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Finite-time bearing-only formation control

Minh Hoang Trinh†, Dwaipayan Mukherjee‡, Daniel Zelazo‡, and Hyo-Sung Ahn†

Abstract— **This paper presents two bearing-only control laws that guarantee almost global convergence of the desired formation in finite time. For each control law, the equilibrium set is firstly studied. Then, we provide rigorous analysis on asymptotic convergence as well as finite time convergence of the system to the desired equilibrium. Finally, numerical simulations are provided to validate our analysis.**

I. Introduction

Recently, formation control has received much research attention [1], [2]. The task of achieving a target formation shape in a multi-robot system is fundamental to many applications in decentralized cooperative control. It is well-known that sensing variables, controlling variables, and graph topological requirements can be used to classify the existing works on formation control in the literature [2]. Figure 1 briefly surveys the classification of formation control problems presented in [2]. Based on this classification, this paper studies a bearing-only formation control problem, where the sensing and controlling variables are both based on bearings.

Note that bearing information can be given as the bearing vector or as the angle between two bearing vectors [3], [4]. Consequently, there are two main approaches in solving bearing-based formation control problems. The first approach involves controlling the bearing angles, for example, the early works on three-, and four-agent formations [5], [6]. Extension to n-agent systems can be found in [7]–[9]. However, it is difficult to extend the bearing angle approach to control formations in a three dimensional space.

Another approach is based on the bearing rigidity theory in $\mathbb{R}^d$ [10]–[13]. Considering a planar formation with undirected sensing topologies, the authors in [14] proposed a control law to stabilize undirected planar formations to a desired formation shape. Under the bearing-only control law proposed in [12], all infinitesimally bearing rigid frameworks can be almost globally exponentially achieved. For directed cases, the leader-first follower formation [15], [16] in $\mathbb{R}^d$ and the directed cycle formation in $\mathbb{R}^2$ [17], [18] have been studied. Finally, it is worth noting that bearing rigidity in $SE(2)$ and $SE(3)$ have been recently developed and applied to formation control and network localization problems, as in [19]–[21].

This paper considers the problem of stabilizing a formation in finite-time using only bearing measurements. The motivation for designing finite-time controllers is as follows. Consider a system of satellites achieving a desired formation using only bearing measurements. The distance between satellites are generally in kilometers, and since the desired formation shape is defined by a set of bearing vectors, small errors in bearing could lead to a relatively large formation shape variance. Controllers with asymptotic stability performance normally take a long time to achieve a good formation shape and are vulnerable to noises. On the other hand, finite-time controllers not only stabilize the formation faster, but also enable us to estimate an upper-bound on the formation's settling time. For related works, the theory of finite time convergence for continuous systems was developed in [22], [23]. Applications of finite-time control laws on consensus and formation control problems can be found in [24], [25], and [8], [26], [27], respectively.

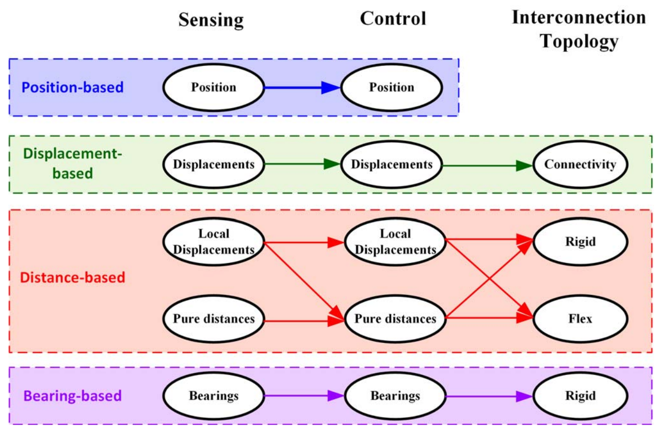

Fig. 1: Classification of formation control problems based on sensing variables, control variables and interconnection topology [2].

Along this avenue, this paper proposes two families of finite-time bearing-only controllers for stabilizing formations of n-single integrator agents in $\mathbb{R}^d$. These proposed control laws are modified from the control laws in [12] and, to the best of our knowledge, are the first ones to be based only on the bearing vectors. The analysis in this paper is divided into two parts: almost globally asymptotic stability of the desired formation, and finite-time convergence of the desired formation under both of the proposed control laws. The convergence of the desired formation is established based on the bearing rigidity theory and the theory of finite-time convergence [23].

Consequently, the contributions of this paper can be summarized as follows. First, we propose two families of

†DCAS Lab., Gwangju Institute of Science and Technology (GIST), Gwangju, Republic of Korea. E-mail: {trinhhoangminh,hyosung}@gist.ac.kr

‡Faculty of Aerospace Eng., Technion - Israel Institute of Technology Haifa 32000, Israel. E-mail: {dwaipayan.mukherjee2@gmail.com, dzelazo@technion.ac.il}

decentralized formation stabilizing controllers that use only bearing information. Second, we characterize the equilibrium set and prove almost global stability of the desired equilibrium under both control laws. Third, finite-time convergence of the system is established under the two control laws and upper bounds on convergence time are derived. Finally, we provide simulations to support the analysis and compare the performance between the unmodified control law and our proposed ones.

The remainder of this paper is organized as follows. In Section II, some preliminary results on finite-time convergence and bearing rigidity are presented. The main problem is formulated and two bearing-only finite-time formation control laws are proposed in Section III. Section IV presents the main stability results. Simulations are provided in Section V. Finally, we summarize the conclusions and outline several further research directions in Section VI.

II. Preliminaries

A. Finite-time convergence theory

Let $\mathbf{x} = [x_1, \dots, x_d]^T$ denote a column vector in $\mathbb{R}^d$. We denote $|\mathbf{x}| = [|x_1|, \dots, |x_d|]^T$. Let $\|\cdot\|$ denote the 2-norm or Euclidean norm, $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$. Further, we use $\|\mathbf{x}\|_1$ to denote the 1-norm, $\|\mathbf{x}\|_1 = \sum_{i=1}^d |x_i|$.

For $\alpha \in \mathbb{R}$, the function $sig(\cdot)^\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is defined as $sig(\mathbf{x})^\alpha = [sign(x_1)|x_1|^\alpha, \dots, sign(x_d)|x_d|^\alpha]^T$ [23]. The following inequality will be used in this paper.

Lemma 1: [28] If $\xi_1, \dots, \xi_d \geq 0$ and $0 \leq p \leq 1$, then

$$\left(\sum_{i=1}^d \xi_i\right)^p \leq \sum_{i=1}^d \xi_i^p.$$

A condition for finite-time convergence of a continuous time system is given by the following lemma.

Lemma 2: [23] Suppose there exists a continuous function $V(\mathbf{x}) : \mathcal{D} \to \mathbb{R}$ such that the following conditions hold

i) $V(\mathbf{x})$ is positive definite,

ii) If there exist $\kappa > 0$, $\alpha \in (0, 1)$, and an open neighborhood $\mathcal{U}_0 \in \mathcal{D}$ of the origin such that

$$\dot{V}(\mathbf{x}) + \kappa(V(\mathbf{x}))^\alpha \leq 0, \forall \mathbf{x} \in \mathcal{U}_0 \setminus \{\mathbf{0}\},$$

then $V(\mathbf{x})$ will reach zero in finite time with the settling time $T \leq V(0)^{1-\alpha}/(\kappa(1-\alpha))$.

B. Bearing rigidity theory

Consider a framework $\mathcal{G}(\mathbf{p})$ in $\mathbb{R}^d$ ($d \geq 2$). Here $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denotes an undirected graph with $|\mathcal{V}| = n$ vertices and $|\mathcal{E}| = m$ edges, and $\mathbf{p} = [\mathbf{p}_1^T, \dots, \mathbf{p}_n^T]^T \in \mathbb{R}^{dn}$ is a configuration of $\mathcal{G}$ in $\mathbb{R}^d$.

For each edge $(i,j) \in \mathcal{E}$, we define a corresponding displacement vector from the configuration as $\mathbf{z}_{ij} = \mathbf{p}_j - \mathbf{p}_i$. For an arbitrary labeling of the edges in $\mathcal{E}$, let $\mathbf{H} \in \mathbb{R}^{m\times n}$ denote the corresponding incidence matrix. Then, the stacked displacement vector[1] is defined as $\mathbf{z} = [\mathbf{z}_1^T, \dots, \mathbf{z}_m^T]^T = (\mathbf{H} \otimes \mathbf{I}_d)\mathbf{p} = \bar{\mathbf{H}}\mathbf{p} \in \mathbb{R}^{dm}$, where $\mathbf{I}_d$ denotes the $d \times d$ identity matrix, and $\otimes$ denotes the Kronecker product.

[1] Although both $\mathbf{z}_{ij}$ and $\mathbf{z}_k$ are used to denote displacement vectors, the notation will be clear from the context in each part of the paper.

Suppose that $\mathbf{p}_i \neq \mathbf{p}_j$. The bearing vector $\mathbf{g}_{ij}$ is the unit vector pointing from $\mathbf{p}_i$ to $\mathbf{p}_j$ [3], i.e.,

$$\mathbf{g}_{ij} = \frac{\mathbf{p}_j - \mathbf{p}_i}{\|\mathbf{p}_j - \mathbf{p}_i\|} = \frac{\mathbf{z}_{ij}}{\|\mathbf{z}_{ij}\|}. \quad (1)$$

The orthogonal projection matrix corresponding to $\mathbf{g}_{ij}$ is defined by $\mathbf{P}_{\mathbf{g}_{ij}} = \mathbf{I}_d - \mathbf{g}_{ij}\mathbf{g}_{ij}^T$. Note that $\mathbf{P}_{g_{ij}}$ is symmetric, idempotent, and positive semidefinite, i.e. $\mathbf{P}_{g_{ij}} = \mathbf{P}_{\mathbf{g}_{ij}}^T = \mathbf{P}_{\mathbf{g}_{ij}}^2 \geq 0$. Furthermore, $\mathbf{P}_{\mathbf{g}_{ij}}$ has eigenvalues $\{0, 1, \dots, 1\}$ and its null space is given by $\mathcal{N}(\mathbf{P}_{\mathbf{g}_{ij}}) = \text{span}\{\mathbf{g}_{ij}\}$.

With the same labeling of edges in $\mathcal{E}$, we denote the stacked bearing vector $\mathbf{g} = [\mathbf{g}_1^T, \dots, \mathbf{g}_m^T]^T \in \mathbb{R}^{dm}$. The bearing rigidity matrix is defined as follows [12],

$$\mathbf{R}(\mathbf{p}) = \frac{\partial \mathbf{g}}{\partial \mathbf{p}} = \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial \mathbf{p}} = \text{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{z}_k\|}\right)\bar{\mathbf{H}} \in \mathbb{R}^{dm\times dn}, \quad (2)$$

where the calculation uses the fact that

$$\frac{\partial \mathbf{g}_{ij}}{\partial \mathbf{z}_{ij}} = \frac{\mathbf{P}_{\mathbf{g}_{ij}}}{\|\mathbf{z}_{ij}\|},\ \frac{\partial \mathbf{g}_{ij}}{\partial \mathbf{p}_i} = -\frac{\mathbf{P}_{\mathbf{g}_{ij}}}{\|\mathbf{z}_{ij}\|},\ \frac{\partial \mathbf{g}_{ij}}{\partial \mathbf{p}_j} = \frac{\mathbf{P}_{\mathbf{g}_{ij}}}{\|\mathbf{z}_{ij}\|}.$$

Further, for any bearing rigidity matrix, span$\{\text{Range}(\mathbf{1} \otimes \mathbf{I}_d),\ \mathbf{p}\} = \text{span}\{\text{Range}(\mathbf{1} \otimes \mathbf{I}_d),\ \mathbf{p} - \mathbf{1} \otimes \bar{\mathbf{p}}\} \subseteq \mathcal{N}(\mathbf{R}(\mathbf{p}))$, where $\mathbf{1} = [1, \dots, 1]^T \in \mathbb{R}^n$ denotes a vector of all ones and $\bar{\mathbf{p}} = \frac{1}{n}\sum_{i=1}^n \mathbf{p}_i$ is the formation centroid. Consequently, rank$(\mathbf{R}(\mathbf{p})) \leq dn - d - 1$. A framework $\mathcal{G}(\mathbf{p})$ is said to be *infinitesimally bearing rigid* (IBR) if and only if rank$(\mathbf{R}(\mathbf{p})) = dn - d - 1$. We have the following lemma.

Lemma 3: [12, Theorem 4] A framework $\mathcal{G}(\mathbf{p})$ is infinitesimally bearing rigid if and only if $\mathcal{N}(\mathbf{R}(\mathbf{p})) = \text{span}\{\text{Range}(\mathbf{1} \otimes \mathbf{I}_d), \mathbf{p} - \mathbf{1} \otimes \bar{\mathbf{p}}\}$, where $\bar{\mathbf{p}} = \frac{1}{n}\sum_{i=1}^n \mathbf{p}_i = \frac{1}{n}(\mathbf{1} \otimes \mathbf{I}_d)^T\mathbf{p}$.

It may be remarked that an infinitesimally bearing rigid framework can be uniquely determined up to a translation and a scaling factor.

III. Problem formulation and the proposed control laws

A. Problem formulation

Consider a system consisting of n autonomous agents in $\mathbb{R}^d$ ($d \geq 2$). The dynamics of an agent i is governed by the single-integrator dynamics

$$\dot{\mathbf{p}}_i = \mathbf{u}_i, \quad (3)$$

where $\mathbf{p}_i, \mathbf{u}_i \in \mathbb{R}^d$ are respectively the position and control input of agent i, $i = 1, \dots, n$. The agents in the system aim to achieve a desired/target formation shape. The desired formation is specified by a set of bearing vectors $\{\mathbf{g}_{ij}^*\}_{(i,j)\in\mathcal{E}}$. Suppose that there exists a framework $\mathcal{G}(\mathbf{p}^*)$ realized from the set $\{\mathbf{g}_{ij}^*\}_{(i,j)\in\mathcal{E}}$, and $\mathcal{G}(\mathbf{p}^*)$ is infinitesimally bearing rigid. Furthermore, the n-agent system satisfies the following assumptions.

Assumption 1: All agents have information about a common reference frame.

Assumption 2: Each agent in the system can sense the bearing vectors with regard to its neighboring agents and the sensing topology is given by a fixed undirected graph $\mathcal{G}$.

We aim to solve the following problem in this paper.

Problem 1: Under the Assumptions 1-2, design a bearing-only control law for each agent using only bearing vectors that achieves the desired formation in finite time.

B. Proposed control laws

The following two control laws, for each agent i, are proposed to solve Problem 1:

$$\mathbf{u}_i = -\sum_{j\in\mathcal{N}_i} \frac{\mathbf{P}_{\mathbf{g}_{ij}}\mathbf{g}_{ij}^*}{\|\mathbf{P}_{\mathbf{g}_{ij}}\mathbf{g}_{ij}^*\|^\alpha}, \tag{4}$$

and

$$\mathbf{u}_i = -\sum_{j\in\mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}} sig(\mathbf{P}_{\mathbf{g}_{ij}}\mathbf{g}_{ij}^*)^\alpha, \tag{5}$$

where $0 < \alpha < 1$ is a positive constant. Observe that for $0 < \alpha < 1$, (4) and (5) are continuous control laws. Both of the proposed control laws are modified from the bearing-only control law commonly used in the literature [12]. The finite-time modifications are inspired from previous works on finite-time consensus [24], [25] and distance-based formation control [27].

We can write the n-agent system under each control law (4) and (5) in the compact forms

$$\dot{\mathbf{p}} = \mathbf{f}_1(\mathbf{p}) = \bar{\mathbf{H}}^T \mathrm{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^*, \tag{6}$$

$$\dot{\mathbf{p}} = \mathbf{f}_2(\mathbf{p}) = \bar{\mathbf{H}}^T \mathrm{diag}(\mathbf{P}_{\mathbf{g}_k}) sig(\mathrm{diag}(\mathbf{P}_{\mathbf{g}_k})\mathbf{g}^*)^\alpha, \tag{7}$$

where $\mathbf{g}^* = [\mathbf{g}_1^{*T}, \ldots, \mathbf{g}_m^{*T}]^T \in \mathbb{R}^{dm}$ is the stacked vector of all desired bearing vectors.

We study the two systems (6) and (7) in the next section and prove that they asymptotically converge to the desired formation in a finite time.

IV. Analysis

In this section, we study the n-agent system under the two control laws (4) and (5). First, we study the equilibrium sets of the systems (6) and (7). Then we prove that under each control law, the desired formation is almost globally asymptotically stable. Finally, we prove that the desired formation can be achieved in finite-time, given that the initial agent configuration does not correspond to an undesired equilibrium.

A. The equilibrium sets

In this subsection, we follow techniques similar to those in [12] to analyze the equilibrium set of systems (6) and (7). Let $\bar{\mathbf{p}} = \frac{1}{n}\sum_{i=1}^n \mathbf{p}_i = \frac{1}{n}(\mathbf{1}\otimes\mathbf{I}_d)^T\mathbf{p}$ be the centroid and $s = \sqrt{\frac{1}{n}\sum_{i=1}^n \|\mathbf{p}_i - \bar{\mathbf{p}}\|^2} = \frac{1}{\sqrt{n}}\|\mathbf{p} - \mathbf{1}\otimes\bar{\mathbf{p}}\|$ be the scale of formation, respectively.

Lemma 4: The formation's centroid and scale are invariant under the control laws (6) and (7).

Proof: Under the control law (4), the system (6) can be rewritten as

$$\begin{aligned}\dot{\mathbf{p}} &= \bar{\mathbf{H}}^T \mathrm{diag}(\mathbf{P}_{\mathbf{g}_k})\mathrm{diag}\left(\frac{\mathbf{I}_d}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^* \\ &= \tilde{\mathbf{R}}^T \mathrm{diag}\left(\frac{\mathbf{I}_d}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^*. \end{aligned}\tag{8}$$

From (2), it follows that $(\mathrm{diag}(\|\mathbf{z}_k\|)\otimes\mathbf{I}_d)\tilde{\mathbf{R}} = \mathbf{R}$. Consequently, we have $\mathcal{N}(\tilde{\mathbf{R}}) = \mathcal{N}(\mathbf{R})$. As a result, it follows from Lemma 3 that $\dot{\mathbf{p}} \perp span\{\mathrm{Range}(\mathbf{1}\otimes\mathbf{I}_d), \mathbf{p}\}$. Since

$$\dot{\bar{\mathbf{p}}} = \frac{1}{n}(\mathbf{1}\otimes\mathbf{I}_d)^T\dot{\mathbf{p}} = \mathbf{0},$$

the formation centroid is invariant. Also, the formation scale is invariant because

$$\dot{s} = \frac{1}{\sqrt{n}}\frac{(\mathbf{p} - \mathbf{1}\otimes\bar{\mathbf{p}})^T}{\|\mathbf{p} - \mathbf{1}\otimes\bar{\mathbf{p}}\|}\dot{\mathbf{p}} = 0.$$

Next, under the control law (5), we rewrite the system (7) as follow:

$$\dot{\mathbf{p}} = \tilde{\mathbf{R}}^T sig(\mathrm{diag}(\mathbf{P}_{\mathbf{g}_k})\mathbf{g}^*)^\alpha. \tag{9}$$

By using similar arguments on the null space of $\tilde{\mathbf{R}}$, we can also conclude that the formation's centroid and scale are invariant under the control law (5). ■

Next, we find the equilibrium sets of the systems driven by control laws (6) and (7).

Lemma 5: The system (6) and (7) have two isolated equilibria, $\mathbf{p}^*$ corresponding to $\mathbf{g}_k = \mathbf{g}_k^*, \forall k = 1, \ldots, m$, and $\mathbf{p}'$ corresponding to $\mathbf{g}_k = -\mathbf{g}_k^*, \forall k = 1, \ldots, m$.

Proof: Let $\mathcal{Q} = \{\mathbf{p} \in \mathbb{R}^{dn} | \ \dot{\mathbf{p}} = 0\}$. Under both control laws, $\dot{\mathbf{p}} = 0$ implies $\mathrm{diag}(\mathbf{P}_{\mathbf{g}_k})\mathbf{g}^* = 0$, which is equivalent to $\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^* = \mathbf{0}$, $\forall k = 1, \ldots, m$.

Since the desired framework is IBR, and the formation centroid and scale are invariant under both (8) and (9), the claim follows from a similar proof as in [12, Theorem 10]. ■

Let $\boldsymbol{\delta}_i = \mathbf{p}_i - \mathbf{p}_i^*$, and $\boldsymbol{\delta} = [\boldsymbol{\delta}_1^T, \ldots, \boldsymbol{\delta}_n^T]^T$. Since $\dot{\boldsymbol{\delta}}_i = \dot{\mathbf{p}}_i$, equations (6) and (7) can be rewritten as

$$\dot{\boldsymbol{\delta}} = \mathbf{f}_1(\boldsymbol{\delta}) = \bar{\mathbf{H}}^T \mathrm{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^*, \tag{10}$$

$$\dot{\boldsymbol{\delta}} = \mathbf{f}_2(\boldsymbol{\delta}) = \bar{\mathbf{H}}^T \mathrm{diag}(\mathbf{P}_{\mathbf{g}_k}) sig(\mathrm{diag}(\mathbf{P}_{\mathbf{g}_k})\mathbf{g}^*)^\alpha, \tag{11}$$

where $0 < \alpha < 1$.

Let $\mathbf{r}(t) = \mathbf{p}(t) - \mathbf{1}\otimes\bar{\mathbf{p}}(t)$ and $\mathbf{r}^* = \mathbf{p}^* - \mathbf{1}\otimes\bar{\mathbf{p}}^*$. Since the centroid of the formation is invariant (Lemma 4), it follows that $\boldsymbol{\delta}(t) = \mathbf{r}(t) - \mathbf{r}^*$. Under both control laws (10) and (11), $\frac{d}{dt}\|\mathbf{r}(t)\| = \sqrt{n}\dot{s} = 0$, i.e. $\|\mathbf{r}(t)\| = \|\mathbf{r}^*\|$ holds. Thus, from $\boldsymbol{\delta}(t) = \mathbf{r}(t) - \mathbf{r}^*$, we have $\|\boldsymbol{\delta}(t) + \mathbf{r}^*\| = \|\mathbf{r}(t)\| = \|\mathbf{r}^*\|$. As a result, the systems (10) and (11) evolve on the surface of the sphere

$$\mathcal{S} = \{\boldsymbol{\delta} \in \mathbb{R}^{dn} | \ \|\boldsymbol{\delta} + \boldsymbol{r}^*\| = \|\boldsymbol{r}^*\|\}.$$

We observe that both systems (10) and (11) have two equilibrium points $\boldsymbol{\delta} = \mathbf{0}$ and $\boldsymbol{\delta} = -2(\mathbf{p}^* - \mathbf{1}\otimes\bar{\mathbf{p}}^*) = -2\mathbf{r}^*$ [12, Theorem 10]. In the next subsection, we will show that $\boldsymbol{\delta} = \mathbf{0}$ is asymptotically stable while $\boldsymbol{\delta} = -2(\mathbf{p}^* - \mathbf{1}\otimes\bar{\mathbf{p}}^*) = -2\mathbf{r}^*$ is unstable.

B. Almost global asymptotic stability analysis

This subsection establishes almost global convergence of the desired formation under both control laws (4) and (5). By almost global convergence, we mean that the target formation is asymptotically achieved from all initial configurations in $\mathbb{R}^{dn}\setminus\mathcal{A}$, where $\mathcal{A} \subset \mathbb{R}^{dn}$ is a set of Lebesgue measure zero.

Theorem 1: Under the Assumptions 1–2 and the control law (4), the desired equilibrium $\mathbf{p} = \mathbf{p}^*$ of system (6) is asymptotically stable.

Proof: Consider the Lyapunov candidate function $V = \frac{1}{2}\|\boldsymbol{\delta}\|^2 = \frac{1}{2}\|\mathbf{p} - \mathbf{p}^*\|^2$, which is positive definite, radially unbounded, and continuously differentiable. The derivative of V along a trajectory of (10) is

$$\begin{aligned}\dot{V}(\mathbf{p}) &= (\mathbf{p} - \mathbf{p}^*)^T \bar{\mathbf{H}}^T \operatorname{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^* \\ &= (\mathbf{z} - \mathbf{z}^*)^T \operatorname{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^* \\ &= -\mathbf{z}^{*T} \operatorname{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^* \\ &= -\mathbf{g}^{*T} \operatorname{diag}\left(\|\mathbf{z}_k^*\|\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^* \\ &= -\sum_{k=1}^m \|\mathbf{z}_k^*\| \frac{\mathbf{g}_k^{*T}\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha} \\ &= -\sum_{k=1}^m \|\mathbf{z}_k^*\| \|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^{2-\alpha}. \end{aligned} \tag{12}$$

As a result, for $0 < \alpha < 1$, $\dot{V} \leq 0$ holds for all $\mathbf{p} \in \mathbb{R}^{dn}$. Moreover, $\dot{V} = 0$ if and only if $\mathbf{p} = \mathbf{p}^*$ or $\mathbf{p} = \mathbf{p}'$. Based on LaSalle invariance principle, any solution of (6) asymptotically converges to one of the two formations $\mathbf{p}^*$ or $\mathbf{p}'$.

Consider a neighborhood of $\mathbf{p} = \mathbf{p}^*$ which does not contain $-\mathbf{p}'$. We have $\dot{V} < 0$ for $\mathbf{p} \neq \mathbf{p}^*$ in this region. Thus, $\mathbf{p} = \mathbf{p}^*$ is (locally) asymptotically stable. ■

Theorem 2: Under the Assumptions 1-2 and the control law (5), the desired equilibrium $\mathbf{p} = \mathbf{p}^*$ of (7) is asymptotically stable.

Proof: The proof is similar to that of Theorem 1 and is omitted due to length restrictions. ■

Lemma 6: The undesired equilibrium of (4) (resp., (5)) corresponding to $\mathbf{g} = -\mathbf{g}^*$ is unstable.

Proof: Consider the Lyapunov function $V = \frac{1}{2}\|\mathbf{p} - \mathbf{p}'\|^2$, where $\mathbf{p}' \in \mathbb{R}^{dn}$ corresponds to the undesired formation at which $\mathbf{g} = -\mathbf{g}^*$. Obviously, V is positive definite, radially unbounded, and continuously differentiable. Further, it follows from $\bar{\mathbf{H}}\mathbf{p}' = -\mathbf{z}'$ that

$$\begin{aligned}\dot{V}(\mathbf{p}) &= (\mathbf{p} - \mathbf{p}')^T \bar{\mathbf{H}}^T \operatorname{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^* \\ &= (\mathbf{z} + \mathbf{z}^*)^T \operatorname{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^* \\ &= \mathbf{z}^{*T} \operatorname{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^\alpha}\right)\mathbf{g}^* \\ &= \sum_{k=1}^m \|\mathbf{z}_k^*\| \|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^{2-\alpha}. \end{aligned} \tag{13}$$

As a result, $\dot{V} > 0$ in a neighborhood of $\mathbf{p}'$. From Chetaev instability theorem [29], the undesired equilibrium $\mathbf{p} = \mathbf{p}'$ of (4) is unstable.

By similar arguments, we can establish the instability of the undesired equilibrium of (5). ■

Theorem 3: Under the control law (4) (the control law (5)), the desired equilibrium $\mathbf{p}^*$ of (6) (resp. (7)) is almost globally asymptotically stable.

Proof: The result follows directly from Theorem 1 (resp., Theorem 2), Lemma 6 and observing that in the proof of Theorem 1 (resp., Theorem 2) $\dot{V} < 0$ everywhere in $\mathbb{R}^{dn}$, except at $\mathbf{p}'$, which is a set of measure zero in $\mathbb{R}^{dn}$. ■

C. Finite time convergence analysis

In order to prove finite time convergence, we need the following useful lemma.

Lemma 7: Under the Assumptions 1-2, and under both control laws (6) and (7), the following inequality holds

$$\|\mathbf{z}_k\| \leq 2s\sqrt{n-1}, \quad \forall k = 1, \ldots, m,$$

where s is the formation scale.

Proof: The proof of this lemma is similar to the proof of [12, Corollary 2] and is omitted. ■

The next two theorems establish the finite time convergence of the formations under the two newly proposed control laws (4) and (5).

Theorem 4: Under the control law (4), starting from an initial formation $\mathbf{p}(0)$ differing from the undesired equilibrium, $\mathbf{p}$ converges to the desired formation $\mathbf{p}^*$ in a finite time.

Proof: Let $\epsilon = \min_{k=1,\ldots,m} \|\mathbf{z}_k^*\|$, from equation (12), we have

$$\begin{aligned}\dot{V} &= -\sum_{k=1}^m \|\mathbf{z}_k^*\| \|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^{2-\alpha} \\ &\leq -\epsilon \sum_{k=1}^m \|\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*\|^{2-\alpha} \\ &\leq -\epsilon \sum_{k=1}^m (\mathbf{g}_k^{*T}\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^*)^{\frac{2-\alpha}{2}}. \end{aligned} \tag{14}$$

Note that $\mathbf{g}_k^{*T}\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^* = \mathbf{g}_k^{*T}(\mathbf{I}_d - \mathbf{g}_k\mathbf{g}_k^T)\mathbf{g}_k^* = \mathbf{I}_d - \mathbf{g}_k^{*T}\mathbf{g}_k\mathbf{g}_k^T\mathbf{g}_k^* = \mathbf{I}_d - \mathbf{g}_k^T\mathbf{g}_k^{*T}\mathbf{g}_k^*\mathbf{g}_k = \mathbf{g}_k^T(\mathbf{I}_d - \mathbf{g}_k^*\mathbf{g}_k^{*T})\mathbf{g}_k = \mathbf{g}_k^T\mathbf{P}_{\mathbf{g}_k^*}\mathbf{g}_k$. Substituting this into (14) yields

$$\begin{aligned}\dot{V} &\leq -\epsilon \sum_{k=1}^m (\mathbf{g}_k^T\mathbf{P}_{\mathbf{g}_k^*}\mathbf{g}_k)^{\frac{2-\alpha}{2}} \\ &\leq -\epsilon \sum_{k=1}^m \left(\frac{1}{\|\mathbf{z}_k\|^2}\mathbf{z}_k^T\mathbf{P}_{\mathbf{g}_k^*}\mathbf{z}_k\right)^{\frac{2-\alpha}{2}}. \end{aligned}$$

From Lemma 7, we know that $\|\mathbf{z}_k\| \leq 2s\sqrt{n-1}$, $\forall k = 1, \ldots, n$. Thus,

$$\begin{aligned}\dot{V} &\leq -\epsilon \sum_{k=1}^m \left(\frac{1}{(2s\sqrt{n-1})^2}\mathbf{z}_k^T\mathbf{P}_{\mathbf{g}_k^*}\mathbf{z}_k\right)^{\frac{2-\alpha}{2}} \\ &\leq -\frac{\epsilon}{(2s\sqrt{n-1})^{2-\alpha}} \sum_{k=1}^m \left(\mathbf{z}_k^T\mathbf{P}_{\mathbf{g}_k^*}\mathbf{z}_k\right)^{\frac{2-\alpha}{2}} \\ &\leq -\gamma_1 \left(\sum_{k=1}^m \mathbf{z}_k^T\mathbf{P}_{\mathbf{g}_k^*}\mathbf{z}_k\right)^{\frac{2-\alpha}{2}}, \end{aligned} \tag{15}$$

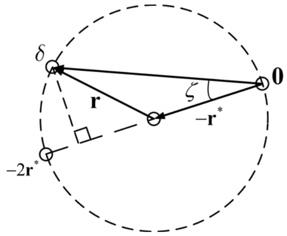

Fig. 2: Illustration of proof of Theorem 4: $\boldsymbol{\delta}$ and ζ.

where $\gamma_1 = \frac{\epsilon}{(2s\sqrt{n-1})^{2-\alpha}}$ and the last inequality follows by applying Lemma 1 with $\frac{1}{2} < \frac{2-\alpha}{2} < 1$.

We further rewrite (15) as follows.

$$\begin{aligned}\dot{V} &\leq -\gamma_1 \left(\sum_{k=1}^{m} (\mathbf{z}_k - \mathbf{z}_k^*)^T \mathbf{P}_{\mathbf{g}_k^*} (\mathbf{z}_k - \mathbf{z}_k^*) \right)^{\frac{2-\alpha}{2}} \\ &\leq -\gamma_1 \left((\mathbf{p} - \mathbf{p}^*)^T \bar{\mathbf{H}}^T \text{diag}(\mathbf{P}_{\mathbf{g}_k^*}) \bar{\mathbf{H}} (\mathbf{p} - \mathbf{p}^*) \right)^{\frac{2-\alpha}{2}} \\ &\leq -\gamma_1 \left(\boldsymbol{\delta}^T \mathbf{R}^{*T} \mathbf{R}^* \boldsymbol{\delta} \right)^{\frac{2-\alpha}{2}}. \end{aligned} \tag{16}$$

Since the desired formation is IBR, the matrix $\mathbf{M}^* = \mathbf{R}^{*T}\mathbf{R}^*$ has $d+1$ zeros eigenvalues. Denote λ_{d+2} as the smallest nonzero eigenvalue of $\mathbf{M}^*$. Since $\boldsymbol{\delta} \perp \text{Range}(\mathbf{1} \otimes \mathbf{I}_d)$, and $\mathcal{N}(\mathbf{R}^{*T}\mathbf{R}^*) = \mathcal{N}(\mathbf{R}^*) = \text{span}\{\text{Range}(\mathbf{1} \otimes \mathbf{I}_d),\ \mathbf{r}^*\}$ (Lemma 3), let ζ be the angle between $\boldsymbol{\delta}$ and $-\mathbf{r}^*$, we have

$$\begin{aligned}\boldsymbol{\delta}^T \mathbf{R}^{*T} \mathbf{R}^* \boldsymbol{\delta} &\geq \lambda_{d+2} \sin^2 \zeta \|\boldsymbol{\delta}\|^2 \\ &\geq \lambda_{d+2} \sin^2 \zeta_0 \|\boldsymbol{\delta}\|^2,\end{aligned}$$

where $\zeta_0 \leq \zeta(t)$ due to $\mathbf{p} \to \mathbf{p}^*$ as $t \to \infty$ as illustrated in Fig. 2. It follows from (16) that

$$\dot{V} \leq -\gamma_1 \left(\lambda_{d+2} \sin^2 \zeta_0 \|\boldsymbol{\delta}\|^2 \right)^{\frac{2-\alpha}{2}} \leq -\kappa_1 V^{\frac{2-\alpha}{2}}, \tag{17}$$

where $\kappa_1 = \gamma_1 (2\lambda_{d+2} \sin^2 \zeta_0)^{\frac{2-\alpha}{2}}$. Note that for $0 < \alpha < 1$, we have $\frac{1}{2} < \frac{2-\alpha}{2} < 1$. Combining Lemma 2, Theorem 3, and equation (17), we conclude that $V \to 0$ in finite time. Hence, $\mathbf{p} = \mathbf{p}^*$ is almost globally finite time stable with the settling time $T_1 \leq V(0)^{1-\frac{2-\alpha}{2}} / (\kappa_1 (1 - \frac{2-\alpha}{2})) = 2V(0)^{\frac{\alpha}{2}} / (\kappa_1 \alpha)$. ■

Following a similar analysis, we have the following theorem.

Theorem 5: Under the control law (5), starting from an initial formation $\mathbf{p}(0)$ differing from the undesired equilibrium, $\mathbf{p}$ converges to the desired formation $\mathbf{p}^*$ in a finite time.

Remark 1: In the proof of Theorem 5, we will have to invoke the inequality in Lemma 1 twice to estimate γ_2, which is a term similar to γ_1 in the proof of Theorem 4. Thus, the upper-bound on the settling time in Theorem 5 is usually more conservative than in Theorem 4. Therefore, in case the upper bounds on the settling times for the two control laws (4) and (5) are equal, the actual convergence time of control law (5) is expected to be smaller.

V. Simulation

Consider a system comprising four agents in $\mathbb{R}^2$. The agents' objective is to form a square. To accomplish this task, the agents sense the bearing toward other agents and the sensing topology is given as shown in Fig. 3. The edge set is given by $\mathcal{E} = \{(1,2),(1,3),(1,4),(2,3),(3,4)\}$. The corresponding set of desired bearing vectors are given as follows: $\mathbf{g}_{12}^* = -\mathbf{g}_{21}^* = \mathbf{g}_{34}^* = \mathbf{g}_{43}^* = [1,0]^T$, $\mathbf{g}_{23}^* = -\mathbf{g}_{32}^* = \mathbf{g}_{14}^* = -\mathbf{g}_{41}^* = [0,1]^T$, and $\mathbf{g}_{13}^* = -\mathbf{g}_{31}^* = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$. It can be verified that the desired formation graph is infinitesimally bearing rigid.

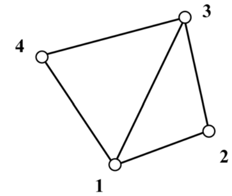

Fig. 3: The sensing topology of the four-agent system.

We simulate the four-agent system under three control laws: (i) - the control law $\mathbf{u}_i = -\sum_{j \in \mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}} \mathbf{g}_{ij}^*$ as used in [12], (ii) the control law (4) with $\alpha_1 = \frac{1}{2}$, and (iii) the control law (5) with $\alpha_2 = \frac{1}{2}$.

The simulation results are shown in Figs. 4-6. In all three cases, the initial positions of the four agents are the same. The formation converges to the desired formation. However, convergence rate is different in each case. It can be observed that in case (ii) and case (iii), the desired formation is achieved in finite time.

It is worth noting that the upper-bound of the settling time computed in both cases derived from Theorem 4 and Theorem 5 are the same. However, the second and third simulations in Figs. 5-6 show that the actual settling time under the control law (4) (about $8s$) is longer than the settling time under the control law (5) (about $6.5s$). Thus, this observation is consistent with Remark 1.

VI. Conclusions

In this paper, two finite-time bearing-only control laws have been proposed. We proved that under both of the proposed control laws, the desired formation shape can be almost globally achieved in a finite time. The upper bounds of the convergence time are also estimated. As finite-time convergence often leads to large control efforts, though the control law (4) is expected to have a longer settling time than (5), it could be more realistic in implementation.

The current paper only deals with undirected formations. Thus, finite-time bearing only formation control of directed formations is left for further investigations. Moreover, noises are always presented in bearing measurements and could contaminate into the system dynamics. Studying the detrimental effects of noises are thus essential for assessing the system performance. Finally, studying finite-time bearing only formation control with more general agents' dynamics is also an interesting research direction.

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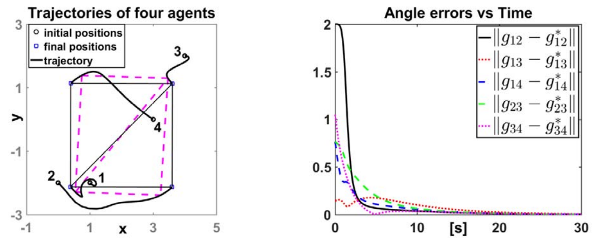
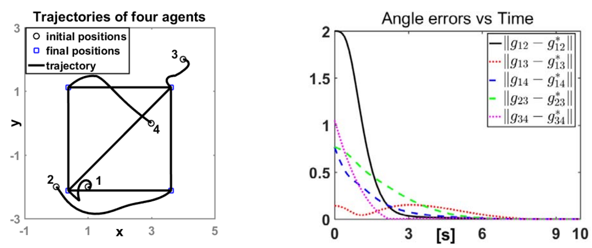


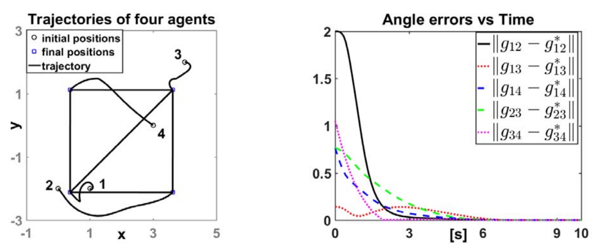
(a) Trajectories. (b) Bearing angle errors vs. Time.

Fig. 4: A simulation using the *unmodified* bearing only control law $\mathbf{u}_i = -\sum_{j\in\mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}}\mathbf{g}_{ij}^*$ in [12]. The formation shape at $t = 10s$ is depicted in magenta color.



(a) Trajectories. (b) Bearing angle errors vs. Time.

Fig. 5: A simulation using the control law (4) with $\alpha = \frac{1}{2}$.



(a) Trajectories. (b) Bearing angle errors vs. Time.

Fig. 6: A simulation using the control law (5) with $\alpha = \frac{1}{2}$.

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REFERENCES

[1] B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx, "Rigid graph control architectures for autonomous formations," *Control Systems Magazine*, vol. 28, no. 6, pp. 48–63, 2008.

[2] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, 2015.

[3] M. H. Trinh, G.-H. Ko, V. H. Pham, K.-K. Oh, and H.-S. Ahn, "Guidance using bearing-only measurements with three beacons in the plane," *Control Engineering Practice*, vol. 51, pp. 81–91, 2016.

[4] T. Eren, W. Whiteley, and P. N. Belhumeur, "Using angle of arrival (bearing) information in network localization," in *Decision and Control, 45th IEEE Conference on*. IEEE, 2006, pp. 4676–4681.

[5] M. Basiri, A. N. Bishop, and P. Jensfelt, "Distributed control of triangular formations with angle-only constraints," *Systems & Control Letters*, vol. 59, no. 2, pp. 147–154, 2010.

[6] A. N. Bishop, "Distributed bearing-only formation control with four agents and a weak control law," in *Proc. of the 9th IEEE International Conference on Control & Automation*, 2011, pp. 30–35.

[7] A. N. Bishop, M. Deghat, B. D. O. Anderson, and Y. Hong, "Distributed formation control with relaxed motion requirements," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 17, pp. 3210–3230, 2015.

[8] S. Zhao, F. Lin, K. Peng, B. M. Chen, and T. H. Lee, "Finite-time stabilization of cyclic formations using bearing-only measurements," *International Journal of Control*, vol. 87, no. 4, pp. 715–727, 2014.

[9] ——, "Distributed control of angle-constrained cyclic formations using bearing-only measurements," *Systems & Control Letters*, vol. 63, pp. 12–24, 2014.

[10] A. Franchi and P. R. Giordano, "Decentralized control of parallel rigid formations with direction constraints and bearing measurements," in *Proc. of the 51st IEEE Conference on Decision and Control, USA*, Dec. 10-13 2012, pp. 5310–5317.

[11] R. Tron, L. Carlone, F. Dellaert, and K. Daniilidis, "Rigid components identification and rigidity control in bearing-only localization using the graph cycle basis," in *Prof. of the American Control Conference*, 2015, pp. 3911–3918.

[12] S. Zhao and D. Zelazo, "Bearing rigidity and almost global bearing-only formation stabilization," *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1255–1268, 2015.

[13] ——, "Localizability and distributed protocols for bearing-based network localization in arbitrary dimensions," *Automatica*, vol. 69, pp. 334–341, 2016.

[14] E. Schoof, A. Chapman, and M. Mesbahi, "Bearing-compass formation control: A human-swarm interaction perspective," in *Proc. of the 2014 American Control Conference*, 2014, pp. 3881–3886.

[15] T. Eren, "Formation shape control based on bearing rigidity," *International Journal of Control*, vol. 85, no. 9, pp. 1361–1379, 2012.

[16] M. H. Trinh, K.-K. Oh, K. Jeong, and H.-S. Ahn, "Bearing-only control of leader first follower formations," in *Proc. of the 14th IFAC Symposium on Large Scale Complex Systems: Theory and Applications*, 2016, pp. 7–12.

[17] M. H. Trinh, D. Mukherjee, D. Zelazo, and H.-S. Ahn, "Planar bearing-only cyclic pursuit for target capture," in *Proc. of the 20th IFAC World Congress, Toulouse, France*, 2017, pp. 10 553–10 558.

[18] ——, "Formations of cycle digraph with bearing-only measurements," 2017, *accepted to International Journal of Robust and Nonlinear Control*, 2017.

[19] D. Zelazo, P. R. Giordano, and A. Franchi, "Formation control using a SE(2) rigidity theory," in *Proc. of the 54th IEEE Conference on Decision and Control (CDC)*, 15-18 2015, pp. 6121–6126.

[20] F. Schiano, A. Franchi, D. Zelazo, and P. R. Giordano, "A rigidity-based decentralized bearing formation controller for groups of quadrotor UAVs," in *Proc. of the 2016 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS 2016)*, 2016.

[21] G. Michieletto, A. Cenedese, and A. Franchi, "Bearing rigidity theory in SE(3)," in *Proc. of the 55th IEEE Conference on Decision and Control (CDC' 2016), Las Vegas, USA*, 2016.

[22] S. P. Bhat and D. S. Bernstein, "Continuous finite-time stabilization of the translational and rotational double integrators," *IEEE Transactions on Automatic Control*, vol. 43, no. 5, pp. 678–682, 1998.

[23] ——, "Finite-time stability of continuous autonomous systems," *SIAM Journal of Control and Optimization*, vol. 38, no. 3, pp. 751–766, 1998.

[24] J. Cortés, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, pp. 1993—-2000, 2006.

[25] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 950—-955, 2010.

[26] M.-C. Park, Z. Sun, K.-K. Oh, B. D. O. Anderson, and H.-S. Ahn, "Finite-time convergence control for acyclic persistent formations," in *Intelligent Control (ISIC), 2014 IEEE International Symposium on*. IEEE, 2014, pp. 1608–1613.

[27] Z. Sun, S. Mou, M. Deghat, B. D. O. Anderson, and A. S. Morse, "Finite time distance-based rigid formation stabilization and flocking," in *Proc. of the 19th IFAC World Congress, Vol. 47, Iss. 3*, 2014, pp. 9138–9189.

[28] G. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed. Cambridge University Press: Cambridge Mathematical Library, 1952.

[29] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.