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Growing Optimally Rigid Formations

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***Abstract*— This work extends the theory on rigid frameworks for formation keeping in multi-agent systems. We introduce the $\mathcal{H}_2$ performance measure for relative sensing networks where the underlying sensing graph is rigid. The first contribution shows that the optimal $\mathcal{H}_2$ sensing graph must be a minimally rigid graph. We then describe a variation of the Herrenberg construction for generating rigid graphs in the plane by adding performance requirements and sensing constraints, leading to the $\mathcal{H}_2$ optimal vertex addition and edge splitting procedures. These results are then used to derive a centralized algorithm for generating an $\mathcal{H}_2$ optimally rigid relative sensing network.**

I. Introduction

Formation control is one of the fundamental problems for multi-agent systems consisting of mobile and dynamically decoupled agents. Many applications depend on accurate and robust formation control to achieve team-level objectives, such as interferometry in deep space, distributed sensing for environmental monitoring, and surveillance and reconnaissance missions [1], [2], [3], [4]. As a well defined problem, it encompasses many of the challenges associated with the more general problem of distributed control and decision making for multi-agent systems.

One of the challenges surrounding formation control is the alignment of theoretical and analytical tools with the constraints of real-world systems. Indeed, assumptions leading to problem simplifications, such as full-state information and all-to-all communication, are not realizable in practice, or at least without significant cost. Consequently, this area has been intensely pursued in the controls community.

In this direction, work in formation control has generated two different approaches: position-based and distance-based control. In position-based formation control, each agent requires either the inertial positions or relative positions of its neighbors [5], [6], [7], [8]. A significant challenge with these strategies is the implicit assumption that all agents either have a common inertial frame or sensors capable of measuring position and velocity in a relative coordinate frame. In distance-based control strategies, only the relative distances between neighboring agents are required for formation control [9], [10], [11]. These strategies have an advantage from an implementation standpoint, as relative distance measurements are independent of any reference frame. However, distance measurements alone may not be sufficient to specify a formation.Indeed, specifying a formation using only distance measurements requires a minimum number of distances to uniquely construct the formation. This concept can be formalized using tools from graph theory and the notion of *rigidity*. Graph rigidity and rigid frameworks have recently been identified as an important architectural tool for the specification and maintenance of formations [12], [13], [14], [15].

The contribution of this work is the inclusion of system performance measures in the specification and generation of rigid frameworks for formation control. The notion of optimally rigid formations in the context of formation control has, thus far, not been widely considered. In [16], rigid graphs were formed in such a way that the distance between agents were minimized. This was justified by relating communication costs to inter-agent distances. Other works have focused on operations for joining or splitting formations in a way that preserves rigidity with no requirements on system performance; see, for example, [12], [14] for an overview.

The specification of system performance in the construction of rigid formations, however, has not been considered. Performance measures, such as the $\mathcal{H}_2$ norm, provide an important characterization of how the system will behave in the presence of disturbances and exogenous inputs. A formation that is less sensitive to disturbances has direct implications for the performance of the corresponding formation control laws. For relative sensing networks, it was shown in [17] that the overall system performance can be described in terms of properties of the underlying sensing graph. We extend these results in this work in the context of performance requirements for rigid frameworks. In this direction, we provide an algorithm for generating rigid formations that are optimal from an $\mathcal{H}_2$ standpoint. In particular, we show that for agents with heterogeneous and linear dynamics, optimally rigid formations can be constructed using a greedy algorithm. The main step of the algorithm relies on an extension of the Herrenberg construction for generating rigid graphs in the plane [18]. We modify this procedure to include sensing constraints of the agents and performance requirements in terms of the $\mathcal{H}_2$ norm.

The outline of this paper is as follows. Some preliminaries and notations will be introduced in the next sub-section. The performance of formations will be described in §II. A review of rigidity theory and Henneberg constructions is given in §III. In §IV, the main results are given with a simulation example in §V. Finally, some concluding remarks are offered in §VI.

A. Preliminaries and Notations

Graphs and the matrices associated with them will be widely used in this work [19]. We denote an undirected graph with node set $\mathcal{V}$ and edge set $\mathcal{E}$ as the pair, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We

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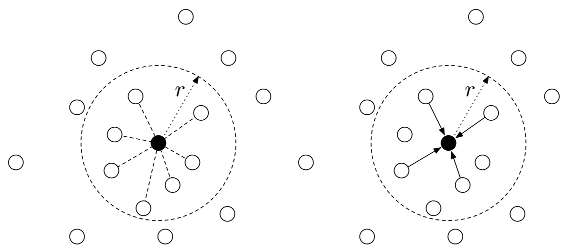


(a) Communication range of an agent with all possible edges. (b) Agents must decide how many neighbors to communicate with.

Fig. 1. Each agent can establish sensing and communication links with other agents inside the sensing range.

employ the notation $v_i \sim v_j$ to describe incidence relations between vertices; i.e. when $\{v_i, v_j\} \in \mathcal{E}$. The incidence matrix $E(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$ is a $\{0, \pm 1\}$-matrix defined in the usual way [19]. The degree of vertex i, d_i, is the cardinality of the set of vertices adjacent to it.

II. Formations and Performance

A. Relative Sensing Networks

We consider a collection of N agents, indexed by the set $\mathcal{V} = \{1, 2, \ldots, N\}$. Each agent is assumed to have heterogeneous linear and time-invariant dynamics,

$$\Sigma_i \begin{cases} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + \Gamma_i w_i(t) \\ z_i(t) &= C_i^z x_i(t) + D_i^{11} u_i(t) + D_i^{12} w_i(t) \\ y_i(t) &= C_i^y x_i(t) + D_i^{21} u_i(t) + D_i^{22} w_i(t), \end{cases} \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$ is the state of agent i, $u_i(t) \in \mathbb{R}^m$ the control, $w_i(t) \in \mathbb{R}^p$ the external disturbances, $z_i(t) \in \mathbb{R}^r$ the controlled variable, and $y_i(t) \in \mathbb{R}^m$ the measured output. We further assume that all motions are restricted to the plane; that is the position of each agent is in $\mathbb{R}^2$. In particular, we denote the position of each agent as $q_i(t) \in \mathbb{R}^2$, with

$$q_i(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} x_i(t) = C_r x_i(t).$$

Each agent is also equipped with a sensor having the ability to measure and communicate relative state information between other agents. The sensor is assumed to have finite range, and can only measure and communicate with agents within that range. We assume identical sensors on each agent, and the range is given by the parameter r; this is visualized in Figure 1(a). We define the *neighborhood* of an agent at time t as the set of agents that are within its sensing range, denoted as

$$\mathcal{N}(v_i, t) = \{v_j \in \mathcal{V} \mid \|q_i(t) - q_j(t)\| \leq r\}.$$

It is also assumed that there is a cost 'c' associated with establishing a sensing and communication link with another agent. In this way, communication links are modeled as directed edges in the graph; see Figure 1(b).

The decision to establish an edge between agents, therefore, induces a graph. Denote the set of edges that agent i establishes as $\mathcal{E}_i \subseteq \mathcal{V} \times \mathcal{V}$. Each edge is also assigned a weight, defined to be the Euclidian distance between the two agents, $w_{ij}(t) = \|q_i(t) - q_j(t)\|^2$. The union of all the edges, $\mathcal{E} = \cup_i \mathcal{E}_i$, together with their weights w and the set of agents $\mathcal{V}$, defines the weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$. Associated with the graph $\mathcal{G}$ is its incidence matrix, denoted $E(\mathcal{G}) \in \mathbb{R}^{N\times|\mathcal{E}|}$. We also define the incidence matrix associated with each agent as $E(\mathcal{G}_i) \in \mathbb{R}^{N\times|\mathcal{E}_i|}$, where $\mathcal{G}_i = (\mathcal{V}, \mathcal{E}_i) \subseteq \mathcal{G}$ is the subgraph induced by the edges agent i establishes. By construction, each sub-graph $\mathcal{G}_i$ will be a *star graph*, and the oriented incidence matrix will have all edges entering the center node; see Figure 1(b).

With the notion of a relative sensing graph in place, each agent is able to use the sensed information in the implementation of a control law used to achieve high-level group objectives (such as formations). In particular, the dynamics of each agent is augmented with the relative sensed information,

$$\Sigma_i(\mathcal{G}_i) \begin{cases} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + \Gamma_i w_i(t) \\ z_i(t) &= C_i^z x_i(t) + D_i^{11} u_i(t) + D_i^{12} w_i(t) \\ y_i(t) &= C_i^y x_i(t) + D_i^{21} u_i(t) + D_i^{22} w_i(t) \\ g_i(t) &= (E(\mathcal{G}_i)^T \otimes C_r) x(t). \end{cases}$$

The aggregate dynamics of the entire system can now be described in state-space form as

$$\Sigma(\mathcal{G}) \begin{cases} \dot{x}_i(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) + \mathbf{\Gamma}w(t) \\ z(t) &= \mathbf{C}^z x(t) + \mathbf{D}^{11} u(t) + \mathbf{D}^{12} w(t) \\ y(t) &= \mathbf{C}^y x(t) + \mathbf{D}^{21} u(t) + \mathbf{D}^{22} w(t) \\ g(t) &= (E(\mathcal{G})^T \otimes C_r) x(t), \end{cases} \quad (2)$$

where the bold-face matrices represent the diagonal concatenation of each agent's state-space matrices (e.g., $\mathbf{A} = \mathbf{diag}\{A_1, \ldots, A_N\}$). The notation $\otimes$ denotes the Kronecker product [20]

A *formation* is the specification of inter-agent distances; that is, it is the assignment of weights w_{ij} to each edge in the graph $\mathcal{G}$. We denote the set of weights defining the formation as $\mathcal{F}$, and the formation graph as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$. As alluded to in the introduction, the assignment of inter-agent distances alone for an arbitrary graph is not sufficient to specify a unique formation. This concept will be refined later in the context of rigid graphs.

B. Performance of Relative Sensing Networks

The system-theoretic performance of multi-agent systems, such as in (2), can be examined from the standard state-space perspective. A more enlightening approach, however, is to consider graph-theoretic descriptions of the performance. In this way, a direct connection between the role of the system interconnections and its affects on the overall system performance can be established. This leads to both a graph-theoretic understanding of the system, and the opportunity to include graph properties in the synthesis of these systems. In the work [17], the authors derived graph-theoretic performance bounds for relative sensing networks of the form (2).

One of the main results from [17] was a characterization of the $\mathcal{H}_2$ performance for the system in (2) from the exogenous input $w(t)$ to the sensed output $g(t)$.

Theorem 2.1 ([17]): The $\mathcal{H}_2$ performance of the relative sensing network (2) is given as

$$\|\Sigma(\mathcal{G})\|_2 = \|QE(\mathcal{G})\|_F, \quad (3)$$

where $Q = \textbf{diag}\{\|\Sigma_1\|_2, \ldots, \|\Sigma_N\|_2\}$, $\|\cdot\|_F$ denotes the Frobenius norm of a matrix, $\|\cdot\|_2$ denotes the operator or matrix 2-norm depending on the argument. When the RSN is comprised of homogeneous dynamic agents, i.e. when $Q = \|\Sigma\|_2 I_N$, the performance simplifies to

$$\|\Sigma(\mathcal{G})\|_2 = \|E(\mathcal{G})\|_F \|\Sigma\|_2. \quad (4)$$

Qualitatively, Theorem 2.1 indicates that the $\mathcal{H}_2$ performance of (2) is related to the degree of each agent (or equivalently, the number of edges in the graph). For example, consider the case of constructing an RSN of heterogeneous agents with the requirement that the graph is connected and the $\mathcal{H}_2$ norm of the entire system is minimized. In this case, the *minimum weight spanning tree* is the optimal graph for such a system [17].

Regarding the earlier description of the communication capabilities of each agent, the decision for an agent to establish a link to another agent can now be cast as a performance problem. Each agent should make these decisions in such a way that the overall system performance of (2) is minimized. This will be discussed in more detail in the sequel.

III. Graph Rigidity

In this section we review the fundamental concepts of graph rigidity. This discussion will culminate with the presentation of the *Henneberg constructions* for building rigid graphs in the plane. For a detailed treatment on graph rigidity and Henneberg constructions, the reader is referred to [21], [22].

The notion of graph rigidity begins with what is known as a *d-dimensional bar-and-joint framework*. A framework is the pair $(\mathcal{G}, p)$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph, and $p : \mathcal{V} \mapsto \mathbb{R}^d$ maps each vertex to a point in $\mathbb{R}^d$; in this work we restrict our attention to $d = 2$. Each edge in the graph is assigned a *weight*, which is defined to be the Euclidean distance between two nodes mapped into the Euclidean space by p; for the edge $(u, v) \in \mathcal{E}$, its weight is $w_{uv} = \|p(u) - p(v)\|$. We now provide some basic definitions.

Definition 3.1: Frameworks $(\mathcal{G}, p_0)$ and $(\mathcal{G}, p_1)$ are *equivalent* if $\|p_0(u)-p_0(v)\| = \|p_1(u)-p_1(v)\|$ for all $(u, v) \in \mathcal{E}$.

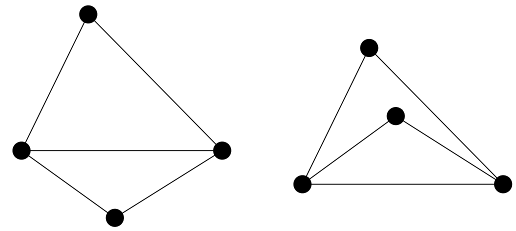
Definition 3.2: Frameworks $(\mathcal{G}, p_0)$ and $(\mathcal{G}, p_1)$ are *congruent* if $\|p_0(u)-p_0(v)\| = \|p_1(u)-p_1(v)\|$ for all $u, v \in \mathcal{V}$.

Definition 3.3: $(\mathcal{G}, p_0)$ is *globally rigid* if every framework which is equivalent to $(\mathcal{G}, p_0)$ is congruent to $(\mathcal{G}, p_0)$.

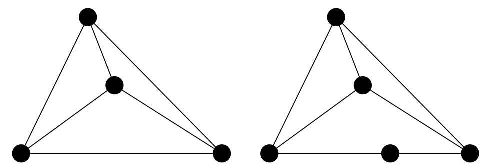
Definition 3.4: $(\mathcal{G}, p_0)$ is *rigid* if there exists an $\epsilon > 0$ such that every framework $(\mathcal{G}, p_1)$ which is equivalent to $(\mathcal{G}, p_0)$ and satisfies $\|p_0(v) - p_1(v)\| < \epsilon$ for all $v \in \mathcal{V}$, is congruent to $(\mathcal{G}, p_0)$.

Definition 3.5: A *minimally rigid graph* is a rigid graph such that the removal of any edge results in a non-rigid graph.

Figure 2 gives examples of rigid graphs. The graphs in Figure 2(a) are both minimally rigid and are equivalent to each other, but are not globally rigid. By adding an additional edge, as in Figure 2(b), the graph becomes globally rigid. The key feature of global rigidity, therefore, is that the distances between *all* node pairs are maintained, and not just those defined by the edge set.



(a) Two equivalent rigid graphs.



(b) A globally rigid graph. (c) Not infinitesimally rigid.

Fig. 2. Examples of rigid, globally rigid, and infinitesimally rigid.

Consider a framework $(\mathcal{G}, p)$ and trajectories $q_i(t)$ of each agent, where each trajectory is associated with a node in the graph. The *trajectory of the framework* is *edge-consistent* if $\|q_i(t) - q_j(t)\| = \|p(v_i) - p(v_j)\|$ for all time.

In rigidity theory, this notion is formalized by considering *infinitesimal motions* of the mapped vertices $p(v_i)$; these are assignments of velocity vectors ξ_i to each vertex v_i such that

$$(\xi_i - \xi_j)^T(p(v_i) - p(v_j)) = 0, \ \forall (v_i, v_j) \in \mathcal{E}. \quad (5)$$

If the mapping p is further parameterized by a positive scalar representing time, then we can consider the infinitesimal motions at each time, and define $\dot{p}(v_i, t) = \xi_i$ and (5) is satisfied for all t then the framework is *infinitesimally rigid*. Observe that the graph in Figure 2(b) is infinitesimally rigid, but the graph in Figure 2(c) is not. Additionally, infinitesimal rigidity implies rigidity.

Infinitesimal rigidity is an important property that allows to consider rigidity as a property of only the graph, as opposed to rigidity of a framework. This is established through the notion of a *generically rigid graph*, which can be thought of as a realization of a particular framework. Therefore, one has that a graph is generically rigid if it has an infinitesimally rigid realization [4].

Finally, we recall an important result from Laman, that expresses rigidity as a purely graph-theoretic concept.

Theorem 3.1 (Laman [23]): A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| \geq 2$ vertices embedded in $\mathbb{R}^2$ is minimally rigid if and only if $|\mathcal{E}| = 2|\mathcal{V}| - 3$ and each induced subgraph $\mathcal{H} = (\mathcal{V}', \mathcal{E}') \subseteq \mathcal{G}$ satisfies $|\mathcal{E}'| \leq 2|\mathcal{V}'| - 3$.

A. Henneberg Constructions

Henneberg provided a constructive method for generating all minimally rigid graphs in the plane beginning with a graph containing two vertices and a single edge between them [18]. The general construction is based on two fundamental operations termed *vertex addition* and *edge splitting*.

Proposition 1 (vertex addition): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with two distinct vertices v_i and v_j, and let $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ be the graph obtained by connecting a new vertex

$u \notin \mathcal{V}$ with edges (v_i, u) and (v_j, u) to the graph $\mathcal{G}$. The $\mathcal{G}$ is infinitesimally rigid if and only if $\mathcal{G}^*$ is infinitesimally rigid.

Proposition 2 (edge splitting): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with three distinct vertices v_i, v_j, and v_k such that $(v_i, v_j) \in \mathcal{E}$. Let $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ be the graph obtained by (i) deleting the edge (v_i, v_j) and (ii) connecting a new vertex $u \notin \mathcal{V}$ with edges (v_i, u), (v_j, u), and (v_k, u). Then $\mathcal{G}$ is infinitesimally rigid if and only if $\mathcal{G}^*$ is infinitesimally rigid.

Propositions 1 and 2 represents essential procedures for constructing rigid frameworks in the plane. Note that these procedures will also lead to minimally rigid graphs if the base graph $\mathcal{G}$ is minimally rigid. This is a key feature for joining graphs in a rigid way, and is discussed in much of the literature related to formation keeping and rigidity. While very simple, this procedure does not indicate *which* nodes, if there are many possible nodes to attach to, to connect to.

IV. GROWING OPTIMALLY RIGID GRAPHS

In this section we discuss how to build rigid graphs in the plane with an optimality cretieria. The objective is to augment the Henneberg construction presented in §III-A with a notion of optimality. In particular, we modify propositions 1 and 2 to consider the overall $\mathcal{H}_2$ performance of the network in addition to the sensing range of the new node in the system. In the following, we will refer to RSNs with a sensing graph that is rigid and minimizes the $\mathcal{H}_2$ norm as the $\mathcal{H}_2$ *optimally rigid graph.*

In this direction, we first formally state the problem we consider in the form of an optimization problem. The objective is to minimize the $\mathcal{H}_2$ norm of an RSN parameterized by the underlying sensing graph. Additionally, we want to minimize the communication costs (i.e., the number of edges) and also guarantee that the resulting graph is rigid.

$$\begin{aligned} \min_{\mathcal{G}} \quad & \|\Sigma(\mathcal{G})\|_2 + c|\mathcal{E}| \\ \text{s.t.} \quad & \mathcal{G} \text{ is rigid.} \end{aligned} \tag{6}$$

Having already established in §III that rigidity can be equated to a purely graph-theoretic property, we can make further statements about the optimality of rigid graphs from the $\mathcal{H}_2$ perspective. The first result relates the $\mathcal{H}_2$ optimally rigid graphs to minimally rigid graphs.

Theorem 4.1: The $\mathcal{H}_2$ optimally rigid graph is minimally rigid.

Proof: For homogeneous RSNs, the $\mathcal{H}_2$ norm is expressed in terms of the Frobenius norm of the incidence matrix. The Frobenius norm can be expressed in terms of the number of edges in the graph, $\|E(\mathcal{G})\|_F^2 = 2|\mathcal{E}|$. Therefore, a rigid graph with the minimum number of edges, i.e. a minimally rigid graph, will minimize the RSN system norm.

For heterogeneous RSNs, observe that $\|Q_2 E(\mathcal{G})\|_F^2 = \sum_i d_i \|\Sigma_i\|_2^2$. Next, adding an edge e to the RSN must strictly increase the norm, as $\|Q_2 \begin{bmatrix} E(\mathcal{G}) & e \end{bmatrix}\|_F^2 = \|Q_2 E(\mathcal{G})\|_F^2 + \|Q_2 e\|_F^2$. Therefore, a rigid graph with the minimum number of edges will minimize the RSN $\mathcal{H}_2$ norm. ■

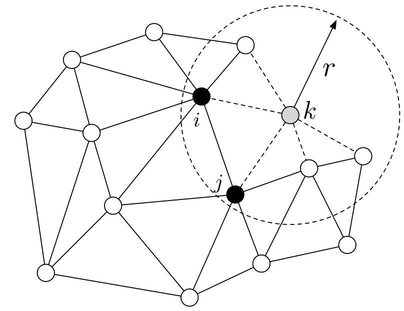


Fig. 3. Sensing range limits available nodes for Henneberg constructions.

Having established the minimally rigid graphs are optimal for RSNs, we can focus on growing minimally rigid graphs making the Henneberg construction suitable. In the general vertex addition or edge splitting step of the Henneberg constructions, a new node can attach to *any* node in the network. However, when considering dynamic systems with limited sensing and communication, the nodes that are available for attachment are limited. For example, in Figure 3(a), the node v_k can only choose between 5 nodes for establishing a link.

Therefore, the Henneberg constructions must be modified to include both the "range" of a new vertex, and an optimality criteria for establishing new edges. In this regard, we first present a result for adding a new vertex to an RSN that is already an $\mathcal{H}_2$ optimally rigid graph. Before presenting the result, it should be noted that this procedure leads to a *locally* $\mathcal{H}_2$ optimal RSN. The vertex addition is limited to the nodes within the sensing range of the new agent. If, for example, the sensing range included all the other agents in the RSN, then, in fact, this procedure would produce the globally optimal RSN.

Proposition 3 ($\mathcal{H}_2$ Optimal Vertex Addition): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the $\mathcal{H}_2$ optimally rigid sensing graph for the relative sensing network $\Sigma(\mathcal{G})$ in (2). Consider a new dynamic agent Σ_u not in the RSN, represented by a node $u \notin \mathcal{V}$ with position q_u. Additionally, assume that $|\mathcal{N}(u,t)| \geq 2$. Let $\mathcal{G}^* = (\mathcal{V} \cup u, \mathcal{E}^*)$ be the graph obtained by connecting the new vertex u with edges (v_i, u) and (v_j, u) where $v_i, v_j \in \mathcal{N}(u,t)$ and $d_i\|\Sigma_i\|_2$ and $d_j\|\Sigma_j\|_2$ are the agents with smallest weighted-$\mathcal{H}_2$-norm in the set $\mathcal{N}(u,t)$. Then the graph $\mathcal{G}^*$ is infinitesimally rigid and the RSN $\Sigma(\mathcal{G}^*)$ is a (locally) $\mathcal{H}_2$ optimally rigid graph.

Proof: The proof is a direct consequence of Proposition 1 and Theorem 4.1. ■

Remark 4.2: The procedure requires that the new agent communicate with every other agent in its neighborhood in order to obtain the weighted norm of its neighbors. The agent must then perform a sorting operation in order to decide which edges to establish. In the case where there in not a unique solution, the agent can arbitrarily choose amongst the possible solutions.

Proposition 4 ($\mathcal{H}_2$ Optimal Edge Splitting): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the $\mathcal{H}_2$ optimally rigid sensing graph for the relative sensing network $\Sigma(\mathcal{G})$ in (2). Consider a new dynamic agent Σ_u not in the RSN, represented by a node $u \notin \mathcal{V}$

with position q_u. Additionally, assume that $|\mathcal{N}(u,t)| \geq 3$ and at least two nodes in the neighborhood set are directly connected. Let $\mathcal{G}^* = (\mathcal{V} \cup u, \mathcal{E}^*)$ be the graph obtained by connecting the new vertex u with edges (v_i, u), (v_j, u), and (v_k, u) where $v_i, v_j, v_k \in \mathcal{N}(u,t)$, removing the edge $(v_i, v_j) \in \mathcal{E}$, and $d_k\|\Sigma_k\|_2$ is the agent with smallest weighted-$\mathcal{H}_2$-norm in the set $\mathcal{N}(u,t)$. Then the graph $\mathcal{G}^*$ is infinitesimally rigid and the RSN $\Sigma(\mathcal{G}^*)$ is a (locally) $\mathcal{H}_2$ optimally rigid graph.

Proof: The proof is a direct consequence of Proposition 2 and Theorem 4.1. ∎

Remark 4.3: The edge splitting procedure requires a similar communication round as described in Remark 4.2.

Propositions 3 and 4 can be used to describe an algorithm for growing $\mathcal{H}_2$ optimally rigid graphs. The algorithm we present here is a centralized algorithm inspired by the celebrated *Kruskal's Algorithm* for finding a minimum weight spanning tree [24]. Although not considered in this work, the minimum weight spanning tree can also be found using distributed algorithms [25], [26], which can lead to distributed versions of the presented algorithm.

Before proceeding, we first discuss what will be an important operation in the algorithm. At each stage in the algorithm, an agent must decide to join the network by using either the vertex addition or edge splitting operation. This decision will be based on which operation leads to a better performance. Assume that the assumptions of Proposition 3 and 4 on the set $\mathcal{N}(u,t)$ are met, and denote $\mathcal{G}_v^*$ as the graph obtained using the vertex addition operation, and $\mathcal{G}_e^*$ as the graph obtained using the edge splitting operation. It is straight forward to verify that

$$\begin{aligned} \|\Sigma(\mathcal{G}_v^*)\|_2^2 &= \|\Sigma(\mathcal{G})\|_2^2 + 2\|\Sigma_u\|_2^2 + \|\Sigma_i\|_2^2 + \|\Sigma_j\|_2^2, \\ \|\Sigma(\mathcal{G}_e^*)\|_2^2 &= \|\Sigma(\mathcal{G})\|_2^2 + 3\|\Sigma_u\|_2^2 + \|\Sigma_k\|_2^2. \end{aligned}$$

Therefore, deciding which procedure to use involves selecting the minimum between the following expression,

$$\min\{2\|\Sigma_u\|_2^2 + \|\Sigma_i\|_2^2 + \|\Sigma_j\|_2^2,\ 3\|\Sigma_u\|_2^2 + \|\Sigma_k\|_2^2\}. \quad (7)$$

Note that in both cases, a net of 2 new edges is added, so the cost due to edges is identical for both operations.

With this selection criteria in place, we are now ready to formally state the algorithm.

Remark 4.4: The termination of this algorithm requires a basic assumption on the connectivity of the underlying graph with all possible candidate edges. Indeed, a necessary condition for the termination of this algorithm is that the graph is connected, each node has degree at least two, and the graph is rigid.

Remark 4.5: The correctness of this algorithm is due to Propositions 3 and 4. The algorithm begins with the minimum weight edge, and vertices are added based on their ordered $\mathcal{H}_2$ norm. Due to the sensing constraint on each node, as already discussed, this algorithm will produce a locally optimally rigid graph. We also note that the solution may not be unique.

Algorithm 1: $\mathcal{H}_2$ Optimally Rigid Graph Algorithm

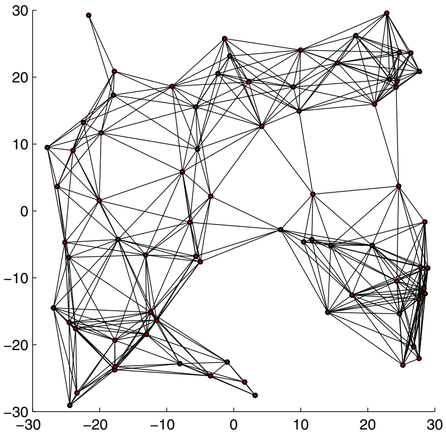
Data: A set of N dynamic agents of form (1), indexed by the set $\mathcal{V} = \{v_1, \ldots, v_n\}$. Each agent has $\mathcal{H}_2$ norm $\|\Sigma_i\|_2$ and identical sensing radius r.
Result: An $\mathcal{H}_2$ optimally rigid graph.
begin
 ·Sort and relabel each agent according to their $\mathcal{H}_2$ norm such that $\|\Sigma_1\|_2^2 \leq \|\Sigma_2\|_2^2 \leq \cdots \leq \|\Sigma_N\|_2^2$
 ·Assign weights, sort, and label candidate edges† such that $w(e_1) \leq \cdots \leq w(e_{|\mathcal{E}|})$, where $e_i = (v_k, v_l) \in \mathcal{E}$ and $w(e_i) = \|\Sigma_k\|_2^2 + \|\Sigma_l\|_2^2$.
 ·Set $\mathcal{G}^* := (\mathcal{V}^*, \mathcal{E}^*)$ with $\mathcal{V}^* = \{v_a, v_b\}$, $\mathcal{E}^* = \{e_1 = (v_a, v_b)\}$.
 while $\mathcal{V}^* \neq \mathcal{V}$ **do**
  ·Set $\Omega = \{v \in \mathcal{V} \,|\, |\mathcal{V}^* \cap \mathcal{N}(v,t)| \geq 2\}$ and select the node $u = \arg\min_{i\in\Omega} \|\Sigma_i\|_2^2$
  if $|\mathcal{N}(u,t)| = 2$ **then**
   ·**do** $\mathcal{H}_2$ Optimal Vertex Addition (new edges e_a, e_b)
   ·Set $\mathcal{G}^* = (\mathcal{V}^* \cup \{u\}, \mathcal{E}^* \cup \{e_a, e_b\})$
  else
   ·Evaluate (7) for candidate edges
   ·**do** $\mathcal{H}_2$ Optimal Vertex Addition or $\mathcal{H}_2$ Optimal Edge Splitting based on (7) (new edges $\{e_a, e_b, e_c\}$ and deleted edge e_d)
   ·Set $\mathcal{G}^* = (\mathcal{V}^* \cup \{u\}, \mathcal{E}^* \cup \{e_a, e_b\})$ or $\mathcal{G}^* = (\mathcal{V}^* \cup \{u\}, \mathcal{E}^* \cup \{e_a, e_b, e_c\} - e_d)$

 † The candidate edges are all possible edges an agent can establish within its sensing range.

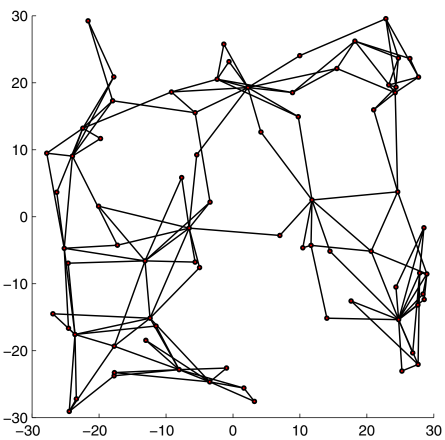
V. Simulation Example

In this section we provide a simulation example demonstrating the Algorithm 1. The scenario considered here is a mission related to the *Autonomous NanoTechnology Swarm* project, or ANTS, currently under investigation by NASA [27]. One component of the ANTS mission involves the deployment of 1,000 pico-satellites to the asteroid belt for observational study. En-route to the asteroid belt the spacecraft must organize into smaller teams that will coordinate to search for various resources and materials. Maintaining a formation can be considered a desirable objective and the satellites should therefore establish a sensing graph that is $\mathcal{H}_2$ optimally rigid.

For this example, we consider 75 agents uniformly distributed on the plane. Each agent is assigned a random and stable SISO state-space model in MATLAB. The sensing radius of each agent is $r = 15$. Figure 4(a) shows the position of each agent along with every possible edge (i.e., all the edges that the agents can establish based on the sensing radius). Based on this initial condition, Algorithm 1 was applied, and the resulting $\mathcal{H}_2$ optimally rigid formation is shown in Figure 4(b). The graph with all possible edges has an $\mathcal{H}_2$ norm of 55.8082 with 409 edges, while the optimally



(a) All possible edges.



(b) The $\mathcal{H}_2$ optimally rigid graph.

Fig. 4. Example of Algorithm 1.

rigid graph produced a performance of 23.8166 and 147 edges (satisfying one of the conditions of Theorem 3.1). An interesting result of this algorithm is that it suggests that certain agents are more advantageous to establish many connections with in terms of the overall system performance.

VI. Concluding Remarks

This work extended the general theory of graph rigidity for formations by introducing system performance metrics. Using graph-theoretic descriptions of the $\mathcal{H}_2$ performance of relative sensing networks, the basic Henneberg constructions operations of vertex addition and edge splitting were augmented with an optimality criteria. It was also shown that $\mathcal{H}_2$ optimally rigid graphs are minimally rigid. Finally, we presented an algorithm for growing an optimally rigid graph. Future work will focus on developing distributed versions of the algorithm and additional performance metrics, such as $\mathcal{H}_\infty$.

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