Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 50-1 (2017) 10136–10141

# Planar Bearing-only Cyclic Pursuit for Target Capture

**Minh Hoang Trinh** *, **Dwaipayan Mukherjee** **, **Daniel Zelazo** **, **Hyo-Sung Ahn** *

* *Gwangju Institute of Science and Technology, Gwangju, Republic of Korea (emails:{trinhhoangminh,hyosung}@gist.ac.kr),*
** *Aerospace Engineering, Technion - Israel Institute of Technology, Haifa, Israel (e-mails: dwaipayan.mukherjee2@gmail.com, dzelazo@technion.ac.il)*

**Abstract:** This paper investigates the stability of formations around a target using bearing-only measurements for agents in cyclic pursuit. A control law is proposed for every agent that uses bearing information of its leader and the target. It is shown that this control law is locally asymptotically stable with respect to a desired arbitrary formation around the target. A detailed analysis of the equilibrium formations is also provided. Simulations support the theoretical results.



*Keywords:* cyclic pursuit, bearing-only formation, co-operative target tracking

## 1. INTRODUCTION

Cyclic pursuit is a well known strategy for multi-agent systems where every agent, indexed $i$, receives information about its leader, agent $i + 1$ (modulo $n$), and chooses its control law based on this information. From a graph theoretic perspective, this means that cyclic pursuit is represented by a directed cycle graph, whose vertices/nodes represent the agents and the directed edges depict the information flow. This is illustrated in Fig. 1. A lot of work related to the consensus problem, for agents in cyclic pursuit, such as by Behroozi and Gagnon (1979); Marshall et al. (2004); Sinha and Ghose (2006); Mukherjee and Ghose (2015) have been reported. This has further led to several applications of cyclic pursuit, such as target capture (Hara et al., 2008; Ma and Hovakimyan, 2013; Mukherjee and Ghose, 2016), boundary tracking (Mukherjee et al., 2014) and vehicular formations (Marshall et al., 2004) emerging, with encouraging results. This paper broadly looks at one such problem: co-operative capture of a target by multi-agent systems. This problem is generally tackled by having a desired formation of vehicles around the target (Tanner, 2007; Mukherjee et al., 2017). Thus, it is imperative to develop strategies that ensure stable desired formations around the target point. This naturally leads to the problem of formation control.

Formation control has been a widely investigated subject in the domain of multi-agent systems such as by Lin et al. (2005); Oh et al. (2015); Sun et al. (2016) and the references therein. The present paper considers a bearing-only formation control around a target, while the agents are in cyclic pursuit. Of late, bearing-only formation control has received significant attention from researchers (Bishop et al., 2015; Zhao and Zelazo, 2015b,c) since it can be realized from vision-based techniques (Montijano et al., 2016). However, most of them consider undirected communication topology, while using bearing-only information.

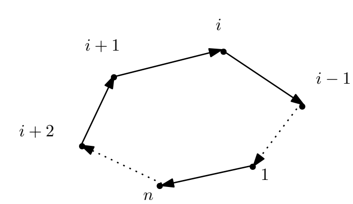



Fig. 1. Information flow in cyclic pursuit.

The formations with directed leader-follower graphs were studied in (Trinh et al., 2014, 2016). In (Zhao and Zelazo, 2015a) a directed graph was considered for exchanging bearing information and sufficient conditions for the stability of formations were derived. However, information about relative distances were also used therein. Another approach is based on bearing rigidity in $SO(2)$ and $SO(3)$ with requirements on inter-agent communication, see (Zelazo et al., 2015; Schiano et al., 2016; Michieletto et al., 2016). Thus, study of stable bearing-only formation control over agents communicating via general digraphs is still an open problem. Towards that end, this paper considers bearing-only formation control over a specific directed topology, the cycle digraph. Inspired by the works such as by (Kim and Sugie, 2007; Hara et al., 2008), the present paper also casts the problem of target capture as one of achieving a desired formation around the target using bearing-only measurements from the target and a leader.

The possibility of achieving *any* desired formation around a target, under the cyclic pursuit paradigm, is the main focus in this work. Thus, the present paper proceeds with a two-fold objective. On the one hand, a specific directed information exchange topology (directed cycle) is used to study the stability of desired formations in $\mathbb{R}^2$ and on the other hand, the cyclic pursuit problem is cast with bearing-only measurements and tailored for target capture. It should be noted that formations of bearing-only planar

2405-8963 © 2017, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2017.08.1759

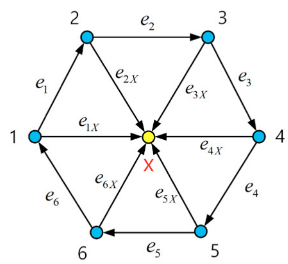



Fig. 2. A group consisting of six agents and a stationary target $\mathbf{X}$.

cyclic pursuit have been recently studied by Trinh et al. (2017), but no target was considered there.

The rest of the paper is organized in the following manner. In Section 2, some preliminary results on bearing-only formation are described and the main problem is formulated. Section 3 presents the main stability results pertaining to target capture using bearing-only formation control for agents in planar cyclic pursuit. Simulations in Section 4 vindicate the theoretical developments. Finally, Section 5 concludes the paper.

## 2. PROBLEM FORMULATION

Consider a group of $n$-autonomous agents modeled by single integrator dynamics, as follows:

$$\dot{\mathbf{p}}_i = \mathbf{u}_i. \tag{1}$$

Here, $\mathbf{p}_i \in \mathbb{R}^2$ and $\mathbf{u}_i \in \mathbb{R}^2$ are the position and the control input of agent $i$, $i = 1, \ldots, n$, respectively. Throughout this paper the agent indices are in modulo $n$. Suppose that there is a stationary target $\mathbf{X}$, such that $\dot{\mathbf{p}}_X = \mathbf{0}$ (see Fig. 2 for an example).

Let $\mathbf{p} = [\mathbf{p}_1^T, \ldots, \mathbf{p}_n^T]^T \in \mathbb{R}^{2n}$ denote the vector containing the co-ordinates of the agents in $\mathbb{R}^2$. Further, we define the displacement vectors $\mathbf{z}_i = \mathbf{p}_{i+1} - \mathbf{p}_i$ and $\mathbf{z}_{iX} = \mathbf{p}_X - \mathbf{p}_i$, $i = 1, \ldots, n$. We denote $d_i = \|\mathbf{z}_i\|$ ($d_{iX} = \|\mathbf{z}_{iX}\|$) as the distance between agent $i$ and agent $i+1$ (the target $\mathbf{X}$, respectively). Similarly, for $j \neq i+1, i-1$, denote the distance between agents $i$ and $j$ as $d_{ij}$.

The bearing vector $\mathbf{g}_i$ is defined as the unit vector pointing from agent $i$ to agent $i+1$, given by:

$$\mathbf{g}_i = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{\|\mathbf{p}_{i+1} - \mathbf{p}_i\|} = \frac{\mathbf{z}_i}{\|\mathbf{z}_i\|}. \tag{2}$$

Suppose the system of $n-$agents and the target satisfies the following assumptions:

*Assumption 1.* All agents have access to a global reference frame in $\mathbb{R}^2$. The positions of the agents, $\mathbf{p}_i \in \mathbb{R}^2$, are initially non-collocated, i.e., $\mathbf{p}_i(0) \neq \mathbf{p}_j(0)$, for all $1 \leq i \neq j \leq n$.

*Assumption 2.* Each agent $i$ can sense the bearing vectors with respect to agent $i+1$ and the target $\mathbf{X}$. The sensing topology of the agents is thus a directed cycle graph with $n$ nodes and an additional node whose information is sensed by all other nodes.

Assumption 2 means that the overall graph may be given as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V} = \{v_1, \ldots, v_n, v_X\}$, $|\mathcal{V}| = n+1$, $\mathcal{E} = \{(v_i, v_{i+1}), (v_i, v_X) | i \in \mathcal{V}\}$ and $|\mathcal{E}| = 2n$. We define a feasible formation based on the above assumptions.

*Definition 1.* The set $\mathcal{B}_n = \{\mathbf{g}_i^*, \mathbf{g}_{iX}^*\}_{i=1,\ldots,n}$ is called a feasible bearing vector set if and only if the following conditions hold for each $i = 1, \ldots, n$:

(1) $\mathbf{g}_i^* \neq \pm\mathbf{g}_{i+1}^*$, $\mathbf{g}_i^* \neq \pm\mathbf{g}_{iX}^*$, $\mathbf{g}_{i-1}^* \neq \pm\mathbf{g}_{iX}^*$, and there exist $d_i^* > 0$ such that $\sum_{i=1}^n d_i^* \mathbf{g}_i^* = \mathbf{0}$, and
(2) There are $d_{iX}^* > 0$ such that $d_i^* \mathbf{g}_i^* - d_{iX}^* \mathbf{g}_{1X}^* + d_{i+1,X}^* \mathbf{g}_{i+1,X}^* = \mathbf{0}$.

The constraint $\sum_{i=1}^n d_i^* \mathbf{g}_i^* = \mathbf{0}$ ensures that the desired formation is a closed polygon, since $d_i^* \mathbf{g}_i^*$ is the edge of the desired polygon joining agent $i$ with agent $i+1$. Similarly, the desired formation does not include the possibility of having any three successive agents $i-1$ through $i+1$ collinear. Neither do we consider desired formations such that any two successive agents $i$ and $i+1$ are collinear with the target $\mathbf{X}$. Thus, $\mathbf{g}_{i+1,X}^* \neq \pm\mathbf{g}_{iX}^*$. The second part of Definition 1 implies that every agent $i$, its leader $i+1$ and the target $\mathbf{X}$ form a triangle. For brevity, we stack all bearing vectors in a column vector as $\mathbf{g} = [\mathbf{g}_1^T, \ldots, \mathbf{g}_n^T, \mathbf{g}_{1X}^T, \ldots, \mathbf{g}_{nX}^T]^T \in \mathbb{R}^{4n}$. Similarly, we stack all the desired bearing vectors in a column vector as $\mathbf{g}^* = [\mathbf{g}_1^{*T}, \ldots, \mathbf{g}_n^{*T}, \mathbf{g}_{1x}^{*T}, \ldots, \mathbf{g}_{nx}^{*T}]^T$.

Next, we will restate the definitions of bearing equivalence and bearing congruency as stated by Zhao and Zelazo (2015b) because these will aid us in establishing the relationship between bearing congruency and bearing equivalence for our problem. However, to understand these definitions, we first need to take a closer look at the orthogonal projection matrix $\mathbf{P}_\mathbf{v} = \mathbf{I}_2 - \frac{\mathbf{v}\mathbf{v}^\mathbf{T}}{\mathbf{v}^T\mathbf{v}} \in \mathbb{R}^{2\times 2}$, corresponding to a nonzero vector $\mathbf{v} \in \mathbb{R}^2$. The projection matrix $\mathbf{P}_\mathbf{v}$ is symmetric, positive semidefinite, and idempotent, that is, $\mathbf{P}_\mathbf{v}^2 = \mathbf{P}_\mathbf{v}$. Moreover, we have $\mathbf{P}_\mathbf{v}\mathbf{v} = \mathbf{0}$. Thus, the right null space of $\mathbf{P}_\mathbf{v}$ is spanned by $\mathbf{v}$.

*Definition 2.* (Zhao and Zelazo (2015b)). (Bearing Equivalency) Frameworks $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ are *bearing equivalent* if $\mathbf{P}_{(\mathbf{p}_i - \mathbf{p}_j)}(\mathbf{p}_i' - \mathbf{p}_j') = \mathbf{0}$ for all $(i,j) \in \mathcal{E}$.

*Definition 3.* (Zhao and Zelazo (2015b)). (Bearing Congruency) Frameworks $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ are *bearing congruent* if $\mathbf{P}_{(\mathbf{p}_i - \mathbf{p}_j)}(\mathbf{p}_i' - \mathbf{p}_j') = \mathbf{0}$ for all $i, j \in \mathcal{V}$.

*Lemma 1.* Given two formations $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ with the sensing graph satisfying Assumption 2, if $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ are bearing equivalent, they are also bearing congruent. Moreover, $d_{ij}/d_{ij}' = \zeta > 0$, for all $i, j \in \mathcal{V}$, $i \neq j$.

**Proof.** Suppose $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ are bearing equivalent. We have $\mathbf{g}_i^* = \mathbf{g}_i'^*$ and $\mathbf{g}_{iX}^* = \mathbf{g}_{iX}'^*$ for all $i = 1, \ldots, n$. Observe that

$$d_i \mathbf{g}_i^* - d_{iX} \mathbf{g}_{iX}^* + d_{i+1,X} \mathbf{g}_{i+1,X}^* = \mathbf{0}, \tag{3}$$

$$d_i' \mathbf{g}_i'^* - d_{iX}' \mathbf{g}_{iX}'^* + d_{i+1,X}' \mathbf{g}_{i+1,X}'^* = \mathbf{0}. \tag{4}$$

Moreover, since $\mathbf{g}_i^* = \mathbf{g}_i'^*$, $\mathbf{g}_{iX}^* = \mathbf{g}_{iX}'^*$, $\mathbf{g}_{i+1,X}^* = \mathbf{g}_{i+1,X}'^*$ and $\mathbf{g}_i^* \neq \mathbf{g}_{iX}^*$, $\mathbf{g}_i^* \neq \mathbf{g}_{i+1,X}^*$, it follows that:

$$\frac{d_i'}{d_i} = \frac{d_{iX}'}{d_{iX}} = \frac{d_{i+1,X}'}{d_{i+1,X}} = \zeta, \quad \forall i = 1, \ldots, n. \tag{5}$$

This is because any two vectors out of $\mathbf{g}_i^*$, $\mathbf{g}_{i+1}^*$ and $\mathbf{g}_{iX}^*$ form a linearly independent set in $\mathbb{R}^2$ (and thus a basis). So the representation of the third in terms of the other two is unique. For any bearing unit vector $\mathbf{g}_{ij}$, $j \neq i+1, i-1$, pointing from agent $i$ to agent $j$, we may write

$$\mathbf{g}_{ij}^* = \frac{1}{d_{ij}}(d_{iX}\mathbf{g}_{iX}^* - d_{jX}\mathbf{g}_{jX}^*), \quad (6)$$

$$\mathbf{g}_{ij}'^* = \frac{1}{d_{ij}'}(d_{iX}'\mathbf{g}_{iX}'^* - d_{jX}'\mathbf{g}_{jX}'^*). \quad (7)$$

Since $\mathbf{g}_{iX}^* = \mathbf{g}_{iX}'^*$, $d_{iX}' = \zeta d_{iX}$, from (7) we may arrive at

$$\mathbf{g}_{ij}'^* = \frac{1}{d_{ij}'}(\zeta d_{iX}\mathbf{g}_{iX}^* - \zeta d_{jX}\mathbf{g}_{jX}^*) = \zeta\frac{d_{ij}}{d_{ij}'}\mathbf{g}_{ij}^*. \quad (8)$$

It follows $\mathbf{g}_{ij}^* = \mathbf{g}_{ij}'^*$ and $d_{ij}' = \zeta d_{ij}$. Thus, the two frameworks $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ are bearing equivalent and their shapes are different by a scale factor $\zeta$. ■

We are now equipped to state the main problem.

*Problem.* Given a group of $n$-agents in cyclic pursuit and a target, satisfying Assumptions 1-2, design bearing-only control laws for the agents such that the agents asymptotically attain a desired formation shape, described by a feasible bearing vector set as in Definition 1, around the target.

## 3. MAIN RESULTS

### *3.1 Proposed control law*

The proposed bearing-only control law for each agent $i$ ($1 \le i \le n$), in cyclic pursuit, is

$$\mathbf{u}_i = -\mathbf{P}_{\mathbf{g}_i}\mathbf{g}_i^* - \mathbf{P}_{\mathbf{g}_{iX}}\mathbf{g}_{iX}^*, \quad (9)$$

where $\mathbf{P}_{\mathbf{g}_i} = \mathbf{I}_2 - \mathbf{g}_i\mathbf{g}_i^T$, $\mathbf{P}_{\mathbf{g}_{iX}} = \mathbf{I}_2 - \mathbf{g}_{iX}\mathbf{g}_{iX}^T$ are orthogonal projection matrices as described earlier. In $\mathbb{R}^2$, we also have $\mathbf{P}_{\mathbf{g}_i} = \mathbf{g}_i^{\perp}(\mathbf{g}_i^{\perp})^T$, where $\mathbf{g}_i^{\perp} = \mathbf{J}\mathbf{g}_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{g}_i$ is a unit vector, perpendicular to $\mathbf{g}_i$. Let $\mathbf{z}_i = \mathbf{p}_{i+1} - \mathbf{p}_i$, $\mathbf{z}_{iX} = \mathbf{p}_X - \mathbf{p}_i$, $d_i = \|\mathbf{z}_i\|$, $d_{iX} = \|\mathbf{z}_{iX}\|$, for $i = 1, \dots, n$.

The following lemma is concerning the equilibrium of the cyclic pursuit system driven by (9).

*Lemma 2.* The system under control law (9) has two types of equilibria which are symmetric about the target's position: the desired equilibrium $\mathbf{p}^*$ corresponding to $\mathbf{g} = \mathbf{g}^*$ and the undesired equilibrium $\mathbf{p}'$ corresponding to $\mathbf{g} = -\mathbf{g}^*$.

**Proof.** From Definition 1, there exist positive scalars $d_i^*, d_{i,X}^*$ and $d_{i+1,X}^*$ such that

$$d_i^*\mathbf{g}_i^* - d_{i,X}^*\mathbf{g}_{i,X}^* + d_{i+1,X}^*\mathbf{g}_{i+1,X}^* = \mathbf{0},\ i = 1, \dots, n. \quad (10)$$

The equilibria of (9) satisfy

$$-\mathbf{P}_{\mathbf{g}_i}\mathbf{g}_i^* - \mathbf{P}_{\mathbf{g}_{iX}}\mathbf{g}_{iX}^* = \mathbf{0},\ i = 1, \dots, n. \quad (11)$$

Premultiplying by $\mathbf{g}_i^T$ on both side of Eq. (11) we get

$$\mathbf{g}_i^T\mathbf{P}_{\mathbf{g}_{iX}}\mathbf{g}_{iX}^* = 0. \quad (12)$$

Equation (12) is satisfied if and only if:

$$\text{either} \quad \mathbf{g}_i = \pm\mathbf{g}_{iX}, \quad (13)$$

$$\text{or} \quad \mathbf{g}_{iX} = \pm\mathbf{g}_{iX}^*, \quad (14)$$

for all $i = 1, \dots, n$. We consider the following cases:

**Case 1**: Suppose some agents satisfy (13) while others satisfy (14). Then there exists some $i \in \{1, \dots, n\}$ such that condition (13) holds for agent $i$ while condition (14) holds for agent $i+1$, i.e., $\mathbf{g}_i = \pm\mathbf{g}_{iX}$ and $\mathbf{g}_{i+1,X} = \pm\mathbf{g}_{i+1,X}^*$.

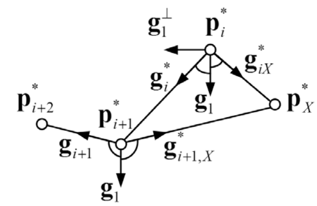



Fig. 3. Illustration of the proof for Case 2.

The condition $\mathbf{g}_i = \pm\mathbf{g}_{iX}$ implies that agent $i$, agent $i+1$ and the target $\mathbf{X}$ are collinear. Thus, $\mathbf{P}_{\mathbf{g}_i} = \mathbf{P}_{\mathbf{g}_{iX}}$. Substituting this in (11), we have

$$\mathbf{P}_{\mathbf{g}_i}(\mathbf{g}_i^* + \mathbf{g}_{iX}^*) = \mathbf{0}, \quad (15)$$

which happens if and only if $\mathbf{g}_{iX}^* + \mathbf{g}_i^* = k\mathbf{g}_i$ since the null space of $\mathbf{P}_{\mathbf{g}_i}$ is spanned by $\mathbf{g}_i$. Equivalently, we may write

$$\mathbf{g}_{iX}^* = k\mathbf{g}_i - \mathbf{g}_i^* \quad (16)$$

where $k$ is a nonzero constant. Moreover, since agent $i$, agent $i+1$ and the target are collinear, we have

$$\mathbf{g}_i = \pm\mathbf{g}_{i+1,X} = \pm\mathbf{g}_{i+1,X}^*. \quad (17)$$

Substituting (16) and (17) into (10), we obtain

$$d_i^*\mathbf{g}_i^* - d_{iX}^*(k\mathbf{g}_i - \mathbf{g}_i^*) \pm d_{i+1,X}^*\mathbf{g}_i = \mathbf{0},$$

or equivalently,

$$(d_i^* + d_{iX}^*)\mathbf{g}_i^* + (-kd_{iX}^* \pm d_{i+1,X}^*)\mathbf{g}_i = \mathbf{0}. \quad (18)$$

From (18) and (16) it follows that $\mathbf{g}_i^* = \pm\mathbf{g}_i$ and they are both aligned with $\mathbf{g}_{iX}^*$. But according to Definition 1, $\mathbf{g}_i^* \neq \pm\mathbf{g}_{iX}^*$. Hence, the contradiction.

**Case 2**: Suppose for all $i = 1, \dots, n$, the condition $\mathbf{g}_i = \pm\mathbf{g}_{iX}$ holds. Thus, all agents and the target are collinear and we have

$$\mathbf{g}_1 = \dots = \pm\mathbf{g}_n = \pm\mathbf{g}_{1X} = \dots = \pm\mathbf{g}_{n,X}. \quad (19)$$

It follows that

$$\mathbf{P}_{\mathbf{g}_1} = \dots = \mathbf{P}_{\mathbf{g}_n} = \mathbf{P}_{\mathbf{g}_{1X}} = \dots = \mathbf{P}_{\mathbf{g}_{nX}}. \quad (20)$$

Substituting this in (11), we get

$$\mathbf{g}_{iX}^* + \mathbf{g}_i^* = k_i\mathbf{g}_i, \quad i = 1, \dots, n, \quad (21)$$

or equivalently, upon combining this with (19),

$$\mathbf{g}_1 = \dots = \pm\mathbf{g}_n = \frac{\mathbf{g}_{1X}^* + \mathbf{g}_1^*}{k_1} = \dots = \frac{\mathbf{g}_{nX}^* + \mathbf{g}_n^*}{k_n}. \quad (22)$$

where $k_i$ are scalars. Let $\mathbf{g}_1^{\perp} = \mathbf{J}\mathbf{g}_1$ be the unit vector perpendicular to $\mathbf{g}_1$. From (10) and (22), we have

$$\begin{aligned} d_i^*\mathbf{g}_i^* + d_{iX}^*\mathbf{g}_i^* &= d_{iX}^*\mathbf{g}_{iX}^* - d_{i+1,X}^*\mathbf{g}_{i+1,X}^* + d_{iX}^*\mathbf{g}_i^* \\ (d_i^* + d_{iX}^*)\mathbf{g}_i^* &= d_{iX}^*(\mathbf{g}_{iX}^* + \mathbf{g}_i^*) - d_{i+1,X}^*(k_{i+1}\mathbf{g}_1 - \mathbf{g}_{i+1}^*) \\ (d_i^* + d_{iX}^*)\mathbf{g}_i^* &= (k_i d_{iX}^* - k_{i+1}d_{i+1,X}^*)\mathbf{g}_1 + d_{i+1,X}^*\mathbf{g}_{i+1}^* \end{aligned} \quad (23)$$

Premultiplying both sides of (23) with $(\mathbf{g}_1^{\perp})^T$ and using the fact that $(\mathbf{g}_1^{\perp})^T\mathbf{g}_1 = 0$ yields

$$(d_i^* + d_{iX}^*)(\mathbf{g}_1^{\perp})^T\mathbf{g}_i^* = d_{i+1,X}^*(\mathbf{g}_1^{\perp})^T\mathbf{g}_{i+1}^*, \quad (24)$$

which further implies that

$$\mathrm{sgn}((\mathbf{g}_1^{\perp})^T\mathbf{g}_i^*) = \mathrm{sgn}((\mathbf{g}_1^{\perp})^T\mathbf{g}_{i+1}^*). \quad (25)$$

Due to the assumption that $\mathbf{g}_i^* \neq \pm\mathbf{g}_{i+1}^*$, we may conclude that $(\mathbf{g}_1^{\perp})^T\mathbf{g}_i^* \neq 0\ \forall i$. Hence, by applying the principle of mathematical induction on (25), we may infer that

$$\mathrm{sgn}((\mathbf{g}_1^{\perp})^T\mathbf{g}_1^*) = \dots = \mathrm{sgn}((\mathbf{g}_n^{\perp})^T\mathbf{g}_n^*) \neq 0. \quad (26)$$

Now, we know that $\sum_{i=1}^{n}\mathbf{z}_i^* = 0$. Hence, we obtain the following:

$$0 = (\mathbf{g}_1^{\perp})^T \sum_{i=1}^{n} \mathbf{z}_i^* = \sum_{i=1}^{n} d_i^* (\mathbf{g}_1^{\perp})^T \mathbf{g}_i^*. \tag{27}$$

Equation (26) implies that either $(\mathbf{g}_1^{\perp})^T\mathbf{g}_i^* > 0$ or $(\mathbf{g}_1^{\perp})^T\mathbf{g}_i^* < 0$ holds for all $i$. Since in both cases, the right hand side of (27) cannot be zero, we have a contradiction. This contradiction implies that Case 2 is also unfeasible.

**Case 3**: Suppose some agents satisfy the condition $\mathbf{g}_{iX} = \mathbf{g}_{iX}^*$ while others satisfy $\mathbf{g}_{iX} = -\mathbf{g}_{iX}^*$. Again, we may conclude that for some $j$, $\mathbf{g}_{jX} = \mathbf{g}_{jX}^*$ and $\mathbf{g}_{j+1,X} = -\mathbf{g}_{j+1,X}^*$. But, clearly we have a positive solution (in terms of the distances) for

$$\mathbf{g}_j^* = \frac{d_{jX}^*}{d_j^*}\mathbf{g}_{jX}^* - \frac{d_{j+1,X}^*}{d_j^*}\mathbf{g}_{j+1,X}^* \tag{28}$$

up to a scaling factor (due to feasibility of formation shape) and $d_j\mathbf{g}_j - d_{jX}\mathbf{g}_{jX} + d_{j+1,X}\mathbf{g}_{j+1,X} = \mathbf{0}$ always holds. Plugging in the values at the equilibrium, it turns out that we need positive solutions for the equation

$$\mathbf{g}_j^* = \pm\left[\frac{d_{jX}'^*}{d_j'^*}\mathbf{g}_{jX}^* + \frac{d_{j+1,X}'^*}{d_j'^*}\mathbf{g}_{j+1,X}^*\right]. \tag{29}$$

But, in $\mathbb{R}^2$ the representation of every vector in terms of two basis vectors ($\mathbf{g}_{jX}^*$ and $\mathbf{g}_{j+1,X}^*$ form the basis set as they are linearly independent due to Def. 1) is unique. Thus, we have a contradiction as in (28) the coefficients of $\mathbf{g}_{jX}^*$ and $\mathbf{g}_{j+1,X}^*$ in the representation of $\mathbf{g}_j^*$ have opposite signs whereas in (29) they are of the same sign.

Using the relation (11), the system satisfies $\mathbf{g}_i = \mathbf{g}_i^*$ or $\mathbf{g}_i = -\mathbf{g}_i^*$, $\forall i = 1,\ldots,n$, respectively, at these equilibria. From Definition 1, the existence of a desired formation $\mathbf{p}^*$ where $\mathbf{g} = \mathbf{g}^*$ is guaranteed. This formation is a desired equilibrium of (9). Moreover, there exists a formation $\mathbf{p}'$ which is symmetric with $\mathbf{p}^*$ about the target. The formation $\mathbf{p}'$ is an undesired formation where the bearing vectors satisfy $\mathbf{g} = -\mathbf{g}^*$. This completes the proof. ■

### *3.2 Stability analysis*

To analyze the stability of the desired formation we define the following sets

$\mathcal{Q} := \{\mathbf{p} \in \mathbb{R}^{2n} | \mathbf{g}_i = \pm\mathbf{g}_i^* \text{ and } \mathbf{g}_{iX} = \pm\mathbf{g}_{iX}^*,\ i = 1,\ldots,n\}$,

$\mathcal{D} := \{\mathbf{p} \in \mathbb{R}^{2n} | \mathbf{g}_i = \mathbf{g}_i^* \text{ and } \mathbf{g}_{iX} = \mathbf{g}_{iX}^*,\ i = 1,\ldots,n\}$,

$\mathcal{U} := \{\mathbf{p} \in \mathbb{R}^{2n} | \mathbf{g}_i = -\mathbf{g}_i^* \text{ and } \mathbf{g}_{iX} = -\mathbf{g}_{iX}^*,\ i = 1,\ldots,n\}$.

$\mathcal{Q}$ is the set of all equilibria of the system driven by the control law (9). Clearly, $\mathcal{Q}$ can be partitioned into $\mathcal{D}-$ the set of desired equilibria, and $\mathcal{U}-$ the set of undesired equilibria as implied by Lemma 2. This equilibrium partition is inspired from work in distance-based formation control (Cao et al., 2011).

Consider a directed cycle formation in $\mathbb{R}^2$. Let $\alpha_i$ be the magnitude of the angle between $\mathbf{g}_i$ and $\mathbf{g}_i^*$ such that $0 \le \alpha_i \le \pi$. Further, let us similarly define $\beta_i$, $\phi_i$ and $\gamma_i$ as the magnitudes of the angles between $\mathbf{g}_{iX}$ and $\mathbf{g}_{iX}^*$, $\mathbf{g}_i$ and $\mathbf{g}_{iX}$, and $\mathbf{g}_i$ and $\mathbf{g}_{i+1}$, respectively, as shown in Fig. 4. Since we will be investigating local stability, we shall be primarily concerned with the behaviour of the system in the vicinity of the equilibrium. Each equilibrium $\mathbf{p}^* \in \mathcal{D}$ corresponds to $\alpha_i = \beta_i = 0,\ i = 1,2,\ldots,n$.

As described earlier, we denote $d_i = \|\mathbf{p}_{i+1} - \mathbf{p}_i\|$ and $d_{iX} = \|\mathbf{p}_{iX} - \mathbf{p}_i\|$. We are now in a position to derive the dynamics of the agents in terms of the angles defined above. We have,

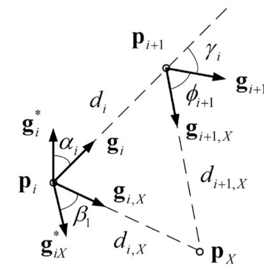



Fig. 4. Illustration for proof of local stability

$$\cos\beta_i = (\mathbf{g}_{iX}^*)^T\mathbf{g}_{iX}, \tag{30}$$

and thus, upon differentiating both sides with respect to time, we get

$$\sin\beta_i\dot{\beta}_i = -(\mathbf{g}_{iX}^*)^T\dot{\mathbf{g}}_{iX} = -(\mathbf{g}_{iX}^*)^T\frac{\mathbf{P}_{\mathbf{g}_{iX}}}{d_{iX}}(-\dot{\mathbf{p}}_i). \tag{31}$$

We then use the relation $\mathbf{P}_{\mathbf{g}_{iX}} = \mathbf{g}_{iX}^{\perp}(\mathbf{g}_{iX}^{\perp})^T$ in (31) to get

$$\begin{aligned} d_{iX}\sin\beta_i\dot{\beta}_i = &-(\mathbf{g}_{iX}^*)^T\mathbf{g}_{iX}^{\perp}(\mathbf{g}_{iX}^{\perp})^T\mathbf{g}_{iX}^{\perp}(\mathbf{g}_{iX}^{\perp})^T\mathbf{g}_{iX}^* \\ &-(\mathbf{g}_{iX}^*)^T\mathbf{g}_{iX}^{\perp}(\mathbf{g}_{iX}^{\perp})^T\mathbf{g}_i^{\perp}(\mathbf{g}_i^{\perp})^T\mathbf{g}_i^* \\ =& -\sin^2\beta_i + (\pm\sin\beta_i)(\cos\phi_i)(\pm\sin\alpha_i). \end{aligned}$$

Thus, the dynamics in terms of the angle $\beta_i$ may be explicitly written as

$$\dot{\beta}_i = -\frac{\sin\beta_i}{d_{iX}} \pm \frac{\sin\alpha_i\cos\phi_i}{d_{iX}}. \tag{32}$$

By the same token, we may obtain the following relation:

$$\begin{aligned} \dot{\alpha}_i = &-\frac{\sin\alpha_i}{d_i} \pm \frac{\sin\alpha_{i+1}\cos\gamma_i}{d_i} \\ &\pm\frac{\sin\beta_{i+1}\cos(\gamma_i\pm\phi_i)}{d_i} \pm \frac{\sin\beta_i\cos\phi_i}{d_i}. \end{aligned} \tag{33}$$

Note that in the control strategies, we controlled $2n$ angle variables $\alpha_i$, $\beta_i$ ($i = 1,\ldots,n$). However, these angle dynamics are dependent on each other due to the existence of a stationary target. In fact, only $2n-1$ independent angle variables are required to specify the desired formation in $\mathbb{R}^2$. As a result, to analyze the stability of the angle dynamics at equilibria, we consider $\boldsymbol{\theta} = [\alpha_1\ldots\alpha_{n-1}\ \beta_1\ldots\beta_n] \in \mathbb{R}^{2n-1}$. The desired equilibria in $\mathcal{D}$, correspond to $\boldsymbol{\theta} = \mathbf{0}_{2n-1}$ and the undesired equilibria in $\mathcal{U}$ correspond to $\boldsymbol{\theta} = \pi\mathbf{1}_{2n-1}$. The following theorem deals with the local stability of the equilibria.

*Theorem 3.* In $\mathbb{R}^2$, the equilibria corresponding to $\mathcal{D}$ are locally asymptotically stable, while those corresponding to $\mathcal{U}$ are unstable.

**Proof.** By linearizing equations (32) and (33) near the corresponding equilibrium, we find that at a desired equilibrium, the perturbed system may be described by

$$\Delta\dot{\boldsymbol{\theta}} = \mathbf{M}\Delta\boldsymbol{\theta} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\Delta\boldsymbol{\theta}, \tag{34}$$

where

$$\mathbf{A}=\begin{bmatrix} -\frac{1}{d_1^*} & \pm\frac{\cos\gamma_1^*}{d_1^*} & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & -\frac{1}{d_{n-2}^*} & \pm\frac{\cos\gamma_{n-2}^*}{d_{n-2}^*} \\ 0 & \cdots & 0 & -\frac{1}{d_{n-1}^*} \end{bmatrix}, \mathbf{C}=\begin{bmatrix} -\frac{1}{d_{1,X}^*} & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & -\frac{1}{d_{n-1,X}^*} \\ 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{0}_{n-1}^T \end{bmatrix}, \mathbf{D}=\text{diag}\left(-\frac{1}{d_{iX}^*}\right)=\begin{bmatrix} \mathbf{D}_1 & \\ & -\frac{1}{d_{n-1,X}^*} \end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix} -\frac{\cos\phi_1^*}{d_1^*} & \pm\frac{\cos\left(\gamma_1^*-\phi_1^*\right)}{d_1^*} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{\cos\phi_{n-1}^*}{d_{n-1}^*} & \pm\frac{\cos\left(\gamma_{n-1}^*-\phi_{n-1}^*\right)}{d_{n-1}^*} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 & \begin{matrix} \mathbf{0}_{n-2} \\ \pm\frac{\cos\left(\gamma_{n-1}^*\pm\phi_{n-1}^*\right)}{d_{n-1}^*} \end{matrix} \end{bmatrix}.$$

$$det(\lambda\mathbf{I}_{2n-1}-\mathbf{M})=\left(\lambda+\frac{1}{d_{n,X}^*}\right)det\left(\begin{bmatrix} \lambda\mathbf{I}_{n-1}-\mathbf{A} & -\mathbf{B}_1 \\ -\mathbf{C}_1 & \lambda\mathbf{I}_{n-1}-\mathbf{D}_1 \end{bmatrix}\right). \quad (35)$$

The eigenvalues of $\mathbf{M}$ are the roots of the polynomial equation in $\lambda$ given by equation (35) obtained from the Laplace expansion. Thus, we have $det(\lambda\mathbf{I}_{2n-1}-\mathbf{M}) = \left(\lambda+\frac{1}{d_{n,X}^*}\right)\prod_{i=1}^{n-1}\left(\left(\lambda+\frac{1}{d_i^*}\right)\left(\lambda+\frac{1}{d_{iX}^*}\right)\pm\frac{(\cos\phi_1^*)^2}{d_1^*d_{1X}^*}\right)$ $=\left(\lambda+\frac{1}{d_{n,X}^*}\right)\prod_{i=1}^{n-1}\left(\lambda^2+\left(\frac{1}{d_1^*}+\frac{1}{d_{iX}^*}\right)\lambda+\frac{1\pm(\cos\phi_1^*)^2}{d_i^*d_{iX}^*}\right)$.

Since $\cos\phi_i^*<1$ at the desired equilibrium, each quadratic equation $\lambda^2+\left(\frac{1}{d_1^*}+\frac{1}{d_{iX}^*}\right)\lambda+\frac{1\pm(\cos\phi_i^*)^2}{d_i^*d_{iX}^*}=0$ has two roots in the open left half plane. Thus, the matrix $\mathbf{M}$ is Hurwitz and local asymptotic stability of the system about any point in $\mathcal{D}$ is guaranteed.

Using similar reasoning as above, the equilibrium corresponding to a point in $\mathcal{U}$ is an unstable one. ■

## 4. SIMULATIONS

We consider a six-agent system with the measurement graph as depicted in Fig. 2. The desired bearing vectors are chosen such that the agents form a regular octahedron around the target. Simulation results are depicted in Fig. 5a–5c. It can be observed in Fig. 5b that the agents asymptotically converge to the desired hexagonal formation around the target. Also, the bearings errors asymptotically decay, as can be seen from Fig. 5c.

Similarly, in another example we considered four agents trying to capture a moving target and obtain a square formation around it. It is assumed that the agents have information about the velocity of the target. Fig. 6 shows the initial formation, the trajectories of the agents and the target along with the bearing errors. Observed that once again the agents manage to capture the moving target while achieving a desired formation around it.

## 5. CONCLUSIONS

In this paper, a bearing-only cyclic pursuit strategy was proposed for target capture. We have first shown that when all desired bearing vectors are satisfied, a target formation shape will be achieved. We also proved that the desired formation is locally asymptotically stable, and the undesired formation is unstable. Even though we carried out our analysis assuming a stationary target, if the agents have information about the target's velocity, they can still capture the target, as shown in our simulations.

## ACKNOWLEDGEMENTS

The work of D. Mukherjee and D. Zelazo was supported in part at the Technion by a fellowship of the Israel Council for Higher Education and the Israel Science Foundation (grant No. 1490/1).

The work of M. H. Trinh and H.-S. Ahn was supported by the National Research Foundation of Korea under Grant NRF-2016M1B3A1A01937575.

## REFERENCES

Behroozi, F. and Gagnon, R. (1979). Cyclic pursuit in a plane. *J. Math. Phys.*, 20, 2212–2216.

Bishop, A.N., Deghat, M., Anderson, B.D.O., and Hong, Y. (2015). Distributed formation control with relaxed motion requirements. *Int. J. Robust Nonlin. Control*, 25(17), 3210–3230.

Cao, M., Morse, A.S., Yu, C., Anderson, B.D., and Dasgupta, S. (2011). Maintaining a directed, triangular formation of mobile autonomous agents. *Communications in Information and Systems*, 11(1), 1–16.

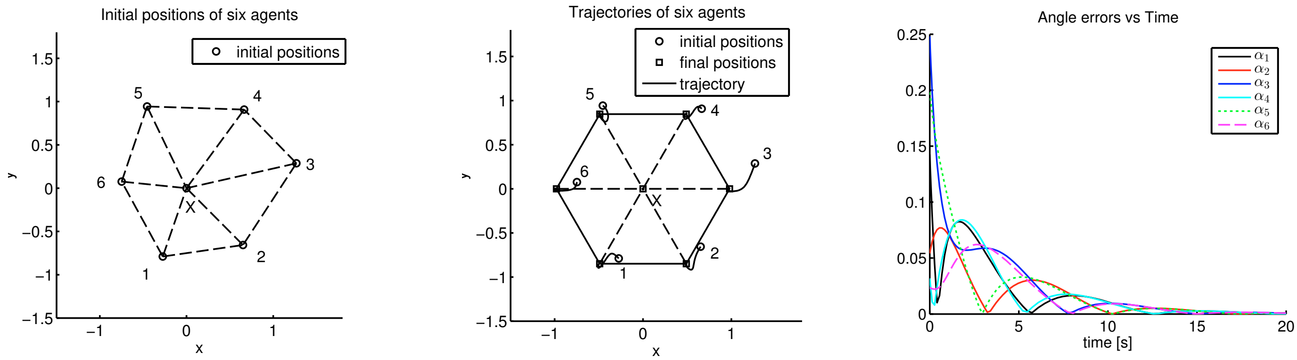
Hara, S., Kim, T.H., and Hori, Y. (2008). Distributed formation control for target-enclosing operations based on a cyclic pursuit strategy. In *Proc. 17th IFAC World Congress*, 6602–6605. IFAC.

Kim, T.H. and Sugie, T. (2007). Cooperative control for target-capturing task based on a cyclic pursuit strategy. *Automatica*, 43, 1426—1431.

Lin, Z., Francis, B., and Magiorre, M. (2005). Necessary and sufficient graphical conditions for formation control of unicycles. *IEEE Trans. Autom. Control*, 50(1), 121–127.

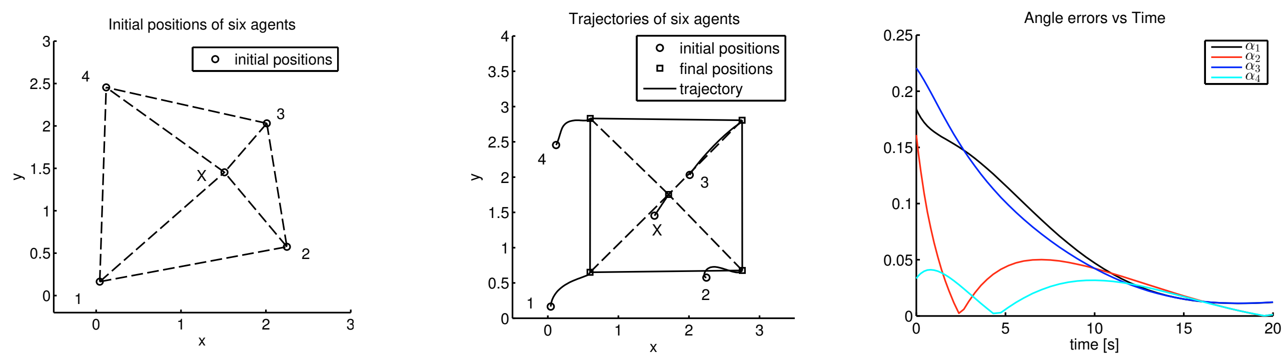
Ma, L. and Hovakimyan, N. (2013). Cooperative target tracking in balanced circular formation: Multiple uavs tracking a ground vehicle. In *Proc. Amer. Control Conf.*, 5386–5391. IEEE.

Marshall, J.A., Broucke, M.E., and Francis, B.A. (2004). Formations of vehicles in cyclic pursuit. *IEEE Trans. Autom. Control*, 49(11), 1963–1974.




(a) The initial formation. (b) Trajectories and the final formation. (c) Angles errors vs. Time [s].

Fig. 5. Simulation results of a three-agent formation under the bearing-only cyclic pursuit control law (9).




(a) The initial formation. (b) Trajectories and the final formation. (c) Angle errors vs. time.

Fig. 6. Capturing a moving target.


Michieletto, G., Cenedese, A., and Franchi, A. (2016). Bearing rigidity theory in se(3). In *Proc. 55th IEEE Conf. Decision Control (CDC' 2016), Las Vegas, USA.*

Montijano, E., Cristofalo, E., Zhou, D., Schwager, M., and Sagues, C. (2016). Vision-based distributed formation control without an external positioning system. *IEEE Transactions on Robotics*, 32(2), 339–351.

Mukherjee, D. and Ghose, D. (2015). Deviated linear cyclic pursuit. *Proc. Royal Soc. A*, 471(2184), 20150682.

Mukherjee, D. and Ghose, D. (2016). Target capturability using agents in cyclic pursuit. *J. Guid. Control Dynam.*, 39(5), 1034—1045.

Mukherjee, D., Menon, P.P., and Ghose, D. (2014). Boundary tracking using a hybrid cyclic pursuit scheme. In *Proc. Amer. Control Conf.*, 153–158. IEEE.

Mukherjee, D., Trinh, M.H., Zelazo, D., and Ahn, H.S. (2017). Bearing-only cyclic pursuit in 2-d for capture of moving target. In *57th Israel Ann. Conf. Aero. Sci.*

Oh, K.K., Park, M.C., and Ahn, H.S. (2015). A survey of multi-agent formation control. *Automatica*, 53, 424–440.

Schiano, F., Franchi, A., Zelazo, D., and Giordano, P.R. (2016). A rigidity-based decentralized bearing formation controller for groups of quadrotor UAVs. In *Proc. IEEE/RSJ Int. Conf. Intel. Robots Systems (IROS 2016)*.

Sinha, A. and Ghose, D. (2006). Generalization of linear cyclic pursuit with application to rendezvous of multiple autonomous agents. *IEEE Trans. Autom. Control*, 51(11), 1819—1824.

Sun, Z., Mou, S., Anderson, B.D.O., and Cao, M. (2016). Exponential stability for formation control systems with generalized controllers: A unified approach. *Systems & Control Letters*, 93(5), 50–57.

Tanner, H.G. (2007). Switched UAV-UGV cooperation scheme for target detection. In *Proc. IEEE Int. Conf. Robotics Autom.*, 3457–3462. IEEE.

Trinh, M.H., Mukherjee, D., Zelazo, D., and Ahn, H.S. (2017). Formations on directed cycles with bearing-only measurements. *under review.*

Trinh, M.H., Oh, K.K., Jeong, K., and Ahn, H.S. (2016). Bearing-only control of leader first follower formations. In *14th IFAC Symp. Large Scale Complex Syst. Theory and Appl.*, 7–12. IFAC.

Trinh, M.H., Oh, K.K., and Ahn, H.S. (2014). Angle-based control of directed acyclic formations with three-leaders. In *Proc. 2014 Int. Conf. Mechatronics Control, China*, 2268–2271.

Zelazo, D., Giordano, P.R., and Franchi, A. (2015). Bearing-only formation control using an SE(2) rigidity theory. In *IEEE 54th Conf. Decision Control*, 6121–6126. IEEE.

Zhao, S. and Zelazo, D. (2015a). Bearing-based formation stabilization with directed interaction topologies. In *IEEE Conf. Decision Control*, 6115–6120. IEEE.

Zhao, S. and Zelazo, D. (2015b). Bearing rigidity and almost global bearing-only formation stabilization. *IEEE Trans. Autom. Control*, 61(5), 1255—1268.

Zhao, S. and Zelazo, D. (2015c). Translational and scaling formation maneuver control via a bearing-based approach. *IEEE Trans. Control Netw. Syst.*, PP(99), 1.