications it is safe to assume that $\mathbf{L} \gg \mathbf{2(p+1)}$. The resulting structure is shown in Figure 3. In order to make the matrix square, $\mathbf{N_a}$**-2** rows must be added.

$$\mathbf{H_t}=\begin{bmatrix} h_0(N_a-1) & h_0(N_a-2) & h_0(N_a-3) & \cdots & h_0(0) & 0 & 0 & 0 & \cdots & 0 \\ h_1(N_a-1) & h_1(N_a-1) & h_1(N_a-3) & \cdots & h_1(0) & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & h_0(N_a-1) & \cdots & h_0(2) & h_0(1) & h_0(0) & 0 & \cdots & \vdots \\ \vdots & \vdots & h_1(N_a-1) & \cdots & h_1(2) & h_1(1) & h_1(0) & 0 & \cdots & \vdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & h_0(1) & h_1(0) \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & h_1(1) & h_0(0) \end{bmatrix}$$

Figure 3. Structure of truncated analysis matrix $\mathbf{H}_t$

In the orthogonal case, the boundary filters can be found by using a Gram-Schmidt based technique as described in [4]. However, this technique can not guarantee any feature beyond orthogonality for the boundary filters. As stated earlier, for a biorthogonal filter bank, only linear independence of the rows are necessary.

For the left boundary we will need b_l boundary filters, and for the right we will need b_r filters, with $b_l + b_r = N_a - 2$. The number of left and right filters can be chosen arbitrarily, but the structure of the problem suggests that they should be as nearly balanced as possible. There are also no constraints on the length of the filters (except that they be <=L), but as their purpose is to analyze the boundaries of the input signal (beginning and end), their support should reflect that. Therefore, we will require that the support of the boundary filters be equal to or less than the support of the analysis filters.

Beyond the requirements specified above, there are no further restrictions for the design of the boundary filters. Therefore, for the left boundary case, any row vector $\mathbf{b}^T$ with support less than or equal to $\mathbf{N}_a$ is suitable provided the following is true:

$$\mathrm{rank}\left(\begin{bmatrix}\begin{bmatrix}\mathbf{b}^T & 0\end{bmatrix}\\ \mathbf{H}_t\end{bmatrix}\right) = \mathrm{rank}\left(\mathbf{H}_t\right)+1 \qquad (5)$$

For the right boundary case, Equation 5 becomes:

$$\mathrm{rank}\left(\begin{bmatrix}\mathbf{H}_t\\ \begin{bmatrix}0 & \mathbf{b}^T\end{bmatrix}\end{bmatrix}\right) = \mathrm{rank}\left(\mathbf{H}_t\right)+1 \qquad (6)$$

Stated another way, the vector $[\mathbf{b}^T\ 0]$ (or $[0\ \mathbf{b}^T]$) must not lie in the row space of $\mathbf{H}_t$ to be considered a candidate (this would violate Equations 5 and 6). If Equation 5 (or 6) is satisfied for a particular $\mathbf{b}^T$, then that row is kept, and the truncated matrix is updated to include the new row. Equation 5 and 6 are repeated for each additional row that is added until $\mathbf{H}_t$ becomes square.

To further simplify the computation, the same boundary filters for the left side can be used on the right side, provided that the non-zero values of the filters do not overlap. This can be easily seen by a simple example. Let $\mathbf{L} = \mathbf{6}$, and the first left boundary filter $\mathbf{b}_{ll}=[\mathbf{1\ 1\ 0\ 0\ 0\ 0}]$, a Haar low-pass filter. The right boundary filter can be chosen to be $\mathbf{b}_{rl}=[\mathbf{0\ 0\ 0\ 0\ 1\ 1}]$. This filter is guaranteed to be independent of the left boundary filter, and if it also satisfies Equation 6, it is a valid candidate. In most practical applications Equations 5 and 6 are easily satisfied.

With this method, the boundary filters can be designed using various FIR design methods, or by choosing established filters that exhibit the desired behavior (frequency selectivity, number of vanishing moments, etc...). The next step is to determine the dual of the now square analysis matrix.

The dual of any square invertable matrix is its inverse. Equations 5 and 6 guarantees that the analysis matrix will have full rank, and therefore it is invertable. If $\mathbf{H}_s$ is the newly created square analysis matrix, then the synthesis matrix $\mathbf{F}_s$ can be expressed as

$$\mathbf{F}_s = \mathbf{H}_s^{-1}\ . \qquad (7)$$

Although conceptually simple, calculating the inverse of a large matrix can be a computational nightmare. However, the bulk of the computation can be avoided due to the structure of $\mathbf{H}_s$. The most important point is that the steady-state portion of the finite synthesis matrix will be identical to its infinite counterpart. This is a consequence of how the analysis matrix was designed. The middle of $\mathbf{F}_s$ is just a truncated version of the infinite matrix F. Furthermore, due to the sparse structure of $\mathbf{H}_s$ and with proper partitioning of $\mathbf{H}_s$ the boundaries of the synthesis matrix can be found with significantly less computational complexity. One side effect of the inverse operation is that the synthesis matrix will have more boundary filters than the analysis matrix, and is directly proportional to the number of analysis boundary filters used and their support.

IV. Design Example

One application where finite length input signals are used is segmentation based audio coding. Using boundary filters, an audio signal can be divided into short frames without overlap or windowing.

Consider a 2-channel filter bank that uses the Daubechies 8-tap (db4) analysis filters. The db4 filters can be used to create an orthogonal filter bank for the infinite input case. The filters also have 4 vanishing moments. For this example, we will analyze the first 1000 samples of the attack transient portion of a piano note, shown in Figure 4.

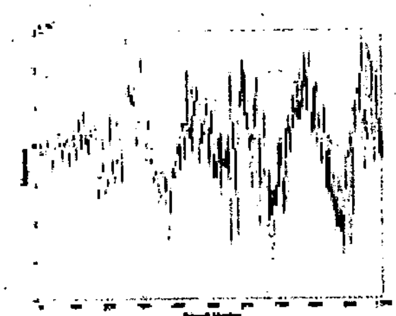

Figure 4. First 1000 samples of a piano note.

The finite analysis filter bank matrix will require 6 boundary filters. For this application, we want the boundary filters to be as identical to the db4 filters as possible in terms of frequency selectivity and number of vanishing moments. Therefore, we need to find two sets of low/high pass filters with the necessary properties to use as boundary filters (note that we can use the same filters for the left and right boundaries as discussed earlier).

The db4 filters come from wavelet analysis. It would be reasonable to choose boundary filters that also come from wavelets because of their similar properties. Therefore, we can choose one set of low/high pass filters to be the Symlet 8-tap analysis filters (Sym4). This filter also has 4 vanishing moments and frequency selectivity identical to the db4 filters. For the second set we can use the db3 analysis filters which has 3 vanishing moments and very similar frequency selectivity. Figure 5 shows the magnitude response of the boundary filters and the main analysis filters.

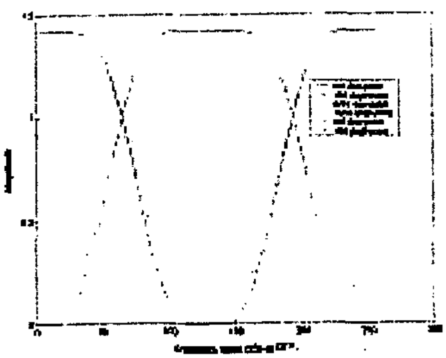

Figure 5. Frequency response of boundary filters and main analysis filters. Frequency selectivity is nearly identical.

Figure 6 shows the sub-band coefficients after analysis is performed. The coefficients are smooth at the boundaries (no large jumps), and the total length of the sub-band coefficients is equal to the original signal length.

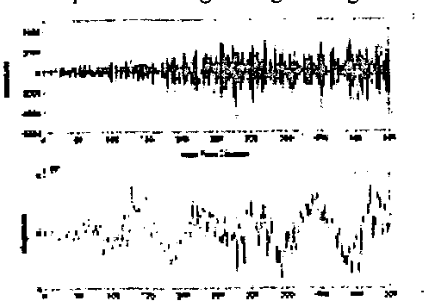

Figure 6. Sub-band coefficients of analyzed piano sample. Upper plot is high pass channel, and lower plot is low pass channel.

The signal can be exactly reconstructed by calculating the inverse of the analysis matrix. The resultant filter bank is biorthogonal, even though the initial filters are orthogonal.

V. Conclusions

In this paper, design methods for boundary filters were presented. For any choice of analysis filters, corresponding boundary filters can be designed to satisfy any criteria including frequency selectivity and number of vanishing moments. The resulting filter bank will be a biorthogonal filter bank. The synthesis bank is calculated by finding the dual of the analysis bank.

VI. Discussion

The advantage of this method lies in the control over the analysis bank. The synthesis bank, however, does not have any freedom of design. No guarantees can be made about the characteristics of the boundary filters added to the synthesis bank.

The methods described can be used in reverse, with the analysis bank resulting from the dual of the synthesis bank. The methods presented are extendable to M channel filter banks with input vectors of arbitrary length. Furthermore, these methods can be extended to time-varying filter banks.

References

[1] A. Mertins, "Boundary Filters for Size-Limited Paraunitary Filter Banks with Maximum Coding Gain and Ideal DC Behavior," *IEEE Trans. Circuits and Systems II*, vol. 48, NO. 2, pp.183-188, Feb. 2001

[2] A. Mertins, "Boundary Filters for Segmentation-Based Subband Coding," in *Proc. Int. Conf. Signal Processingi*, 2000.

[3] A. Mertins, "Time-varying and support preservative filter banks: Design of optimal transition and boundary filters via SVD," in *Proc. IEEE Int. Conf. Acoust. Speech, Signal Processing*, Detroit, pp. 1316-1319, May 1995.

[4] C. Herley, "Boundary filters for finite-length signals and time-varying filter banks," *IEEE Trans. Circuits and Systems II*, vol. 42, pp.102-114, Feb. 1995.

[5] C. Herley, J. Kovacevic, K. Ramchandran, and M. Vetterli, "Tilings of the time-frequency plane," *IEEE Trans. SP* 41 (1993) 3341-3359.

[6] G. Strang and T. Nguyen, "Wavelets and Filter Banks," Wellesley: Wellesley-Cambridge Press, 1996.

[7] K. McGill and C. Taswell, "Length-Preserving Wavelet Transform Algorithms for Zero-Padded and Linearly-Extended Signals," Tech. Rep., Veterans Affairs Medical Center, Palo Alto, CA, 1992.