A Passivity Analysis for Nonlinear Consensus on Balanced Digraphs

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Abstract— **This work deals with the output consensus problem for multiagent systems over balanced digraphs by passivity analysis. As the standard diffusive coupling structure only models the undirected interconnection, we propose a general approach capable of processing directed coupling and performing passivity analysis. To mitigate the complexity arising from the nonlinearity and directed interconnections, we reformulate the output consensus problem as a convergence analysis on a submanifold. We provide passivity analysis and establish a sufficient condition based on passivity for achieving output agreement in multi-agent systems over balanced digraphs. The results are supported by a numerical example.**

I. Introduction

Multi-agent systems (MASs) have received extensive attention in both industrial practice and theoretical research, ranging from smart grids, distributed sensing and transportation networks to control, robotics and computer science [1]–[4]. From a control perspective, one of the most fundamental tasks in this field is the consensus problem [5], the objective of which is to control the dynamics of each agent so that they reach an agreement on some state or trajectory.

Consensus analysis requires understanding the fundamental interplay between agent dynamics, information exchange structures, and interaction protocols in networked systems [6], [7]. Diffusively-coupled networks provide a canonical architecture for studying these relationships [8]. Their inherent structure, composed of symmetric and feedback interconnections, makes passivity theory a natural tool for its analysis [9]. Arcak's seminal work [10] leveraged passivity to characterize network convergence behavior. This approach was later extended into a comprehensive passivity-based cooperative control framework for single-input single-output (SISO) systems [6] and subsequently for multi-input multi-output (MIMO) systems [7]. This framework revealed a profound connection between system trajectories and dual network optimization problems [11].

While the passivity framework has proved very powerful, it relies heavily on the symmetric feedback interconnection of the incidence matrix in the diffusively-coupled networks. This structural requirement limits the framework's applicability to systems with undirected interconnections. Replacing one of the incidence matrices in the structure (detailed in Section II) enables the representation of directed graph topologies but sacrifices the diffusive coupling property due to the loss of symmetry. Moreover, given passive edge controllers, the feedback path in the loop may not preserve passivity for the entire interconnection. These challenges leave the passivity analysis for MASs on digraphs blank. This motivates the development of a more general approach for analyzing network systems with directed information exchanging topologies, which can leverage the advantages of passivity theory to solve consensus problems.

On the other hand, since the diffusive coupling networks and passivity enable the separate analysis of system dynamics and the underlying graphs, we can categorize the consensus problems based on the linearity of the system dynamics and the directionality of graphs. The problem can be classified, in order of increasing complexity, as linear dynamics over undirected graphs (e.g., the standard linear consensus protocol), nonlinear dynamics over undirected graphs (e.g., [6]), linear dynamics over digraphs (e.g., the linear consensus protocol for digraphs), and nonlinear dynamics over digraphs (e.g., [12], [13]). Also, there are two types of consensus behaviors: *average consensus* and *regular consensus*. A system achieves (regular) consensus when the states of all agents converge to the same value, while average consensus requires the converged state to equal the mean of the initial conditions. When applying linear consensus protocols, systems over connected undirected graphs achieve average consensus, whereas systems over digraphs containing a rooted out-branching only achieve regular consensus. However, if the considered digraph is also balanced, the system achieves average consensus. This observation suggests that, in the linear case, balanced digraphs may occupy an intermediate position between directed and undirected topologies, motivating our investigation into balanced digraphs in this paper.

This paper focuses on the hardest consensus problem within the above taxonomy: nonlinear dynamics over digraphs. The works [12], [13] were related to this topic and developed a passivation approach, but they only considered the case where the controllers are linear static maps and didn't provide a general analysis method for network systems with directed coupling. Montenbruck et al. [14] regarded the agreement space as a submanifold and developed powerful analytical tools to establish connections between passivity properties and stabilization around a submanifold, yielding explicit controller synthesis methods. However, they only solved the stabilization problem, which cannot guarantee convergence to the submanifold. Moreover, this analysis considered all controllers as a single entity without examining the passivity of each individual agent or controller.

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Consequently, it did not allow for an in-depth investigation of the interplay among the dynamics of the controllers, the agents, and the underlying digraphs within the MASs.

In this paper, we conduct the passivity analysis for MASs interconnected via balanced digraphs and investigate the relationship between passivity and output consensus behavior in the systems. Our contributions are as follows. We begin by discussing the difference between the diffusively coupled network and its variant for digraphs. Our analysis uncovers a potential loss of passivity in the feedback path of the variant structure for general digraphs, even under the fundamental linear consensus protocol. With these insights, we provide a general approach to handling the directed coupling, enabling passivity analysis for the considered network systems. Then, we transform the output agreement problem to examine the convergence to a submanifold. We derive passivity conditions for agents and controllers that guarantee output agreement in network systems governed by balanced digraphs.

The remainder of the paper is organized as follows. Section II introduces some preliminaries, including digraphs, diffusively-coupled networks, and tools to analyze the convergence to submanifolds. Section III proposes a general approach for analyzing directed coupling and reformulates the output agreement problem. Section IV provides the passivity analysis for the diffusively-coupled network with balanced digraphs. Numerical examples and concluding remarks are given in Sections V and VI.

Notations: The notation $\mathbb{1}_n$ ($\mathbb{0}_n$) denotes the n-dimensional vector of all ones (zeros), and I_n represents the $n \times n$ identity matrix, where the subscript n may be omitted when the dimension is clear from the context. For a set A, its cardinality is denoted by $|A|$. For a linear transformation $T: X \to Y$, we denote the kernel of T by $\ker(T)$. We denote the orthogonal complement of a subspace U by $U^{\perp}$, and the orthogonal projection of some $x \in \mathbb{R}^n$ onto U by $\mathrm{Proj}_U(x)$. For a smoothly embedded submanifold M, the notation $d(x, M)$ denotes the infimal Euclidean distance from all the points in M to x.

Fundamental notions from algebraic graph theory are also used in this paper. A directed graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ comprises of a finite vertex set $\mathbb{V}$ and an edge set $\mathbb{E} \subset \mathbb{V} \times \mathbb{V}$. The incidence matrix $E \in \mathbb{R}^{|\mathbb{V}| \times |\mathbb{E}|}$ is defined as follows. $[E]_{ik} := 1$ if i is the head of edge $e_k = (i, k)$, $[E]_{ik} := -1$ if i is the tail of edge e_k and $[E]_{ik} := 0$ otherwise. The incidence matrix can be represented as the sum of the out-incidence matrix B_o and the in-incidence matrix B_i [15], i.e., $E = B_o + B_i$. The elements of B_o and B_i are defined as: $[B_o]_{ik} := 1$ if i is the head of edge $e_k = (i, k)$ and $[B_o]_{ik} := 0$ otherwise; $[B_i]_{ik} := -1$ if i is the tail of edge e_k and $[B_o]_{ik} := 0$ otherwise. The graph Laplacian of undirected graphs is defined as $L = EE^{\top}$. Similarly, for digraphs, we can define the in-Laplacian matrix $L_i = B_i E^{\top}$ and out-Laplacian matrix $L_o = B_o E^{\top}$.

II. Preliminaries

In this section, we introduce two special digraphs, the diffusively-coupled networks and its variant for digraphs, and some notions related to passivity. Then, we provide an overview of the mathematical tools employed to transform an output consensus problem into an equivalent problem of analyzing convergence to a submanifold.

A. Balanced digraphs and rooted-out branchings

A digraph is called *balanced* if the in-degree equals the out-degree for every node.

Lemma 1: For incidence matrix $E \in \mathbb{R}^{|\mathbb{V}| \times |\mathbb{E}|}$ and in-Laplacian $L_i \in \mathbb{R}^{|\mathbb{V}| \times |\mathbb{V}|}$, the following statements are equivalent:

i) The digraph is balanced;
ii) $E^{\top} \mathbb{1}_{|\mathbb{V}|} = \mathbb{0}_{|\mathbb{E}|}$, and $E \mathbb{1}_{|\mathbb{E}|} = \mathbb{0}_{|\mathbb{V}|}$;
iii) $L_i \mathbb{1}_{|\mathbb{V}|} = \mathbb{0}_{|\mathbb{V}|}$ and $L_i^{\top} \mathbb{1}_{|\mathbb{V}|} = \mathbb{0}_{|\mathbb{V}|}$.

Proof: *i)* $\Leftrightarrow$ *iii)*: We recommend readers to refer to Lemma 6.4 in [1].

i) $\Leftrightarrow$ *ii)*: It is sufficient to show the equivalence between statement *i)* and $E \mathbb{1}_{|\mathbb{E}|} = \mathbb{0}_{|\mathbb{V}|}$. The incidence matrix can be represented as $E = B_{out} + B_{in}$. For row i ($i = 1, \dots, |\mathbb{V}|$) of $B_{in}(B_{out})$, the row-sum of row i is the in(out)-degree of the corresponding node i. Thus, the given digraph is balanced if and only if $E \mathbb{1}_{|\mathbb{E}|} = B_o 1_{|\mathbb{E}|} + B_i 1_{|\mathbb{E}|} = \mathbb{0}_{|\mathbb{V}|}$. ■

We will also use the notion of rooted out-branching in this work.

Definition 1 (Rooted out-branching): A directed graph is called a *rooted out-branching* if there exists a node $r \in \mathbb{V}$, such that: (1) there is a directed path from r to every other node; (2) the in-degree of r is 0; and (3) the in-degree of every other node is 1.

B. Diffusively-coupled networks and passivity

Consider a population of agents interacting over a network $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, where the vertices $\mathbb{V}$ and the edges $\mathbb{E}$ denote the set of agents and edge controllers describing their interaction relations, respectively. Each agent $\{\Sigma_i\}_{i \in \mathbb{V}}$ and controller $\{\Pi_e\}_{e \in \mathbb{E}}$ are described by the SISO nonlinear dynamical systems,

$$\Sigma_i: \quad \begin{aligned} \dot{x}_i(t) &= f(x_i(t), u_i(t)), \\ y_i(t) &= h_i(x_i(t), u_i(t)), \quad i \in \mathbb{V} \end{aligned} \tag{1}$$

$$\Pi_k: \quad \begin{aligned} \dot{\eta}_i(t) &= \phi_k(\eta_k(t), \zeta_k(t)), \\ \mu_k(t) &= \psi_k(\eta_k(t), \zeta_k(t)), \quad k \in \mathbb{E}. \end{aligned} \tag{2}$$

Define the stacked inputs of agents $u(t) = [u_1, \cdots, u_{|\mathbb{V}|}]^{\top}$, and similarly for outputs of agents $y(t)$, inputs of controllers $\zeta(t)$ and outputs of controllers $\mu(t)$.

A network system described by Fig.1 is called *diffusively coupled* if the matrix $\mathcal{E}$ is set to E. In this configuration, the topology of the system is characterized by the undirected counterpart of $\mathcal{G}$, where the edges between the agents represent bidirectional communication links. Under the diffusive coupling, the controller input is the difference between the outputs of adjacent agents, and the control input is the sum of the outputs of the edge controllers, as described by $\zeta(t) = E^T y(t)$, and $u(t) = -E\mu(t)$.

When we substitute $\mathcal{E}$ with either B_o or B_i, the structure describes the networked system interconnected by digraphs.

It may no longer be considered diffusively coupled because the substitution breaks the inherent symmetry of the underlying graph. We use the tuple $(\Sigma, \Pi, \mathcal{G})_E$ to represent the diffusively-coupled networks and denote the other case by $(\Sigma, \Pi, \mathcal{G})_{B_i}$ or $(\Sigma, \Pi, \mathcal{G})_{B_o}$.

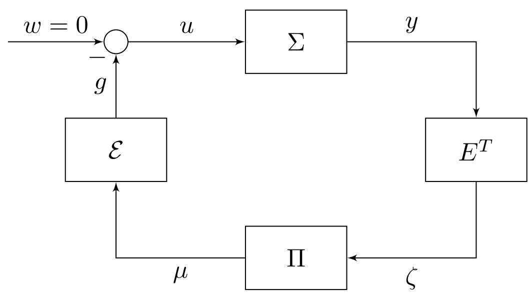


Fig. 1: Block-diagram of networked systems. The structure is diffusive coupling when $\mathcal{E}$ is set to E. When $\mathcal{E}$ is set to B_o or B_i it corresponds to networks over directed graphs.

Now, let's take the system (1) as an example to introduce the definition of passivity.

Definition 2: For the SISO system (1), if there exists a positive semi-definite storage function $V_i(x_i)$ and scalers ε_i and δ_i such that

$$\dot{V}_i(x_i) \leq u_i y_i - \varepsilon_i y_i^2 - \delta_i u_i^2, \quad \forall x_i, u_i, y_i, \tag{3}$$

then, the system (1) is said to be

1) *passive* if $\varepsilon_i = 0$ and $\delta_i = 0$,
2) *output strictly passive* (OP-ε_i) if $\varepsilon_i > 0$ and $\delta_i \geq 0$,
3) *input strictly passive* (IP-δ_i) if $\varepsilon_i \geq 0$ and $\delta_i > 0$.

The maximal δ_i and ε_i are called the *passivity indices* of the system.

Given a passive edge controller, the feedback path (from y to g) of $(\Sigma, \Pi, \mathcal{G})_E$ preserves the passivity while that of $(\Sigma, \Pi, \mathcal{G})_{B_i(o)}$ may lose passivity. We will discuss the differences between the two structures from a passivity perspective in Section III.

C. Tools for analyzing convergence to a submanifold

In this paper, we provide a passivity-based analysis for the output consensus problems of networked systems governed by balanced digraphs. Let S, $S^\perp$, and the vector $y(t)$ denote the *agreement space* $\text{span}(\mathbb{1})$, *disagreement space*, and the output of a system at time t. The system is said to achieve *asymptotic output agreement* if the output satisfies

$$\lim_{t\to\infty} y(t) = c\mathbb{1} \in S, \tag{4}$$

where $c \in \mathbb{R}$ is called the agreement value. On the other hand, if S is regarded as a smooth embedded submanifold, the above definition implies that $y(t)$ *asymptotically converges to the agreement submanifold* S. This allows for converting the output agreement problem into a study of the system's convergence properties on a submanifold.

To analyze the convergence to a submanifold, it is necessary to define a new space with respect to the manifold. Montenbrunk et al. [14] introduced a suitable space $(\mathscr{L}_M^p, \|\cdot\|_{\mathscr{L}_M^p})$ that can be effectively employed in our analysis.

Let M be a smoothly embedded submanifold of $\mathbb{R}^n$. By the *tubular neighborhood theorem* [16, Chapter 10], M has a tubular neighborhood U. Then, we can define the space $\mathscr{L}_M^p$,

$$\mathscr{L}_M^p = \left\{ f : [0,\infty) \to U | f \text{ measurable}, \int_{\mathbb{R}} d(f(t), M)^p \mathrm{d}t < \infty \right\},$$

where "measurable" means Lebesgue measurable and $\mathrm{d}t$ is short for $\mathrm{d}\lambda(t)$, with λ being the Legesgue measure on $\mathbb{R}$. The "norm" $\|\cdot\|_{\mathscr{L}_M^p}$ on $\mathscr{L}_M^p$ is defined as

$$\|\cdot\|_{\mathscr{L}_M^p} : \mathscr{L}_M^p \to \mathbb{R}, f \mapsto \left(\int_{\mathbb{R}} d(f(t), M)^p \mathrm{d}t \right)^{\frac{1}{p}}. \tag{5}$$

It was shown in [14] that the $\|\cdot\|_{\mathscr{L}_M^p}$ is not a norm because it fails to satisfy the triangular inequalities. So, the space $\left(\mathscr{L}_M^p, \|\cdot\|_{\mathscr{L}_M^p}\right)$ does not constitute a normed space, and it cannot be a Banach space. To address this limitation, the paper developed a framework that enabled the application of properties typically enjoyed by normed spaces.

First, a set $\bar{\mathscr{L}}_M^p$ that encompasses all the signals in $\mathscr{L}_M^p$ should be defined. In the context of the considered consensus problem, we have $p = 2$ and $M = S$. By working within tubular neighborhoods $U = \mathbb{R}^n$ of S, we can employ the smooth retraction mapping onto S for signal truncation, which is the orthogonal projection onto S, as described by,

$$r : \ \mathbb{R}^n \to S, \quad x \mapsto \text{Proj}_S(x) = \frac{1}{n}\mathbb{1}\mathbb{1}^\top(x). \tag{6}$$

Then, given $f : \mathbb{R} \to \mathbb{R}^n$ and $T \in \mathbb{R}$ we can define the truncation of f at time T by the retraction r,

$$f_S^T(t) = \begin{cases} f(t), & \text{for } 0 \leq t \leq T, \\ \text{Proj}_S(f(t)), & \text{otherwise.} \end{cases} \tag{7}$$

With these notions, $\bar{\mathscr{L}}_S^p$ can be defined as,

$$\bar{\mathscr{L}}_S^2 = \left\{ f : \ \mathbb{R} \to U | \forall t \in \mathbb{R}, f_S^t \in \bar{\mathscr{L}}_S^2 \right\}. \tag{8}$$

Next, to apply $\left(\mathscr{L}_S^2, \|\cdot\|_{\mathscr{L}_S^2}\right)$ as if it was a Banach space (to use the usual inequalities), define the following map,

$$\Theta_S : \ \bar{\mathscr{L}}_S^p \to \bar{\mathscr{L}}^p, \quad f(t) \mapsto f(t) - r(f(t)), \tag{9}$$

where $\bar{\mathscr{L}}^p$ denotes the case $M = \{0\}$ and $f(t) - r(f(t)) = \text{Proj}_{S^\perp}(f(t)) = (I - \frac{1}{n}\mathbb{1}\mathbb{1}^\top)f(t)$ denotes the projection of $f(t)$ onto the disagreement subspace $S^\perp$. Similarly, define the truncation of Θ_S,

$$(\Theta_S(f(t)))^T = \begin{cases} \text{Proj}_{S^\perp}(f(t)), & \text{for } 0 \leq t \leq T, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Lastly, the following equalities enable the application of standard inequalities in Banach spaces to the space $\left(\mathscr{L}_M^p, \|\cdot\|_{\mathscr{L}_M^p}\right)$. For any signal $f(t) \in \bar{\mathscr{L}}_S^2$,:

$$\begin{aligned} &(\Theta_S(f(t)))^t = \Theta_S(f_S^t(t)) \\ &\|\Theta_S(f_S^t(t))\|_{\mathscr{L}^2} = \|f_S^t(t)\|_{\mathscr{L}_S^2} = \|\text{Proj}_{S^\perp}(f(t))\|_{\mathscr{L}^2}. \end{aligned} \tag{11}$$

Given these notions, demonstrating convergence to the submanifold S is equivalent to proving that $\mathrm{Proj}_{S^\perp}(f(t))$ approaches zero as $t \to \infty$. This equivalence arises from the geometric interpretation of the projection operator: as $\mathrm{Proj}_{S^\perp}(f(t))$ tends to zero, the distance between the signal $f(t)$ and its projection onto the submanifold S diminishes, implying convergence to S. The following definition connects output consensus and convergence to a submanifold.

Definition 3: Consider a network system consisting of a group of agents and edge controllers interconnected as in Fig. 1. Let $y(t)$ be the output of the system. We say that output $y(t)$ *asymptotically converges to the agreement submanifold* S, if

$$\lim_{t\to\infty} \mathrm{Proj}_{S^\perp}(y(t)) = 0. \tag{12}$$

For conciseness, in the following discussion, we adopt the notation $\mathrm{Proj}_{S^\perp}(y)$ in place of $\mathrm{Proj}_{S^\perp}(y(t))$.

III. A General Analysis Approach for Directed Coupling

This section begins with a passivity-based analysis of $(\Sigma, \Pi, \mathcal{G})_E$ and $(\Sigma, \Pi, \mathcal{G})_{B_i(o)}$ under the linear consensus protocol, revealing a potential loss of passivity in the feedback path of $(\Sigma, \Pi, \mathcal{G})_{B_i(o)}$. To address this issue, we propose a general approach for analyzing directed information exchange topologies.

A. Passivity analysis for the linear consensus protocol

The system $(\Sigma, \Pi, \mathcal{G})_{B_i(o)}$ in Fig.1 might be the most straightforward candidate to analyze directed coupling. However, when applying the basic linear consensus protocol for digraphs to this structure, the passivity of the feedback path (from y to g) cannot be guaranteed even though the edge controllers are output-strictly passive. We focus our analysis on $(\Sigma, \Pi, \mathcal{G})_{B_i}$, as the approach and results for the alternative case are analogous.

Consider the linear consensus protocol for digraphs. Here, we take the agent dynamics to be the integrators,

$$\Sigma^l: \begin{cases} \dot{x}(t) &= u(t) \\ y(t) &= x(t) \end{cases}. \tag{13}$$

Note that the integrator dynamics are passive [17]. Meanwhile, the edge controllers are the static map that is output strictly passive,

$$\Pi_k^l: \mu_k = \zeta_k, \tag{14}$$

for non-negative ζ_k. The closed-loop dynamics then yield $\dot{x}(t) = -L_i(\mathcal{G})x(t)$, and the generated trajectories converge to the agreement space, $S = \mathrm{span}(\mathbb{1})$ if and only if the underlying digraph contains a rooted out-branching.

To leverage the benefits of passivity theory for analyzing the diffusively-coupled structure, both the forward and feedback paths should be passive [17]. Consider the system $(\Sigma^l, \Pi^l, \mathcal{G})_{B_i}$, where Σ^l and Π^l are known to be passive. Our objective is to investigate whether the feedback path (from y to g) in Fig.1 is passive. The controllers in this protocol are memoryless functions. Consequently, with input y and output g, the feedback path is passive if $y^\top g \geq 0$ for all y and g [17]. Using the relation $u = -B_i\mu$ and (14), it is equivalent to the spectral analysis of the symmetric part of L_i, denoted by $\frac{(L_i + L_i^\top)}{2}$ [18]. Indeed, if $y^\top g = y^\top B_i E^\top y = y^\top L_i y = y^T \frac{L_i + L_i^\top}{2} y \geq 0$ for all $y \in \mathbb{R}^n$, the feedback path is passive.

Our first result shows that for digraphs with rooted out-branchings, the smallest eigenvalue of L_{sys} is non-positive.

Proposition 1: If $\mathcal{G}$ contains a rooted out-branching, then the smallest eigenvalue of the symmetric part L_{sys} is non-positive.

Proof: Let $s_1, \ldots, s_n$ be the singular values of L_i and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L_{sys}, both arranged in nonincreasing order. The digraph $\mathcal{G}$ containing a rooted out-branching implies that the rank of L_i is $n-1$. It follows that $s_{n-1} > s_n = 0$. To establish the relationship between s_j and λ_j, we apply the Fan-Hoffman [19, Proposition III.5.1]. This proposition implies that $\lambda_j \leq s_j$ for all $j \in [1, n]$. By setting $j = n$, we can deduce that the smallest eigenvalue of L_{sys} is non-positive. ■

The following proposition provides a sufficient and necessary condition for L_{sys} having a zero eigenvalue.

Proposition 2: Let $\mathcal{G}$ contains a rooted out-branching. Then L_i and $L_i^\top$ have the same kernel space if and only if L_{sys} has a zero eigenvalue.

Proof: We first show the sufficiency. Let $q \neq \mathbb{0}$ be a vector in S. Then we have $\frac{1}{2}L_i q + \frac{1}{2}L_i^\top q = \mathbb{0} = \frac{1}{2}(L_i + L_i^\top)q = L_{sys}q$ and $(q, 0)$ is an eigenpair of L_{sys}. To prove the necessity, let $v \neq \mathbb{0}$ be the eigenvector w.r.t. the 0 eigenvalue, i.e., $L_{sys}v = \mathbb{0}$. It follows that $v^\top L_{sys} v = \frac{1}{2} v^\top (L_i + L_i^\top) v = \mathbb{0}$. Since $v^\top L_i v = v^\top L_i^\top v$, we have $v^\top L_{sys} v = v^\top L_i v = v^\top L_i^\top v = \mathbb{0}$. $v \neq 0_n$, so the above equalities are satisfied only when $v \in \ker(L_i)$ and $v \in \ker(L_i^\top)$. The existence of a rooted out-branching implies that the dimensions of the kernel space of L_i and $L_i^\top$ are 1, so $\ker(L_i^\top) = \ker(L_i) = S$. ■

This proposition suggests that for a general digraph where L_i and $L_i^\top$ don't have the same kernel space, the smallest eigenvalue of L_{sys} is negative. Thus, the feedback path may lose passivity, even though the edge controllers are output strictly passive. The following proposition establishes the equivalence between L_i and $L_i^\top$ having the same kernel space and the digraphs being balanced.

Proposition 3: Let $\mathcal{G}$ contain a rooted out-branching. Then, L_i and $L_i^\top$ have same kernel space if and only if $\mathcal{G}$ is balanced.

Proof: To prove sufficiency, we need to show that $L_i\mathbb{1} = L_i^\top \mathbb{1} = \mathbb{0}$ (see subsection II-A). The presence of a rooted out-branching in $\mathcal{G}$ ensures $\ker(L_i) = S$ [8], implying $\mathbb{1} \in \ker(L_i)$ and $\mathbb{1} \in \ker(L_i^\top)$.

For necessity, the given conditions imply that both L_i and $L_i^\top$ have one-dimensional kernel spaces [8] with $\mathbb{1}$ in both $\ker(L_i)$ and $\ker(L_i^\top)$. Consequently, $\ker(L_i) = \ker(L_i^\top) = S$. ■

The above results demonstrate that under linear consensus protocol, only the systems on some specified digraphs can preserve passivity. Moreover, the above passivity analysis only considers the case where the edge controllers follow the simplest dynamics. The passivity analysis may be more tricky if the edge controllers are modeled by more complex dynamics. This suggests we need a more general approach for analyzing MASs on digraphs.

Note that the feedback path of $(\Sigma^l, \Pi^l, \mathcal{G})_E$ preserves the passivity of the controllers, mirroring the behavior observed for balanced digraphs. This serves as another example showing the intermediate position of balanced digraphs between undirected graphs and unbalanced digraphs.

B. A general approach for directed coupling

Recall that the passivity of a system is preserved after being post-multiplied by a matrix and pre-multiplied by its transpose [10]. Also, the incidence matrix can be represented as $E = B_i + B_o$. Equivalently, the incidence matrix for a direcrted graph can be expressed as $B_o = E - B_i$. Inspired by these, we use the decomposition idea to design a structure capable of conducting passivity analysis for MASs over digraphs, as illustrated in Fig.2.

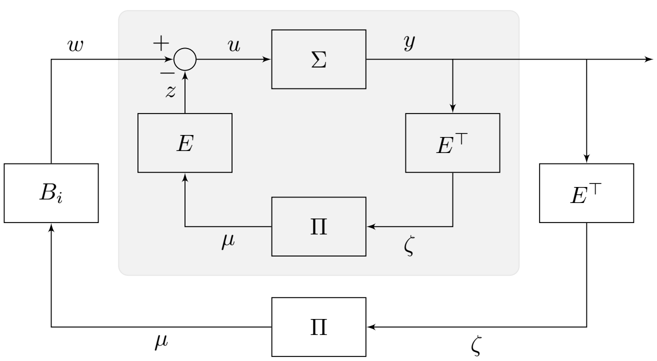


Fig. 2: A loop decomposition for the system $(\Sigma, \Pi, \mathcal{G})_{B_o}$.

To derive the structure depicted in Fig.2, let us begin by examining $(\Sigma, \Pi, \mathcal{G})_{B_o}$. By viewing Fig. 1 from a different perspective, where $u = -B_o\mu = -(E - B_i)\mu$, we can decompose its feedback loop into two distinct branches. The first branch transmits the signals for y to z, and the second branch transmits the signals for y to w. The first branch and the forward path form a feedback connection, as highlighted in the gray box in Fig.2. This sub-structure is essentially the diffusively-coupled network $(\Sigma, \Pi, \mathcal{G})_E$, and the feedback connection is passive, provided that the agents and edge controllers are passive. Indeed, the first branch preserves the passivity of the edge controllers, and the inner product $z^T y$ satisfies

$$z^\top y = \mu^\top E^\top y = \mu^\top \zeta \geq \dot{V},$$

where $V(\eta)$ denotes a continuously differentiable positive semidefinite function known as the storage function. Consequently, by Theorem 6.1 in [17], the feedback connection is passive for all input-output pairs.

With this understanding, we can treat w as an external input that carries directed information to the inner-feedback loop (outlined in grey in Fig. 2). Although the passivity of the overall system $(\Sigma, \Pi, \mathcal{G})_{B_o}$ cannot be guaranteed, we can still exploit the passivity properties preserved in the inner-feedback loop and perform analysis on the feedback interconnection with the input-output pair (w, y).

To differentiate between the structures represented in Fig.1 and Fig.2, we introduce the notation $(\Sigma, \Pi, \mathcal{G}, w)$ to denote the system depicted in Fig.2. Now, we can define new diffusively-coupled relations for $(\Sigma, \Pi, \mathcal{G}, w)$. Let $w(t) = B_i\mu(t)$ and $z(t) = E\mu(t)$. It follows that,

$$u(t) = w(t) - z(t) = -B_o\mu(t) \tag{15}$$

$$\zeta(t) = E^\top y(t), \tag{16}$$

where the structure reduces to $(\Sigma, \Pi, \mathcal{G})_E$ when $w(t) = 0$. This decomposition serves as a general approach to handling directed coupling and allows us to analyze the system's behavior using passivity theory.

Recall that the output consensus problem can be transformed to an equivalent problem of analyzing convergence to a submanifold. We can now define the problem that we will consider.

Problem 1: Consider the network system $(\Sigma, \Pi, \mathcal{G}, w)$. Under what passivity conditions on Σ and Π does the output of the system converge to the agreement submanifold S?

In the following section, we will concentrate on balanced digraphs and present a sufficient condition linked to the passivity that ensures the solvability of Problem 1.

IV. Output consensus for network systems with balanced digraphs

This section focuses on a particular type of digraph, denoted by $\mathcal{G}_o$, which is characterized by being balanced and having a rooted out-branching. We provide passivity analysis and a solution to Problem 1, which also serves as a sufficient condition for the associated consensus problem.

Assume that the agents follow the dynamics

$$\Sigma_i^o: \quad \begin{aligned} \dot{x}_i(t) &= f(x_i(t), u_i(t)), \\ y_i(t) &= h_i(x_i(t)), \quad i \in \mathbb{V} \end{aligned} \tag{17}$$

where f_i and h_i are continuously differentiable functions. Before providing the passivity analysis, we need to introduce the following proposition, which plays a crucial role in the subsequent analysis

Proposition 4: Let $E \in \mathbb{R}^{n\times m}$ be the incidence matrix of $\mathcal{G}_o$. Then, for any $y \in \mathbb{R}^n$, the following equalities hold,

$$E^\top y = \text{Proj}_{S^\perp}(E^\top y) = E^\top \text{Proj}_{S^\perp}(y). \tag{18}$$

Proof: Recall that for the incidence matrix of a balanced digraph, the relation $E^\top \mathbb{1}_n = \mathbb{0}_m$ and $E\mathbb{1}_m = \mathbb{0}_n$ hold. Thus,

$$\begin{aligned} E^\top y &= (I_m - \frac{1}{n}\mathbb{1}_m\mathbb{1}_m^\top)(E^\top y) = \text{Proj}_{S^\perp}(E^\top y) \\ &= E^\top(I_n - \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top)y = E^\top \text{Proj}_{S^\perp}(y). \end{aligned}$$

■

Consider $(\Sigma^o, \Pi, \mathcal{G}_o, w)$, we first explore the relationship between the passivity of individual agents (controllers) and the passivity of the forward (feedback) path in Fig.2. Note that to measure the convergence of output y to the submanifold S, we work within the space $(\mathscr{L}_S^2, \|\cdot\|_{\mathscr{L}_S^2})$ and consider

$\mathrm{Proj}_{S^\perp}(y)$, instead of the real output, to enable the usage of some properties found in Banach spaces. The following result provides a passivity-like inequality for the forward path.

Proposition 5: Consider a group of $|\mathbb{V}|$ SISO agents (17). Assume that for all $i \in \{1, \dots, |\mathbb{V}|\}$, the agents Σ_i^o are OP-ε_i. Then, it follows that

$$u^\top \mathrm{Proj}_{S^\perp}(y) \geq \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i - \frac{1}{\varepsilon}\|u\|_2^2 + \varepsilon \|\mathrm{Proj}_{S^\perp}(y)\|_2^2, \quad (19)$$

where $Q_i(x)$ denote the storage functions and $\varepsilon = \min_i(\varepsilon_i)$.

Proof: We start by summing up the passivity inequalities of all the agents, i.e.,

$$u^\top y \geq \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i + \sum_{i=1}^{|\mathbb{V}|} \varepsilon_i y_i^2 \geq \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i + \varepsilon \|y\|_2^2. \quad (20)$$

Then, consider $u^T \mathrm{Proj}_{S^\perp}(y)$,

$$\begin{aligned} u^T \mathrm{Proj}_{S^\perp}(y) &= u^\top y - \frac{1}{|\mathbb{V}|} u^\top \mathbb{1}\mathbb{1}^\top y \\ &\geq \sum_{i=1}^{n} \dot{Q}_i + \varepsilon \|y\|_2^2 - \frac{1}{|\mathbb{V}|} u^\top \mathbb{1}\mathbb{1}^\top y. \end{aligned} \quad (21)$$

For $-\frac{1}{|\mathbb{V}|} u^\top \mathbb{1}\mathbb{1}^\top y$, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |-\frac{1}{|\mathbb{V}|} u^\top \mathbb{1}\mathbb{1}^\top y| &= \frac{1}{|\mathbb{V}|} |u^\top \mathbb{1}\mathbb{1}^\top y| = \frac{1}{|\mathbb{V}|} |u^\top \mathbb{1}||\mathbb{1}^\top y| \\ &\leq \frac{1}{|\mathbb{V}|} \|u\|_2 \|\mathbb{1}\|_2 \|y\|_2 \|\mathbb{1}\|_2 = \|u\|_2 \|y\|_2. \end{aligned} \quad (22)$$

So, we arrive at $-\frac{1}{|\mathbb{V}|} u^\top \mathbb{1}\mathbb{1}^\top y \geq -\|u\|_2 \|y\|_2$. Due to the output strict passivity of the agents, we have $\|y\|_2 \leq \frac{1}{\varepsilon}\|u\|_2$. It follows that,

$$-\frac{1}{|\mathbb{V}|} u^\top \mathbb{1}\mathbb{1}^\top y \geq -\|u\|_2 \|y\|_2 \geq -\frac{1}{\varepsilon}\|u\|_2^2. \quad (23)$$

Before considering $\|\mathrm{Proj}_{S^\perp}(y)\|_2^2$, recall that $I - \frac{1}{|\mathbb{V}|}\mathbb{1}\mathbb{1}^\top$ is a projection matrix with eigenvalues $\{0, 1^{(|\mathbb{V}|-1)}\}$. Using the properties of the Rayleigh quotient [18, Theorem 4.2.2], we get

$$\|\mathrm{Proj}_{S^\perp}(y)\|_2^2 = y^\top (I - \frac{1}{|\mathbb{V}|}\mathbb{1}\mathbb{1}^\top) y \leq y^\top y. \quad (24)$$

Plug (24) and (23) into (21) and we end the proof. ∎

The following proposition is for the feedback path.

Proposition 6: Consider a group of $|\mathbb{E}|$ SISO edge controllers (2). Assume that for all $k \in \{1, \dots, |\mathbb{E}|\}$, the controllers Π_k are OP-α_i. Then, it follows that

$$z^\top \mathrm{Proj}_{S^\perp}(y) \geq \sum_{k=1}^{|\mathbb{E}|} \dot{W}_k + \alpha \|\mu\|_2^2, \quad (25)$$

where $W_k(\eta)$ denote the storage functions and $\alpha = \min_k(\alpha_k)$.

Proof: Sum up the passivity inequalities of all the controllers and apply Proposition 4,

$$\begin{aligned} \mu^\top \zeta &= \mu^\top E^\top y = z^\top \mathrm{Proj}_{S^\perp}(y) \\ &\geq \sum_{k=1}^{|\mathbb{E}|} \dot{W}_k + \sum_{k=1}^{|\mathbb{E}|} \alpha_k \mu_k^2 \geq \sum_{k=1}^{|\mathbb{E}|} \dot{W}_k + \alpha \|\mu\|_2^2. \end{aligned} \quad (26)$$

∎

For Fig. 2, it has been shown that, from Proposition 5 and 6, both the forward path (from u to y) and the feedback path (from y to z) are output strictly passive. Using Lemma 6.8 in [17], we know that the feedback connection Fig. 2 is finite gain $\mathcal{L}_2$ stable. However, this doesn't guarantee the output agreement of the overall system. In the following discussion, we will provide sufficient conditions for the system to achieve asymptotic output agreement.

Now, let us focus on the overall system. First, we derive a passivity-like inequality with respect to the "input" $w(t)$ and the "output" $\mathrm{Proj}_{S^\perp}(y(t))$. Recall that to demonstrate asymptotic output consensus, it is sufficient to show that $\lim_{t\to\infty} \mathrm{Proj}_{S^\perp}(y(t)) = 0$. With this foundation established, we are ready to present the main result of the paper.

Theorem 1: Consider a diffusively coupled network $(\Sigma^o, \Pi, \mathcal{G}_o, w)$. Suppose the conditions of Proposition 5 and Proposition 6 are met. If $\alpha \geq \frac{\max(D_o)}{\varepsilon}$ where $\max(D_o)$ denotes the maximal out-degree of $\mathcal{G}_o$, then the network achieves output agreement.

Proof: First, we show that the trajectories of $(\Sigma^o, \Pi, \mathcal{G}_o, w)$ are bounded. From Proposition 5 and Proposition 6, both the forward path (from u to y) and the feedback path (from y to z) of Fig.2 are passive. Consequently, it satisfies a global dissipation inequality, where the rate of change of the storage function is bounded by the supply rate. Since the storage function can be chosen to be radially unbounded (i.e., a quadratic storage function), the trajectories must be bounded.

Recall the diffusive coupling in the new structure, $w = z + u$ and $u = -B_o\mu$. Then, the following relation holds,

$$-\|u\|_2^2 = -\mu^T B_o^T B_o \mu \geq \max(D_o)\|\mu\|_2^2, \quad (27)$$

where the last inequality is by the properties of the Geršgorin Disks Theorem [1, Theorem 2.8] and Rayleigh quotient [18, Theorem 4.2.2]. Indeed, observe that the entries of $B_o^\top B_o$ are either 0 or 1, with diagonal elements equal to 1. Furthermore, the largest row sum of this matrix is given by $\max(D_o)$. This implies that the maximal eigenvalue of $B_o^\top B_o$ is less than or equal to $\max(D_o)$.

Now, add the two inequalities (19) and (25) together and use (27),

$$\begin{aligned} w^T \mathrm{Proj}_{S^\perp}(y) \geq & \sum_{k=1}^{|\mathbb{E}|} \dot{W}_k + \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i + \varepsilon \|\mathrm{Proj}_{S^\perp}(y)\|_2^2 \\ & + (\alpha - \frac{\max(D_o)}{\varepsilon})\|\mu\|_2^2. \end{aligned} \quad (28)$$

For any $\alpha \geq \max(D_o)/\varepsilon$, we have that

$$\sum_{k=1}^{|\mathbb{E}|} \dot{W}_k + \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i \leq w^T \mathrm{Proj}_{S^\perp}(y) - \varepsilon \|\mathrm{Proj}_{S^\perp}(y)\|_2^2. \quad (29)$$

Using a similar method to Lemma 6.5 in [17], it can be shown that $\mathrm{Proj}_{S^\perp}(y)$ is bounded for bounded w, as described by,

$$\|y_S^t\|_{\mathscr{L}_S^2} \leq \|w^t\|_{\mathscr{L}_2} + \beta, \quad (30)$$

where

$$\beta = \sqrt{\sum_{k=1}^{|\mathbb{E}|} W(\eta_k(0)) + \sum_{i=1}^{|\mathbb{V}|} Q(x_i(0))}.$$

Now, integrating both sides of (29) over $[0, t]$ for $t > 0$ yields

$$\begin{aligned} &\int_0^t \mathrm{Proj}_{S^\perp}(y(\tau))^T \mathrm{Proj}_{S^\perp}(y(\tau)) \mathrm{d}\tau \\ &\leq \int_0^t \mathrm{Proj}_{S^\perp} y(\tau)^T u(\tau) \mathrm{d}\tau - \sum_{k=1}^{|\mathbb{E}|} W_k(\eta_k(t)) \\ &+ \sum_{k=1}^{|\mathbb{E}|} W(\eta_k(0)) - \sum_{i=1}^{|\mathbb{V}|} Q_i(x_i(t)) + \sum_{i=1}^{|\mathbb{V}|} Q(x_i(0)) \\ &\leq C + \sum_{k=1}^{|\mathbb{E}|} W(\eta_k(0)) + \sum_{i=1}^{|\mathbb{V}|} Q(x_i(0)), \end{aligned} \quad (31)$$

where we use the facts that $\int_0^t \mathrm{Proj}_{S^\perp} y(\tau)^T u(\tau) \mathrm{d}\tau$ is bounded because of (30), and all the terms related to the sum of storage functions are non-negative and bounded due to the boundedness of trajectories.

Next, we demonstrate that $\mathrm{Proj}_{S^\perp}(y(\tau))^T \mathrm{Proj}_{S^\perp}(y(\tau))$ is uniformly continuous by proving that $\dot{y}$ is bounded.Given the dynamics of Σ_i^o, the derivative of y can be expressed as $\dot{y} = \frac{\partial h}{\partial x} f(x, u)$. Since the trajectories are bounded, u and x are confined to compact sets. Consequently, $\frac{\partial h}{\partial x}$ and $f(x, u)$ are also bounded, implying the boundedness of $\dot{y}$ and consequently, $\frac{\partial}{\partial y}(\mathrm{Proj}_{S^\perp}(y)^T \mathrm{Proj}_{S^\perp}(y)) = 2y^\top(I - \frac{1}{|\mathbb{V}|}\mathbb{1}\mathbb{1}^\top)\dot{y}$. Thus, $\mathrm{Proj}_{S^\perp}(y)^T \mathrm{Proj}_{S^\perp}(y)$ is uniformly continuous.

Now, we have satisfied the conditions for applying Barbalat's Lemma [17, Lemma 8.2] to $\mathrm{Proj}_{S^\perp}(y)^T \mathrm{Proj}_{S^\perp}(y)$. Thus, $\mathrm{Proj}_{S^\perp}(y) \to 0$ as $t \to \infty$, implying that the system achieves asymptotic output agreement. ∎

This result implies that $\Pi^l : \mu = w_e \zeta$ serves as the simplest solution for the considered consensus problem.

Theorem 1 presents a passivity-based analysis of network systems with balanced digraphs, establishing a sufficient condition for output consensus in terms of the passivity index of the edge controller dynamics. The theorem provides a lower bound on the passivity index, which has an insightful physical interpretation: it is the ratio between the maximal out-degree of the underlying digraph and the minimal passivity index of the agents. This result guarantees that the system achieves output consensus, but it does not necessarily ensure average consensus.

However, Theorem 1 has some limitations that should be acknowledged. First, the provided condition is only sufficient for achieving output consensus, and a tighter lower bound on the passivity index may exist. Second, the theorem is applicable only to balanced digraphs that contain a rooted out-branching, and it requires the agents to be output strictly passive with their outputs determined solely by their states. We leave finding more general conditions for future research.

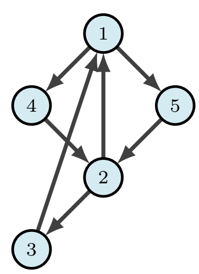
V. CASE STUDY: NEURAL NETWORK

In this section, we consider a continuous neural network on n neurons [5], [20],

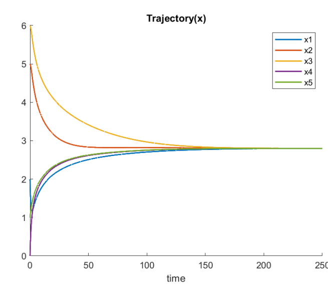
$$\dot{x}_i = -a_i x_i + b \sum_{j \sim i} (\tanh(x_j) - \tanh(x_i)) + \mathrm{w}_i, \quad (32)$$

where x_i and $\frac{1}{a_i} > 0$ denote the voltage on the i-th neuron and the self-correlation time of the neuron, respectively, b is the coupling coefficient and w_i is the exogenous input of the neuron. Note that the agents can be modeled by $\dot{x}_i = -a_i x_i + u_i;\ y_i = \tanh(x_i)$ and the edge controllers follow the linear consensus protocol, i.e., $\mu_k = b\zeta_k$. The agents are OP-a_i and the controllers are OP-b.

We run the system with 5 neurons. The underlying graph of the system, as shown in Fig. 3a, is balanced and contains a rooted out-branching. The maximum out-degree of the graph is 2. In this example, the value $\mathrm{w}_i = 0$ and values a_i, are chosen randomly, $a = [1.66, 3.22, 4.62, 1.5, 2.56]$. The initial conditions are set to $x(0) = [2; 5; 6; 0; 1]$. According to Theorem 1, if $b \geq \frac{\max(D_o)}{\min(a_i)} = \frac{4}{3}$, the system converge to output agreement. In this example, y is determined by x, such that $y \in S$ if and only if $x \in S$. Due to the saturation property of $\tanh(\cdot)$, it suffices to provide only the input values x_i to characterize the output consensus behavior. Fig. 3b presents the trajectories x_i of the system. The state trajectories converge to a common value, so the system achieves output consensus.



(a) A balanced digraph.

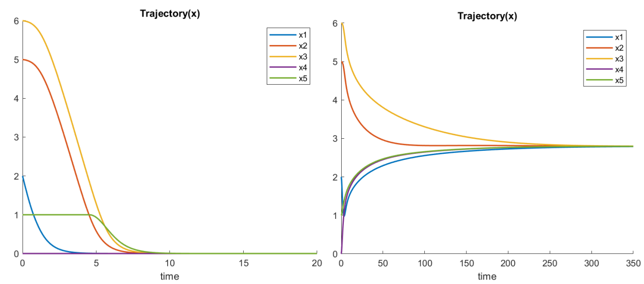


(b) Trajectories for the system with edge controllers $\mu_k = \frac{4}{3}\zeta_k$.

Fig. 3: The underlying graph and trajectories for the states.

Furthermore, the models of the edge controllers can be chosen to be other nonlinear OP-$\frac{4}{3}$ systems, for example, $\mu_k = \frac{4}{3}\max(\zeta_k, 0)$ and $\dot{\eta}_k = -\frac{4}{3}(\eta_k + \eta_k^3) + \zeta_k;\ \mu_k = \eta_k$. Fig. 4 shows the trajectories x_i of the system. We can see that the dynamics in both cases achieve regular consensus.

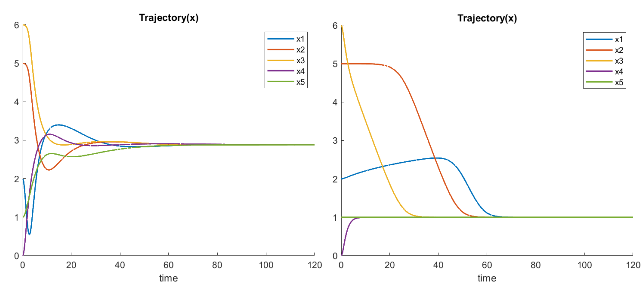
However, Theorem 1 provides a sufficient condition for achieving output agreement, but it is not a necessary condition, as demonstrated by some negative results. Fig. 5a illustrates the trajectories of the states where the edge controllers follow the dynamics $\dot{\eta}_k = -\frac{1}{3}(\eta_k + \eta_k^3) + \zeta_k;\ \mu_k = \eta_k$. The controllers are OP-$\frac{1}{3}$ with the passivity index being smaller than the value required in the theorem, but the system still achieves output (regular) consensus. Furthermore, consider a scenario where the controllers are OP-$\frac{4}{3}$ (i.e., $\dot{\eta}_k = -\frac{4}{3}(\eta_k + \eta_k^3) + \zeta_k;\ \mu_k = \eta_k$) and the underlying digraph is an unbalanced directed tree (i.e., $\mathbb{V} = \{1, 2, 3, 4, 5\}$ and $\mathbb{E} = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$). The output consensus can still be achieved, even though the graph structure does not satisfy the condition specified in Theorem 1.



(a) Trajectories for systems with controllers $\mu_k = \frac{4}{3}\max(\zeta_k, 0)$. (b) Trajectories for systems with 1-order controllers.

Fig. 4: Trajectories for systems with two nonlinear OP-$\frac{4}{3}$ controllers.

The numerical results presented above reveal another important insight: in the case where the underlying digraph of a network system is balanced, the states and the outputs of the system aren't guaranteed to achieve average consensus when the dynamics are nonlinear. This contrasts with the behavior of systems under linear consensus protocols for balanced digraphs, where average consensus is typically achieved.



(a) The dynamics achieve consensus when the edge controllers are OP-$\frac{1}{3}$. (b) The dynamics achieve consensus when the underlying graph is a directed tree.

Fig. 5: Two negative results.

VI. Concluding Remarks

In this work, we first propose a general approach capable of conducting passivity analysis for the network systems with directed coupling. Then, we transform the consensus problem to an equivalent problem of analyzing the convergence to a submanifold. Finally, we provide a passivity-based analysis for network systems over balanced digraphs, which serves as a sufficient condition for achieving output consensus. In future work, we will explore the related network optimization problems for MASs over digraphs and provide passivity analysis for MASs with complex dynamics and interconnections over general digraphs.

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