Formation control from the generic combinatorial viewpoint: edge dynamics and directed sensing

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***Abstract*—We develop a geometric framework for distance-based formation control that separates the evolution of inter-agent distances from its realization by compatible node motions, reducing the stability problem to the edge space. We show that local exponential convergence of the edge dynamics implies local exponential convergence of the formation, and that stability is certified by spectral properties of a linear edge operator. We introduce a hierarchy of generic spectral properties — weak admissibility, admissibility, and strong admissibility — that provide necessary conditions for local exponential stability. Specializing to directed sensing, we obtain a necessary and sufficient spectral condition for local stability at an arbitrary target, together with a quadratic sufficient certificate. These conditions reveal that stability depends jointly on the graph orientation and target geometry, and show that persistence is neither necessary nor sufficient for local convergence. We show that every generically rigid graph admits an admissible orientation and, for acyclic orientations, we give an exact combinatorial characterization of admissibility. Finally, the quadratic certificate leads to a semidefinite program for synthesizing stabilizing edge gains.**

***Index Terms*—formation control; nonlinear control; multi-agent systems; directed sensing; rigidity theory**

I. Introduction

The distance-constrained formation control problem has emerged as one of the canonical problems for multi-agent coordination [1]. The objective is to steer a group of agents to a prescribed geometric shape, specified by inter-agent distances, using only local sensing and distributed control actions. The classical solution, originally proposed by [2], is a gradient-based controller that uses the squared edge length error as a potential in node space and follows the negative gradient. When the sensing graph is generically d-rigid, the gradient controller is generically locally exponentially stable, as shown by [3]. In fact, for certain classes of rigid graphs, one can design decentralized control laws for which all stable equilibria correspond to desired formations, although the presence of undesirable equilibria remains a fundamental challenge [4].

A complementary geometric viewpoint was developed in [5], where attention was shifted from the agent configuration to the space of feasible squared edge lengths, showing that these measurements evolve on a manifold determined by the rigidity constraints. This edge-space perspective is closely related to the approach we will take here, where we further separate the desired evolution of the edge measurements from its realization by compatible motions of the agents. In the undirected setting, these viewpoints are tied together by the symmetric gradient structure of the closed-loop dynamics.

This structure fundamentally changes under *directed sensing*. The resulting interactions become non-reciprocal, leading to closed-loop dynamics that can no longer be considered gradient flows. In particular, the symmetry and positive-semidefinite structure that underlie the local convergence theory for undirected formations are lost. The theory of *persistence* was developed to describe the geometric implications of these directed constraints [6]. Persistence is fundamentally a kinematic notion, as it characterizes whether a directed assignment of distance constraints is compatible with maintaining a rigid formation under admissible agent motions. It does not, however, describe the dynamics induced by a particular feedback law or determine whether the desired formation is a stable equilibrium. Persistence and its variants have nevertheless played a central role in the analysis and design of directed formations [6]–[9].

Consequently, passing from persistence to stability requires additional dynamical analysis. For minimally persistent planar formations with a leader–first-follower structure, [10] constructed agent-dependent gains that locally stabilize the formation. More recently, [11] established nonlinear asymptotic stability of the directed controller for minimally persistent formations constructed by directed Henneberg-type vertex additions, in both two and three dimensions. More generally, several works have obtained convergence guarantees for special classes of directed formations, including acyclic and leader–follower structures [10], [12]–[14], often by exploiting additional structure in the interaction topology. At the same time, it has been shown that, due to the decentralized information flow and the topology of the configuration space, global stabilization is generically impossible for broad classes of directed formations [15].

The central question addressed in this paper is:

Given an arbitrary orientation of a rigid graph and a target formation, when is the resulting directed distance-based controller locally stable, and what structural properties of the orientation promote stability?

We adopt a geometric viewpoint based on the relationship between node-space dynamics and edge-space representations. Specifically, we study formation control through an auxiliary system evolving on the manifold of feasible distance

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measurements, which we then lift to agent dynamics. This construction leads to a general family of compatible formation controllers that includes the classical gradient controller, its directed counterpart, and a canonical minimum-norm lift of the edge-space gradient flow.

The key advantage of our viewpoint is that it allows the convergence problem to be studied directly in edge space. We show that local exponential convergence of the edge dynamics implies local exponential convergence of the agent configuration to a realization congruent to the target. Moreover, local exponential stability is characterized by the linearized edge operator restricted to the tangent space of feasible edge measurements, with positive definiteness of its quadratic form providing a stronger, readily computable sufficient certificate.

The edge-based perspective also allows us to reveal the dependence between the graph orientation and the target configuration in the directed setting. The edge dynamics for the directed case are governed locally by a generally non-symmetric operator, and local stability may therefore depend on the particular target even when the orientation is fixed. To distinguish this target-dependent stability question from generic properties of the orientation, we introduce three nested spectral notions: *weak admissibility*, *admissibility*, and *strong admissibility*. These respectively capture generic full rank of the edge operator, invertibility of its restriction to the feasible edge tangent space, and hyperbolicity of that restriction. Local exponential stability at a generic target implies strong admissibility. We also show that every generically rigid graph admits an admissible orientation.

For the special case of acyclic graphs, we show weak admissibility, admissibility, strong admissibility, and local exponential stability at every generic target are equivalent. We further show that this structure extends beyond acyclic formations. A stable directed rigid subformation may serve as the root of an acyclic extension. These results identify structural classes of directed architectures for which stability can be inferred from generic properties of the orientation rather than tested separately at each target.

We also compare our notions of admissibility to persistence, which we show does not characterize stability of the feedback dynamics considered here. Combined with our positive results, we see that stability of the directed controller depends on the spectral properties of the edge operator and not on the controller's information architecture.

Finally, since the quadratic stability certificate is affine in the edge weights, we formulate a semidefinite program for synthesizing stabilizing gains without altering the underlying sensing architecture.

Notations: We write $\mathbb{R}$ for the set of real numbers and $\mathbb{Q}$ for the set of rational numbers. For a matrix A, $A^\top$ denotes its transpose, $A^\dagger$ its Moore–Penrose pseudoinverse, $\operatorname{im} A$ its image, $\ker A$ its kernel, and $\operatorname{rk} A$ its rank. We write $A \succ 0$ ($A \succeq 0$) for symmetric positive definite (semidefinite). For a subspace W, $W^\perp$ is its orthogonal complement. For a smooth map F, $\mathrm{d}F_x$ denotes the differential at x. All norms are Euclidean unless stated otherwise. For a manifold $\mathcal{M}$, $T_x\mathcal{M}$ denotes the tangent space at x. The quadratic form of a linear operator T is $q_T(x) := \langle x, T(x)\rangle$, and for a T-invariant subspace V, $T|_V : V \to V$ denotes the restriction.

II. A Geometric Framework for Formation Control

The starting point for this work is an ensemble of n agents, modeled by integrator dynamics,

$$\dot{p}_i = u_i, \quad i = 1, \dots, n, \tag{1}$$

where $p_i \in \mathbb{R}^d$ is the state (position) of agent i, and $u_i \in \mathbb{R}^d$ is the control. The aggregate position (control) vector is denoted as $p = \begin{bmatrix} p_1^\top & \cdots & p_n^\top \end{bmatrix}^\top$ (similarly for u). We assume the agents interact by an information exchange network described by the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

The *target configuration* p^* has an associated set of desired inter-agent square distances, m_{ij}^*, for each $ij \in \mathcal{E}$; $m^* \in \mathbb{R}^{|\mathcal{E}|}$ is the aggregate target distance vector. We define the *distance map*, $F : \mathbb{R}^{dn} \to \mathbb{R}^{|\mathcal{E}|}$, by

$$F(p) = \begin{bmatrix} \cdots & \|p_j - p_i\|^2 & \cdots \end{bmatrix}^\top, \tag{2}$$

so that $F(p^*) = m^*$. From now on, the squared edge length measurement $F(p^*)$ of a target formation p^* will be denoted m^*. The distance-based formation control problem is to drive an initial configuration p^0 to a point q congruent and close to p^* (such that $F(q) = m^*$), using, at each time t, the information m^* and $m = F(p)$. We introduce two assumptions that we will use at various points in this paper.

Assumption II.1. *The graph $\mathcal{G}$ is d-rigid and the target configuration p^* is a regular point (preimage of a regular value) of the rigidity map F.*

Assumption II.2. *The graph $\mathcal{G}$ is d-rigid and the target configuration p^* is generic.*

Here generic means that p^* holds over the open, dense complement of an unspecified exceptional set that is defined by polynomials with rational coefficients. Every generic configuration is regular, and if p^* is generic, then $F(p^*)$ is generic in the image of F. A graph $\mathcal{G}$ is d-rigid if, for every configuration p^* so that $F(p^*)$ is smooth in the image of F, the framework $(\mathcal{G}, p^*)$ is locally rigid; in particular, when $\mathcal{G}$ is d-rigid, the frameworks $(\mathcal{G}, p^*)$ for regular p^* will be rigid (see, e.g., [16]).

A. A general family of formation controllers

The formation control objective is specified entirely in terms of the target distance vector m^*. It is natural, therefore, to consider both the node positions p and the distance measurements m as state-variables related by the constraint $F(p) = m$. Rather than designing the node dynamics directly, we instead consider a more general problem of constructing compatible node and edge dynamics that evolve on the constraint manifold.

Since the constraint is that m lies in the image of F, we denote by $\mathcal{Q}$ the regular values of the map F. Because F is a polynomial mapping defined over $\mathbb{Q}$, the semialgebraic Sard theorem [17, p. 234] implies that $\mathcal{Q}$ is a smooth semialgebraic set that is open and dense in the image of F. (The image of

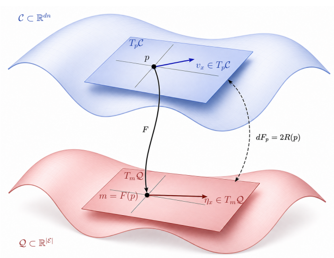


Fig. 1: Geometric structure underlying compatible formation controllers. The distance map F sends a configuration $p \in \mathcal{C}$ to its feasible edge measurement $m = F(p) \in \mathcal{Q}$. At the tangent level, node and edge velocities are related by $\eta_{(p,m)} = 2R(p)\nu_{(p,m)}$, which is the compatibility condition.

F itself is not smooth. We will see presently why we use $\mathcal{Q}$ instead of the smooth locus of $\operatorname{Im} F$.) We also note that $\mathcal{C} := F^{-1}(\mathcal{Q}) \subseteq \mathbb{R}^{dn}$, the set of regular points of F, is a smooth semi-algebraic set of dimension dn. Let $R(p)$ denote the rigidity matrix of $(\mathcal{G}, p)$, so that

$$F(p) = R(p)p, \quad \mathrm{d}F_p = 2R(p) : T_p\mathcal{C} \to T_{F(p)}\mathcal{Q}.$$

At any point $m = F(p)$ in $\mathcal{Q}$, the tangent space $T_m\mathcal{Q}$ is the vector space of differential changes to the squared edge lengths that can arise from the infinitesimal motions of the agents, i.e.,

$$T_m\mathcal{Q} = \operatorname{im} R(p).$$

The manifold $\mathcal{Q}$ inherits the Euclidean inner product from $\mathbb{R}^{|\mathcal{E}|}$, defining a Riemannian metric,

$$g_m(\xi_1, \xi_2) = \xi_1^\top \xi_2, \qquad \xi_1, \xi_2 \in T_m\mathcal{Q}.$$

Let $\Pi(m) : \mathbb{R}^{|\mathcal{E}|} \to T_m\mathcal{Q}$ denote the orthogonal projector onto $T_m\mathcal{Q}$. If m is a regular value of F, for any $p \in F^{-1}(m)$, we have

$$\Pi(m) = R(p)R(p)^\dagger. \tag{3}$$

(Note that if m is smooth but not regular, then the formula for $\Pi(m)$ will be valid for some p in the fiber over m but not others.) In particular, although the expression on the right is written in terms of p, the resulting projector depends only on m.

With the geometry specified above, the state-space for the node-edge system may be defined as

$$\mathcal{S} = \{(p, m) \in \mathcal{C} \times \mathcal{Q} : F(p) = m\}, \tag{4}$$

which is a smooth, irreducible, semi-algebraic set equipped with the natural projections π_1 and π_2. At any point $(p, m) \in \mathcal{S}$, its tangent space is

$$T_{(p,m)}\mathcal{S} = \{(\dot{p}, \dot{m}) \in T_p\mathcal{C} \times T_m\mathcal{Q} : \dot{m} = 2R(p)\dot{p}\}.$$

This setup leads to the central abstraction proposed in this paper. Instead of designing control laws in the node space, we define a formation controller by specifying compatible dynamics on the joint node-edge state space $\mathcal{S}$. The geometric relationship between the node and edge spaces is illustrated in Figure 1.

Definition II.3 (Compatible formation controller)**.** *Let $\mathcal{G}$ be a generically d-rigid graph (which, along with the dimension d, determines, $\mathcal{Q}$ and $\mathcal{S}$). A* compatible formation controller *on $\mathcal{G}$ is specified by a pair of state-dependent linear operators*

$$\nu_{(p,m)} : \mathbb{R}^{|\mathcal{E}|} \to (\mathbb{R}^d)^n, \qquad \eta_{(p,m)} : \mathbb{R}^{|\mathcal{E}|} \to \mathbb{R}^{|\mathcal{E}|},$$

defined smoothly on $\mathcal{S}$, such that

$$(\nu_{(p,m)}x, \eta_{(p,m)}x) \in T_{(p,m)}\mathcal{S}$$

for every $(p, m) \in \mathcal{S}$ and every $x \in \mathbb{R}^{|\mathcal{E}|}$.

The operator $\eta_{(p,m)}$ specifies the desired evolution of the edge measurements, while $\nu_{(p,m)}$ realizes this edge motion in node space. Since

$$(\nu_{(p,m)}x, \eta_{(p,m)}x) \in T_{(p,m)}\mathcal{S},$$

the induced node and edge velocities remain compatible with the constraint $m = F(p)$. Equivalently,

$$\eta_{(p,m)} = 2R(p)\nu_{(p,m)},$$

where the equality is understood as an identity of linear maps.

Given a target formation $(\mathcal{G}, p^*)$ with edge measurements m^* and a starting formation $(\mathcal{G}, p^0)$, the closed-loop dynamics are given by the initial value problem (IVP)

$$\dot{p} = \nu_{(p,m)}(m^* - m) \tag{5a}$$
$$\dot{m} = \eta_{(p,m)}(m^* - m) \tag{5b}$$

with initial conditions $p(0) = p^0$ and $m(0) = m^0 = F(p^0)$.

For technical reasons, we will also allow controllers to be defined only on an open neighborhood in $\mathcal{S}$ of (p^*, m^*). Controllers in this family yield solutions $(p, m) \in \mathcal{S}$. When we discuss the *edge dynamics* of one of these controllers, we mean the flow $\pi_2(p, m) \in \mathcal{Q}$, and the node dynamics are, similarly, $\pi_1(p, m) \in \mathcal{C}$. The edge and node dynamics have the same smoothness as the full solution. For notational ease, we use the shorthand $\eta := \eta_{(p,m)}$ and $\eta^* := \eta_{(p^*,m^*)}$ (similarly for ν).

Because distance measurements determine a rigid formation only up to congruence, stability of the node dynamics is understood relative to the congruence class of the target. This leads to a stability definition we adopt for the remainder of the work.

Definition II.4 (Local exponential stability)**.** *A compatible controller is* locally exponentially stable *at a target configuration $p^* \in \mathcal{C}$ if there is a neighborhood $U \ni p^*$ such that, for all initial conditions $(p^0, F(p^0))$ with $p^0 \in U$ the node dynamics converge exponentially to a configuration $q \in U$ that is congruent to p^*.*

We now present our first main result showing that the edge dynamics actually control the convergence behavior of the node dynamics.

Theorem II.5 (Edge-to-node convergence)**.** *Let Assumption II.1 hold, and let (ν, η) be a compatible formation controller defined on an open neighborhood $V \subseteq \mathcal{S}$ of (p^*, m^*), where ν is smooth. Suppose further there exist constants*

$C, c > 0$ *such that every solution of* (5) *with initial condition* $(p^0, m^0) \in V$ *satisfies*

$$\|m(t) - m^*\| \leq Ce^{-ct}\|m^0 - m^*\|, \qquad t \geq 0. \qquad (6)$$

Then there exists a neighborhood $U \subseteq V$ *such that, for every initial condition* $(p^0, m^0) \in U$*, there is a configuration* q *congruent to* p^* *satisfying*

$$\|p(t) - q\| \leq Ke^{-ct}\|m^0 - m^*\|, \qquad t \geq 0,$$

where $K > 0$ *is independent of the initial condition. In particular, the controller is locally exponentially stable at* p^* *in the sense of Definition II.4.* △

Theorem II.5 shows that local exponential convergence of the edge dynamics implies local exponential convergence of the node dynamics to a configuration congruent to the target.

We will need the following technical "trapping" lemma to bootstrap the main results.

Lemma II.6. *Let Assumption II.1 hold, and let* (ν, η) *be a compatible formation controller defined on an open neighborhood* $V \subseteq \mathcal{S}$ *of* (p^*, m^*)*, where* ν *is smooth. Suppose further there exist constants* $C, c > 0$ *such that every solution of* (5) *with initial condition* $(p^0, m^0) \in V$ *satisfies* (6) *for all times* $t \geq 0$ *such that the trajectory remains in* V*. Then there are neighborhoods* $U \subseteq W \subseteq V$ *of* (p^*, m^*) *such that, for every solution of* (5) *with initial condition* $(p^0, m^0) \in U$*,* $(p(t), m(t))$ *remains in* W *for all times* $t \geq 0$*.*

The proof is given in the Appendix.

Proof of Theorem II.5. Let us fix a target $(p^*, m^*) \in \mathcal{S}$. The goal is to find a neighborhood $U \subseteq \mathcal{S}$ of (p^*, m^*) so that there is a constant $K > 0$ such that, for all initial conditions $(p^0, m^0) \in U$, there is a configuration q congruent to p^*, such that $\|q - p(t)\| \leq K\mathrm{e}^{-ct}$.

It is assumed there are constants $C, c > 0$ and a neighborhood $V \subseteq \mathcal{S}$ of (p^*, m^*) so that, for all initial conditions $(p^0, m^0) \in V$, $\|m(t) - m^*\| \leq C\mathrm{e}^{-ct}\|m^0 - m^*\|$. Because $\mathcal{G}$ is d-rigid and p^* is a regular point of F, we can find a neighborhood U_0 of p^* in $\mathcal{C}$ such that, if $q \in U_0$ and $F(q) = m^*$, then q is congruent to p^*. Let us now consider the full dynamics in the neighborhood $V_0 = (U_0 \times \mathcal{Q}) \cap V$ of the target (p^*, m^*).

By Lemma II.6, we can find neighborhoods $U \subseteq W \subseteq V_0$, such that, for all initial conditions in U, the trajectory remains in W for all time. Consider a sequence $t_i \to \infty$, and let $j > i$. Since the trajectory remains in W, we have

$$\begin{aligned}\|p(t_j) - p(t_i)\| &\leq \int_{t_i}^{t_j} \|\dot{p}(t)\|\, dt \leq \int_{t_i}^{\infty} C' e^{-ct}\|m^0 - m^*\|\, dt \\ &\leq \frac{C'}{c} e^{-ct_i}\|m^0 - m^*\|,\end{aligned}$$

where $C' = (\sup_W \|\nu\|)C$ (we have used that ν is smooth and defined on the closure of W). Hence, the sequence $p(t_i)$ is Cauchy, and has a limit q. A similar estimate gives

$$\|q - p(t)\| \leq \frac{C'}{c} e^{-ct}\|m^0 - m^*\| = Ke^{-ct}\|m^0 - m^*\|,$$

so the convergence is exponential. Since the whole state converges to $(q, m^*) \in \mathcal{S}$, we have $F(q) = m^*$ and $q \in U_0$, so q is congruent to p^*. □

From Theorem II.5, we get two certificates of convergence for compatible controllers.

Theorem II.7 (Certificates of local exponential convergence)**.** *Suppose that Assumption II.1 holds at a target* p^**. Denote by* η^* *the transformation* $\eta_{(p^*, m^*)}$ *and* T *the tangent space* $T_{m^*}\mathcal{Q}$*. Then, if either:*

(a) the transformation $-\eta^*|_T$ *is Hurwitz (i.e. all eigenvalues of* $\eta^*|_T$ *have positive real part);*
(b) the quadratic form $q_{\eta^*}(v) = \langle v, \eta^*(v)\rangle$ *is positive definite on* T*,*

then the controller is locally exponentially stable at p^**. Moreover, (a) is necessary for local exponential stability.* △

Proof. Let us first deal with sufficiency. Because it is easier, we first prove (b). Let A be the matrix of $\eta^*|_T$ in any basis. The signature of q_{η^*} on T is equal to that of $\frac{1}{2}(A + A^\top)$, so $-(A + A^\top) \prec 0$. From the Lyapunov LMI criterion, $-\eta^*|_T$ is Hurwitz. Hence (b) implies (a).

For (a), we use a Lyapunov method. Let A be the matrix of $\eta^*|_T$ in any basis of T. Since $-A$ is Hurwitz, there exists a symmetric $P \succ 0$ such that $\frac{1}{2}(PA + A^\top P) \succ 0$. We can now define a positive-definite operator $\tilde{P}$ on $\mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ by setting $\tilde{P} = P$ on T, and $\tilde{P} = I$ on $T^\perp$. Consider the Lyapunov function

$$W(m) = (m^* - m)^\top \tilde{P}(m^* - m).$$

At (p^*, m^*), the Lyapunov inequality implies that for some $c > 0$ we have the coercive estimate,

$$v^\top \left(\tfrac{1}{2}(PA + A^\top P)\right) v \geq c\|v\|^2, \ v \in T.$$

The main technical step is to extend this estimate from tangent vectors at m^* to the nonlinear error $m^* - m$ for nearby states. In particular, we show there exists a neighborhood V of (p^*, m^*) such that, for $(p, m) \in V$,

$$q_{(p,m)}(m^* - m) \geq \alpha\|m^* - m\|^2, \qquad (7)$$

where $\alpha > 0$ is a constant independent of the state and $q_{(p,m)} := q_{\tilde{P}\eta_{(p,m)}}$ is the quadratic form

$$q_{(p,m)}(x) = x^\top \left(\tfrac{1}{2}(\tilde{P}\eta_{(p,m)} + \eta_{(p,m)}^\top \tilde{P})\right) x.$$

This implies that along trajectories in V,

$$\dot{W}(m) = -2q_{(p,m)}(m^* - m) \leq -2\alpha\|m^* - m\|^2.$$

Since $\tilde{P} \succ 0$, $W(m) \leq \lambda_{\max}(\tilde{P})\|m^* - m\|^2$, it follows that

$$\dot{W}(m) \leq -\tfrac{2\alpha}{\lambda_{\max}(\tilde{P})} W(m). \qquad (8)$$

By Grönwall's inequality [18, Lemma A.1],

$$W(m) \leq W(m^0)\exp\left(-\tfrac{2\alpha}{\lambda_{\max}(\tilde{P})} t\right).$$

Since $W(m) \geq \lambda_{\min}(\tilde{P})\|m^* - m\|^2$, we obtain an estimate

$$\|m^* - m\| \leq Ce^{-ct}\|m^* - m^0\|, \quad 0 \leq t \leq t^*,$$

for trajectories initialized at $(p^0, m^0) \in V$. Here t^* is the supremal time that the trajectory remains in V. Lemma II.6 now implies that there is a neighborhood $U \subseteq V$ of (p^*, m^*) so that trajectories initialized at $(p^0, m^0) \in U$ remain in V for all time. Local exponential convergence then follows from Theorem II.5.

What remains is to construct V on which (7) holds. First, since $q_{(p^*,m^*)}$ is positive definite, there is a $\beta > 0$ such that $q_{(p^*,m^*)}(v) > \beta \|v\|^2$ for all vectors $v \in T$. For $m \neq m^*$ in $\mathcal{Q}$, define the normalized secant

$$e(m) := \frac{m^* - m}{\|m^* - m\|},$$

and decompose $e(m) = v(m) + w(m)$ with $v(m) \in T$, and $w(m) \in T^\perp$. Since $\mathcal{Q}$ is a smooth embedded manifold, as $m \to m^*$ in $\mathcal{Q}$, it follows that $\|w(m)\| \to 0$ and $\|v(m)\| \to 1$, uniformly.

Since compatibility gives $\operatorname{im} \eta^* \subseteq T$, and $\widetilde{P}$ preserves T and $T^\perp$, we have $w^\top \widetilde{P} \eta^* x = 0$ for every $w \in T^\perp$, and $x \in \mathbb{R}^{|\mathcal{E}|}$. Therefore, writing $e = v + w$ as above,

$$q_{(p^*,m^*)}(e) = q_{(p^*,m^*)}(v) + v^\top \widetilde{P} \eta^* w.$$

Using the positive-definiteness estimate on T,

$$q_{(p^*,m^*)}(e) \geq \beta \|v\|^2 - \|\widetilde{P}\eta^*\| \, \|v\| \, \|w\|.$$

Since $\|v\| \to 1$ and $\|w\| \to 0$ as $m \to m^*$ uniformly, there exists a neighborhood U of m^* in $\mathcal{Q}$ such that

$$q_{(p^*,m^*)}(e(m)) \geq \frac{\beta}{2}, \qquad m \in U, \quad m \neq m^*.$$

By smoothness of η, after shrinking to a neighborhood $V \subseteq \mathcal{S}$ of (p^*, m^*) with $m \in U$, we may assume that

$$\big\| \widetilde{P} \eta_{(p,m)} - \widetilde{P} \eta^* \big\| < \frac{\beta}{4}.$$

Hence, for the unit vector $e = e(m)$,

$$\begin{aligned} q_{(p,m)}(e) &= q_{(p^*,m^*)}(e) + \Big\langle e, \widetilde{P}\big(\eta_{(p,m)} - \eta^*\big) e \Big\rangle \\ &\geq \frac{\beta}{2} - \big\| \widetilde{P}\big(\eta_{(p,m)} - \eta^*\big) \big\| \geq \frac{\beta}{4}. \end{aligned}$$

Therefore,

$$q_{(p,m)}(m^* - m) \geq \frac{\beta}{4} \|m^* - m\|^2,$$

for every $(p, m) \in V$. Thus (7) holds with $\alpha = \beta/4$, completing the construction of V and showing that (a) is sufficient.

We now prove necessity. Suppose that the controller is locally exponentially stable at (p^*, m^*). We show that $-\eta^*|_T$ is Hurwitz. Since m^* is a regular value of F, there exists a smooth local section $\sigma : U \subset \mathcal{Q} \to \mathcal{C}$, of F, defined on a neighborhood V of (p^*, m^*) with compact closure so that $F(\sigma(m)) = m$ and $\sigma(m^*) = p^*$. Fix $v \in T$. By smoothness of $\mathcal{Q}$, there exists a smooth curve $s : (-\varepsilon, \varepsilon) \to \mathcal{Q}$ such that $s(0) = m^*$, with $\dot{s}(0) = -v$. For each sufficiently small t, consider the solution of the controller with initial condition $(p_t^0, m_t^0) = (\sigma(s(t)), s(t))$. Let $e(\tau, t) := m^* - m(\tau; t)$ denote the corresponding edge error. By construction, $e(0, t) = tv + o(t)$. Since the vector field is smooth, the solution depends smoothly on its initial condition. Differentiating the edge-error dynamics with respect to t at $t = 0$ therefore gives the variational equation along the equilibrium (p^*, m^*).

The edge error satisfies

$$\dot{e} = -\eta_{(p,m)} e.$$

At (p^*, m^*) we have $e = 0$, so in the linearization all terms involving derivatives of $\eta_{(p,m)}$ are multiplied by the equilibrium error and vanish. Hence, defining

$$z(\tau) := \left. \frac{\partial e(\tau, t)}{\partial t} \right|_{t=0},$$

we obtain $\dot{z} = -\eta^* z, \ z(0) = v$.

Compatibility gives $\operatorname{im} \eta^* \subseteq T$, and hence T is invariant under η^*. Therefore

$$z(\tau) = \exp(-\tau \, \eta^*|_T) \, v.$$

Equivalently, for every fixed $\tau \geq 0$,

$$e(\tau, t) = t \, \exp(-\tau \, \eta^*|_T) \, v + o(t) \qquad \text{as } t \to 0.$$

Since the controller is locally exponentially stable at (p^*, m^*), after shrinking the neighborhood if necessary there exist constants $K, c > 0$ such that, for every sufficiently small initial perturbation, the corresponding node trajectory converges exponentially to a realization congruent to p^*. Since F is locally Lipschitz, there exists $C > 0$ such that the associated edge error satisfies

$$\|e(\tau, t)\| \leq C e^{-c\tau} \|e(0, t)\|, \qquad \tau \geq 0.$$

Substituting the expansions above gives, for each fixed $\tau \geq 0$,

$$\|t \, \exp(-\tau \eta^*|_T) \, v + o(t)\| \leq C e^{-c\tau} \|tv + o(t)\|.$$

Dividing by $|t|$ and letting $t \to 0$ yields

$$\|\exp(-\tau \eta^*|_T) \, v\| \leq C e^{-c\tau} \|v\|.$$

Since $v \in T$ was arbitrary,

$$\|\exp(-\tau \eta^*|_T)\| \leq C e^{-c\tau}, \qquad \tau \geq 0.$$

Thus the linear system $\dot{z} = -\eta^*|_T z$ is exponentially stable. Equivalently, every eigenvalue of $\eta^*|_T$ has positive real part, and therefore $-\eta^*|_T$ is Hurwitz. This proves necessity. □

B. A model compatible controller

Theorem II.7 reduces local convergence of a compatible formation controller to properties of its induced edge dynamics. To develop a geometric understanding of this result, we now consider how one should ideally drive the edge measurements to the target assuming the edge state $m \in \mathcal{Q}$ could be controlled directly. To this end, consider the edge measurement vector $m \in \mathcal{Q}$ as the state of the first-order system

$$\dot{m} = v, \tag{9}$$

where $v \in \mathbb{R}^{|\mathcal{E}|}$ is an edge-space control input. This system should not be interpreted as an independent physical model of the formation. Rather, it specifies the desired evolution of the edge measurements, which must subsequently be realized by compatible node motions. Define the *edge potential* function,

$$V_e(m) = \tfrac{1}{2}\|m - m^*\|^2,$$

which measures the squared deviation of the current edge vector m from the desired one m^*. As we are interested in feasible motions that must remain on $\mathcal{Q}$, the appropriate notion of the gradient is the *Riemannian gradient* on $(\mathcal{Q}, g)$, obtained by projecting $\nabla V_e(m) = m - m^*$ onto the tangent space of $\mathcal{Q}$ at m,

$$\nabla_{\mathcal{Q}} V_e(m) = \Pi(m)\, \nabla V_e(m) = \Pi(m)\,(m - m^*).$$

The *Riemannian gradient flow* associated with V_e is defined by

$$\dot{m} = -\,\nabla_{\mathcal{Q}} V_e(m) = -\,\Pi(m)\,(m - m^*). \tag{10}$$

A solution to (10) describes a trajectory $m(t) \in \mathcal{Q}$ whose velocity $\dot{m}(t)$ is the projection of the negative Euclidean gradient of V_e onto the tangent space $T_{m(t)}\mathcal{Q}$; i.e., $m(t)$ follows the steepest descent of V_e with respect to the Riemannian metric g.

Corollary II.8 (Exponential stability of the edge gradient flow)**.** *Let $m^* \in \mathcal{Q}$. Then m^* is a locally exponentially stable equilibrium of the Riemannian gradient flow* (10)*. In particular, there exist a neighborhood $U_{\mathcal{Q}} \subseteq \mathcal{Q}$ of m^* and constants $C, c > 0$ such that every solution with $m(0) \in U_{\mathcal{Q}}$ satisfies*

$$\|m(t) - m^*\| \le Ce^{-ct}\|m(0) - m^*\|, \qquad t \ge 0.$$

This follows from the fact that $\Pi(m^*)$ is the identity on the tangent space and the proof of Theorem II.7 (the theorem itself deals with the combined dynamics).

What was convenient about this case was that we have an intrinsic description of the edge dynamics that does not use any information about the node state p. This means we can work on $\mathcal{Q}$ instead of $\mathcal{S}$.

To connect the squared edge length flow defined by the Riemannian gradient flow in (10) to node dynamics, we make use of the fact that $\mathcal{Q} = F(\mathcal{C})$. The linear map $\mathrm{d}F_p = 2R(p)$ relates infinitesimal changes of the node positions $\dot{p} = u$ to infinitesimal changes of the squared edge lengths via

$$\dot{m} = \mathrm{d}F_p\, u = 2R(p)\, u.$$

Hence, to produce a prescribed edge-space velocity $v^* \in T_{F(p)}\mathcal{Q}$, we seek a node-space velocity u satisfying

$$\mathrm{d}F_p\, u = v^*.$$

Among the infinitely many u that satisfy this relation, a natural choice is the one with *minimum Euclidean norm*:

$$\min_{u \in \mathbb{R}^{dn}} \|u\|^2 \quad \text{s.t.} \quad \mathrm{d}F_p\, u = v^*. \tag{11}$$

The unique minimizer of (11) is obtained via the Moore–Penrose pseudoinverse,

$$u^* = (\mathrm{d}F_p)^\dagger v^* = \tfrac{1}{2}\, R(p)^\dagger v^*.$$

Consequently, the combined edge–node system can be viewed as a two-level geometric control construction. Substituting $v^* = -\,\Pi(m)\,(m - m^*)$ yields the node-space dynamics

$$\dot{p} = u^* = -\,\tfrac{1}{2}\, R(p)^\dagger\, \Pi(m)\,(m - m^*). \tag{12}$$

The above construction gives us our first general compatible controller, with

$$\begin{cases} \nu = \dfrac{1}{2} R(p)^\dagger, & (13a)\\ \eta = \Pi(m) = 2R(p)\nu. & (13b) \end{cases}$$

We refer to (13) as the *model controller*. This control, while not distributed, and not intended for practical implementation, provides a clear geometric picture of what should be done to drive agents to a desired formation.

Example II.9 (Classical distributed solution as a compatible controller)**.** *We now show how the classical distributed solution to the formation control problem [2] fits our framework. The standard approach to derive this control is through the negative-gradient flow of the potential function $V(p) = \frac{1}{4}\|F(p) - m^*\|^2$, yielding*

$$u = -\tfrac{\partial V(p)}{\partial p} = -R(p)^\top (R(p)p - m^*). \tag{14}$$

This control strategy can also be seen as a compatible formation controller by setting

$$\nu = R(p)^\top \text{ and } \eta = 2R(p)R(p)^\top. \tag{15}$$

These are polynomial functions of the state (p, m), and, hence, are rational over all of $\mathcal{S}$. The column space of η is contained in the column space of $R(p)$, and hence in $T_m\mathcal{Q}$. By construction, we have $2R(p)\nu = \eta$, so the controller is well-defined. The edge controller is no longer the Riemannian gradient: η is not a projection, nor is it constant over fibers of F. However, the matrix $2R(p)R(p)^\top$ is positive definite on $T_m\mathcal{Q}$, which is sufficient for Theorem II.7 to apply, yielding a new proof that the controller is locally exponentially stable for every target p^* such that $(p^*, F(p^*)) \in \mathcal{S}$. ◊

III. Generic admissibility

The main result of Section II showed that local exponential convergence of a compatible formation controller can be certified through the edge dynamics. The corresponding certificates, however, are pointwise conditions on the operator $\eta^*|_{T_{m^*}\mathcal{Q}}$ at a particular target. This motivates the study of properties of η that are *intrinsic* to the controller, rather than to a particular target state (p^*, m^*). We begin with a general definition of a new notion we term *admissibility* for compatible controllers.

Definition III.1 (Admissibile compatible controller)**.** *A compatible formation controller (Definition II.3) defined by rational functions is called*

(a) strongly admissible *if, for every generic target configuration $p^* \in \mathcal{C}$, $\eta^*|_{T_{m^*}\mathcal{Q}}$ is hyperbolic (i.e. all eigenvalues of $\eta^*|_{T_{m^*}\mathcal{Q}}$ have non-zero real part);*

(b) admissible *if $\eta^*|_{T_{m^*}\mathcal{Q}}$ is invertible for every generic target p^*;*

(c) weakly admissible *if the rank of* η^* *is* $dn-\binom{d+1}{2}$ *for every generic target* p^*.

Pointwise, strong admissibility (all eigenvalues have non-zero real parts) implies admissibility (no eigenvalue is zero), implies weak admissibility (since invertibility of $\eta^*|_{T_{m^*}\mathcal{Q}}$ requires η^* to have full rank on $T_{m^*}\mathcal{Q}$). Additionally, weak admissibility and admissibility agree if $\mathcal{G}$ is minimally d-rigid.

A. Genericity

We now show how the admissibility conditions in Definition III.1 can be verified generically. In particular, weak admissibility and admissibility can be certified at a single target, while a Hurwitz condition at one target provides a certificate for strong admissibility.

Definition III.2 (Genericity)**.** *Let* P *be a property of points in* $\mathbb{R}^m$. *If there is a proper algebraic subset,* $X \subseteq \mathbb{R}^m$*, defined over* $\mathbb{Q}$*, such that either for all* $y \notin X$*,* $P(y)$ *holds or,* $y \notin X$*,* $P(y)$ *does not hold, then* P *is called a* generic property.

The motivation for this definition is that, as a proper algebraic set, X is a closed, nowhere dense, subset of $\mathbb{R}^m$. Hence, the points on which a generic property P has atypical behavior are small in measure-theoretic, topological, or geometric terms.

In what follows, we will often work with properties P that are defined by the non-vanishing of a known collection of polynomials with rational coefficients. One example is that a structured matrix $A \in \mathbb{R}[\mathbf{t}]^{m\times n}$ with polynomial entries has rank at least r, where the polynomials are formal minors. Another one is that a configuration lies in $\mathcal{S}$. In this setting, one can show that P is automatically a generic property and that P holds generically if it holds at a single point y.

Theorem III.3 (Admissibility is a generic property)**.** *Fix a generically* d*-rigid graph* $\mathcal{G}$ *with* n *vertices, a compatible formation controller, and a configuration* $p^* \in \mathcal{C}$. *Denote by* m^* *the edge lengths* $F(p^*)$*, so that* $(p^*, m^*) \in \mathcal{S}$*, by* η^* *the linear operator* $\eta_{(p^*,m^*)}$ *and by* T *the tangent space* $T_{m^*}\mathcal{Q}$. *If either of the statements:*

(a) η^* *has rank* $N := dn - \binom{d+1}{2}$*;*

(b) $\eta^*|_T$ *is invertible,*

holds at p^* *then it holds for every generic* $p \in \mathcal{C}$. *Moreover, if* $-\eta^*|_T$ *is Hurwitz, then at every generic* p*, the transformation* $\eta_{(p,F(p))}|_{T_{F(p)}\mathcal{Q}}$ *is hyperbolic.* △

In the proof we will use some technical results about rational maps.

Lemma III.4. *Let* $\alpha(\mathbf{t}) : \mathbb{R}^m \to \mathbb{R}^m$ *be a linear transformation that depends rationally on the variables* $\mathbf{t} = (t_1, \ldots, t_m)$. *Then*

(a) if $\alpha(\mathbf{t})$ *is invertible for some* $\mathbf{t}$*, then it is invertible for generic* $\mathbf{t}$*;*

(b) if, for some $\mathbf{t}$*, no pair of eigenvalues of* $\alpha(\mathbf{t})$ *sums to zero, then the same holds for generic* $\mathbf{t}$.

We remark that we cannot strengthen the conclusion to say that hyperbolicity of α is a generic property. This is why we use Hurwitz as the one-point certificate of generic hyperbolicity.

Lemma III.5. *Let* $\alpha(\mathbf{t}) : \mathbb{R}^m \to \mathbb{R}^m$ *be a linear transformation that depends rationally on the variables* $\mathbf{t} = (t_1, \ldots, t_m)$. *Then the restriction* $\alpha(\mathbf{t})|_{\operatorname{im}\alpha(\mathbf{t})}$ *of* α *to its image also depends rationally on* $\mathbf{t}$.

The proofs of Lemma III.4 and III.5 are given in the Appendix.

Proof of Theorem III.3. Since F is a polynomial map, the state $(p^*, m^*) = (p^*, F(p^*))$ depends polynomially on p^*. Since η is rational in the state, it follows that η^* depends rationally on p^*. From compatibility, $\operatorname{im}\eta^* \subseteq T$, and regularity of m^* implies that $\dim T = N$. Hence η^* has rank N if and only if the induced map $\mathbb{R}^{|\mathcal{E}|}/\ker\eta^* \to T$ is invertible. By Gaussian elimination, there is a complement V to $\ker\eta^*$ such that the inclusion $V \hookrightarrow \mathbb{R}^{|\mathcal{E}|}$ is rational in the state. This shows that the induced map is rational, and (a) follows from Lemma III.4.

Because rank cannot increase under restriction, statement (a) must hold whenever statement (b) does. In that case, we have $T = \operatorname{im}\eta^*$. Statement (b) then follows from Lemma III.4 applied to $\eta^*|_T$, which depends rationally on p^* by Lemma III.5.

Generic hyperbolicity follows similarly, once we note that, if the real parts of every eigenvalue of $\eta^*|_T$ are positive, no pair of them can sum to zero. □

B. Weak admissibility vs admissibility

We can make a precise statement about the relationship between weak admissibility and admissibility of a compatible controller. In what follows, we will often use the condition

$$\operatorname{im}\alpha \cap \ker\alpha = \{0\} \tag{16}$$

for a linear transformation $\alpha : \mathbb{R}^m \to \mathbb{R}^m$.

Theorem III.6 (Weak admissibility upgrade)**.** *A weakly admissible compatible controller is admissible if and only if, at some generic target state* $(p^*, m^*) \in \mathcal{S}$*,* η^* *satisfies* (16). *Moreover, if, at* p^**,* (16) *holds and the eigenvalues in the non-zero spectrum of* η^* *have positive real parts, then* $-\eta^*|_{T_{m^*}\mathcal{Q}}$ *is Hurwitz, and the controller is strongly admissible.* △

Proof. Denote by T the tangent space $T_{m^*}\mathcal{Q}$. By compatibility, $\operatorname{im}\eta^* \subseteq T$, so weak admissibility corresponds to the statement that $\operatorname{im}\eta = T$. Let π be a linear projection $\mathbb{R}^{|\mathcal{E}|} \to T$ and π^* the corresponding inclusion $\operatorname{im}\eta = T \hookrightarrow \mathbb{R}^{|\mathcal{E}|}$. Admissibility is the statement that $\pi \circ \eta \circ \pi^*$ is invertible. By Theorem III.3 and genericity of p^*, we can check the statement pointwise at p^*.

We will show that the controller is not admissible at p^* if and only if (16) does not hold. Suppose first that (16) does not hold. By weak admissibility, $\operatorname{im}\eta = T$, so there is a $v \in T$ such that $\eta^*(v) = 0$. For this v, we have that $\pi^*(v) = v$, so $\pi\circ\eta\circ\pi^*(v) = 0$, certifying that the controller is not admissible at p^*. For the other direction, suppose the controller is not admissible at p^*. Because the controller is not admissible, there is a non-zero $v \in T$ such that $\pi \circ \eta \circ \pi^*(v) = 0$. By weak admissibility $\operatorname{im}\pi^* = \operatorname{im}\eta$, so $w = \pi^*(v) \in \operatorname{im}\eta$. By compatibility, $\pi \circ \eta(w) = \eta(w)$, so $w \in \ker\eta$; i.e., (16) does not hold.

In the case that (16) holds, because $\operatorname{im}\eta = \operatorname{im}\pi \circ \eta \circ \pi^*$, the two maps have the same non-zero eigenspaces. Hence, if the eigenvalues in the non-zero spectrum of η^* have positive real parts, then $-\eta^*|_{T_{m^*}\mathcal{Q}}$ is Hurwitz. □

The condition (16) can be verified locally using a flag of invariant subspaces. To set up some notation, we let $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator and $\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = \mathbb{R}^m$ (we do not require the inclusions to be strict) a flag of α-invariant subspaces. We define $W_j = V_j/V_{j-1}$ and denote by π_j the quotient map and by α_j the induced map $W_j \to W_j$, which is $\pi_j \circ \alpha \circ \pi_j^*$ (where the inclusion π_j^* is any linear section). We say that the flag V satisfies the *decoupling condition* if $\operatorname{im}\alpha \cap V_j = \alpha(V_j)$ for all j. There are several equivalent formulations:

Lemma III.7. *Let $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator and V_i a flag of α-invariant subspaces. The following are equivalent:*
(a) V_i satisfies the decoupling condition;
(b) for all j, if $v \in V_j$ and $\alpha(v) \in V_{j-1}$, then $\alpha(v) \in \alpha(V_{j-1})$;
(c) for all j, $\operatorname{rk}\alpha|_{V_j} = \operatorname{rk}\alpha|_{V_{j-1}} + \operatorname{rk}\alpha_j$;
(d) $\operatorname{rk}\alpha = \operatorname{rk}\alpha_1 + \cdots + \operatorname{rk}\alpha_k$.

The proof is in the appendix. The point of the decoupling condition is that it is a local property that implies the more global (16). We note that the rank inequality $\operatorname{rk}\alpha \geq \operatorname{rk}\alpha_1 + \cdots + \operatorname{rk}\alpha_k$ holds unconditionally for any invariant flag.

Theorem III.8 (Invariant flag certificate of admissibility)**.** *Let $V_{j,(p,m)}$ be a flag of $\eta_{(p,m)}$-invariant subspaces that depends rationally on the state, and suppose that the controller is weakly admissible. If, at some generic target p^*, the flag $V_{j,(p^*,m^*)}$ satisfies the decoupling condition, then the controller is admissible if and only if each induced map η_j^* satisfies* (16)*. If, in addition, the nonzero spectrum of η^* have positive real parts, then $-\eta^*|_{T_{m^*}\mathcal{Q}}$ is Hurwitz and the controller is strongly admissible.* △

The proof uses two further technical lemmas that are proved in the appendix.

Lemma III.9. *Let $\alpha(\mathbf{t}) : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation that depends rationally on $\mathbf{t}$ and $V_i(\mathbf{t})$ a flag of $\alpha(\mathbf{t})$-invariant subspaces that depends rationally on $\mathbf{t}$. Then, the spectrum of $\alpha(\mathbf{t})$ is the union of the spectra of the induced maps $\alpha(\mathbf{t})_j$, and the following are generic properties:*
(a) $V_i(\mathbf{t})$ satisfies the decoupling condition;
(b) the induced maps $\alpha(\mathbf{t})_j$ satisfy (16)*.*

Lemma III.10. *Let $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator and V_i a flag of α-invariant subspaces satisfying the decoupling condition. Then* (16) *holds for α if and only if it holds for each induced map α_j.*

Proof of Theorem III.8. By Theorems III.3 and III.6 and Lemma III.9, all the properties in the hypothesis are generic, and so may be checked pointwise at p^*. The result now follows from Lemma III.10 and then Theorem III.6. □

C. Strong admissibility and dynamics

This next result is immediate from Theorems II.7 and III.3.

Corollary III.11 (Exponential stability implies strong admissibility)**.** *Let Assumption II.2 hold. If a compatible formation controller (Definition II.3) is locally exponentially stable at any target formation (p^*, m^*) the controller is strongly admissible.*

In general, the different admissibility notions are distinct, as we will see in the next section.

IV. Compatible Controllers on Directed Graphs

An appealing feature of the gradient control (14) is that it admits a *distributed* implementation. At the agent level, (14) is equivalent to

$$u_i = \sum_{ij\in\mathcal{E}} (\|p_i - p_j\|^2 - m_{ij}^*)(p_j - p_i). \tag{17}$$

Each agent therefore only requires information from its neighboring agents defined by the graph $\mathcal{G}$. This implementation, however, assumes bidirectional sensing as each edge contributes to the control of both incident agents.

We remove this reciprocity by orienting the edges of $\mathcal{G}$. Given an orientation $\overrightarrow{\mathcal{G}}$, we assign each distance constraint to its tail vertex and consider

$$u_i = \sum_{\overrightarrow{ij}\in\mathcal{E}} (\|p_i - p_j\|^2 - m_{ij}^*)(p_j - p_i), \tag{18}$$

where $\overrightarrow{ij}$ is the directed edge from node i to j. This model was used, for example, in [11]. The aggregate dynamics of (18), which is no longer a gradient flow, becomes

$$\dot{p} = -\overrightarrow{R}(p)^\top (R(p)p - m^*). \tag{19}$$

The matrix $\overrightarrow{R}(p)$ is obtained from $R(p)$ by zeroing out, in the row corresponding to ij, the entries corresponding to the node j for the orientation $\overrightarrow{ij}$, and the entries corresponding to i for the orientation $\overleftarrow{ij}$. We henceforth define (19) to be the *(d-dimensional) directed controller*.

The directed controller (19) naturally fits into the compatible controller framework introduced in Definition II.3. Indeed, from (5), we identify

$$\nu_{(p,m)} = \overrightarrow{R}(p)^\top.$$

Since $\dot{m} = 2R(p)\dot{p}$, the induced edge dynamics are

$$\dot{m} = 2R(p)\overrightarrow{R}(p)^\top (m^* - m).$$

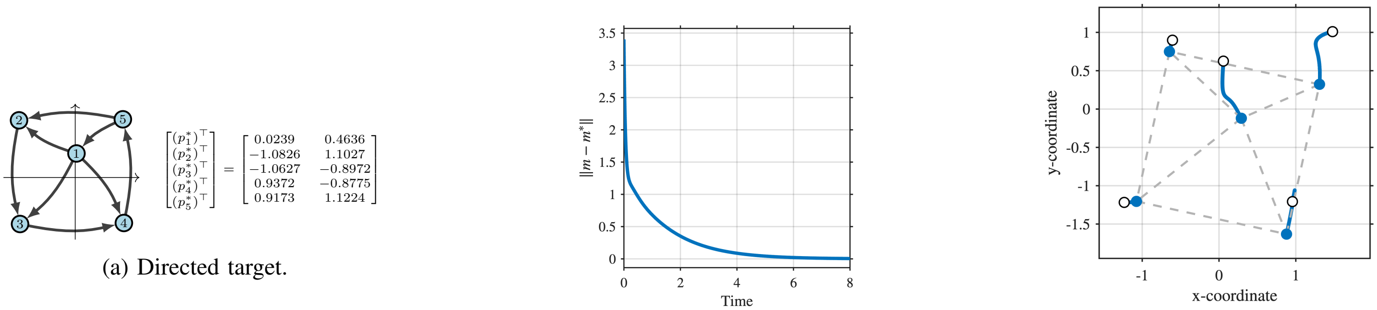
Thus the corresponding edge operator is

$$\eta_{(p,m)} = 2R(p)\overrightarrow{R}(p)^\top,$$

showing that the directed controller is compatible. In this section, it will be useful to have notation for the matrix of the transformations $\eta_{(p,m)}$ and $\eta_{(p,m)}|_{T_m\mathcal{Q}}$. To this end, we define

$$Z = 2R(p)\overrightarrow{R}(p)^\top \text{ and } A = P^\top Z P,$$

where P has orthonormal columns spanning the column space of $R(p)$. These are the matrices of $\eta_{(p,m)}$ in the standard basis and $\eta_{(p,m)}|_{T_m\mathcal{Q}}$ in an orthonormal basis. The latter assertion uses that $(p, m) \in \mathcal{S}$ implies that $\operatorname{im} R(p) = T_m\mathcal{Q}$.



(a) Directed target.

(b) Edge-error norm.

(c) Node trajectories (open circles indicate the initial positions).

Fig. 2: Directed wheel (Example IV.1): $-A$ is Hurwitz but $\frac{1}{2}(A+A^\top)$ is indefinite. The edge error converges to zero and the formation converges to a congruent realization.

A. Local Stability of Directed Compatible Controllers

For the directed controller (19), convergence depends on the target (p^*, m^*). Moreover, there are targets where the certificate of convergence (a) from Theorem II.7 holds but (b) does not. In this section, we illustrate both phenomena with numerical examples.

Example IV.1 (Hurwitz stability without the quadratic certificate)**.** *This example shows that (a) of Theorem II.7 does not imply (b). Consider the directed wheel and target framework shown in Figure 2a. A direct computation gives* $\min \operatorname{Re}\lambda(A) = 0.7221 > 0$*, and* $\min_{\|v\|=1} q(v) = -0.1623$*. Thus* $-A$ *is Hurwitz, while the quadratic form* q *is indefinite. Nevertheless, the edge error converges to zero (Figure 2b), and the corresponding node trajectories converge to a realization congruent to the target (Figure 2c). Hence the quadratic-form condition is sufficient but not necessary for local exponential stability.* $\diamondsuit$

Example IV.2 (Failure of both stability tests)**.** *Consider the directed wheel and target framework shown in Figure 3a. A direct computation gives* $\min \operatorname{Re}\lambda(A) = -0.0598 < 0$*, and* $min_{\|v\|=1} q(v) = -0.2012$*. Thus* $-A$ *is not Hurwitz and the quadratic form* q *is indefinite. The edge error converges to a nonzero value (Figure 3b), and the corresponding node trajectories converge to a configuration that is not congruent to the target (Figure 3c). Together with Example IV.1, this example shows that local stability can depend on the target geometry even when the directed orientation is fixed.* $\diamondsuit$

Theorem II.7 therefore gives a complete stability test for a fixed orientation and target geometry. To obtain information about the orientation independently of a particular target, we now return to the generic admissibility notions introduced in Section III.

B. Admissibility for Directed Orientations

For the directed controller, generic admissibility defines a property of the orientation $\overrightarrow{\mathcal{G}}$. In this case, we refer to $\overrightarrow{\mathcal{G}}$ as a *(weakly/strongly) admissible orientation* of $\mathcal{G}$. Unlike the symmetric controllers of Section II, the three notions of Definition III.1 need not coincide for the directed controllers, which we show with the following example.

Example IV.3 (Weak admissibility does not imply admissibility)**.** *Consider the directed graph in Figure 4. A direct computation gives* $\operatorname{rk} Z = 9 = 2n-3$*. Since* 9 *is the maximal possible rank and the entries of* Z *depend polynomially on the target configuration, it follows that* $\operatorname{rk} Z = 9$ *for every generic target. Hence the controller is weakly admissible. To check admissibility, we compute the eigenvalues of the reduced matrix* $P^\top Z P$ *as* $\{29.6147 \pm 2.8990i,\ 22.1401,\ 14.8599,\ 10.0000,\ 7.7485 \pm 0.1488i,\ 3.2737,\ 0\}$*. The zero eigenvalue shows that the reduced matrix is not invertible. Therefore, the controller is weakly admissible but not admissible.* $\diamondsuit$

We conclude this section by showing that every d-rigid graph has an admissible orientation.

Theorem IV.4 (Admissible orientations exist)**.** *Let* $\mathcal{G}$ *be a generically* d*-rigid graph. Then there is an admissible orientation* $\overrightarrow{\mathcal{G}}$ *of* $\mathcal{G}$*.* $\triangle$

In the proof, we need a simple vanishing result for multiaffine polynomials.

Lemma IV.5. *Let* $A(\mathbf{t})$ *be a multiaffine polynomial*[1] *in variables* $t_1, \ldots, t_m$*. Then* A *vanishes identically if and only if it vanishes on all zero/one vectors* $\mathbf{x} \in \{0,1\}^m$*.*

The proof of the Lemma is given in Appendix G. We also need a specialized sufficient condition for a matrix with polynomial entries to have a multiaffine determinant.

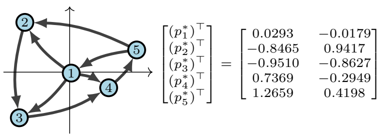
Lemma IV.6. *Let* $A(\mathbf{t}) \in \mathbb{R}[\mathbf{t}]^{m\times m}$ *and* $B \in \mathbb{R}^{m\times p}$ *be matrices with* $p \leq m$*.*[2] *If every* $p \times p$ *minor of* $A(\mathbf{t})$ *is multiaffine then* $\det B^\top A(\mathbf{t}) B$ *is multiaffine.*

The proof of the Lemma is given in Appendix H.

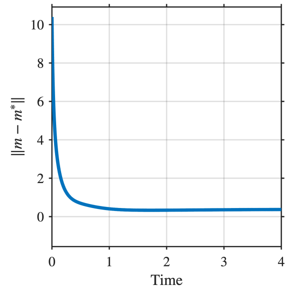
Proof of Theorem IV.4. Let $\mathcal{G}$ be generically d-rigid, and fix a generic configuration p^*. Let P be a matrix with orthonormal columns spanning the column space of R.

[1] A polynomial in $\mathbf{t} = (t_1, \ldots, t_m)$ is *multiaffine* if it has degree at most one in each variable individually.

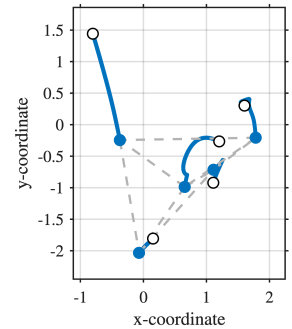
[2] Here, $\mathbb{R}[\mathbf{t}]$, with $\mathbf{t} = (t_1, \ldots, t_k)$, denotes the set of polynomials in $t_1, \ldots, t_k$ with real coefficients. Thus $A(\mathbf{t})$ is a matrix whose entries are polynomials in $\mathbf{t}$.



(a) Directed target.



(b) Edge-error norm.



(c) Node trajectories (open circles indicate the initial positions).

Fig. 3: Directed wheel (Example IV.2): $-A$ is not Hurwitz. The edge error settles at a nonzero value and the formation converges to a non-congruent configuration.

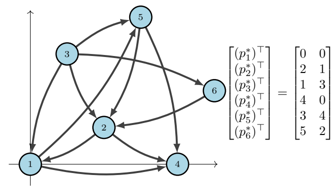


Fig. 4: A directed formation that is weakly admissible but not admissible.

For each edge $ij \in \mathcal{E}$, define r_{ij} to be the row of the rigidity matrix corresponding to the edge ij and $\overrightarrow{r_{ij}}$ the vector obtained by zeroing out the entries corresponding to j in $2r_{ij}$; define $\overleftarrow{r_{ij}}$ similarly so that $r_{ij} = \frac{1}{2}\overrightarrow{r_{ij}} + \frac{1}{2}\overleftarrow{r_{ij}}$ (which is the reason we scaled the directed rows in this proof). Denote by $\overrightarrow{R}$ the matrix with rows $\overrightarrow{r_{ij}}$ and $\overleftarrow{R}$ the matrix with rows $\overleftarrow{r_{ij}}$, ordered compatibly, so that $R = \frac{1}{2}\overrightarrow{R} + \frac{1}{2}\overleftarrow{R}$.

Now we define a matrix $Z(\mathbf{t}) = R[T\overrightarrow{R} + (I - T)\overleftarrow{R}]^\top$, where T is an $|\mathcal{E}|\times|\mathcal{E}|$ diagonal matrix with variables t_{ij} on the diagonal, and then $A(\mathbf{t}) = P^\top Z(\mathbf{t})P$. For each $\mathbf{x} \in \{0,1\}^{|\mathcal{E}|}$, $Z(\mathbf{x})$ corresponds to the orientation $\overrightarrow{\mathcal{G}}_{\mathbf{x}}$ obtained by orienting ij as $\overrightarrow{ij}$ if $x_{ij} = 1$ and as $\overleftarrow{ij}$ if $x_{ij} = 0$. Thus, it suffices to find $\mathbf{x} \in \{0,1\}^{|\mathcal{E}|}$ such that $\det A(\mathbf{x}) \neq 0$, since this implies that $\eta^*|_{T_{m^*}\mathcal{Q}}$ is invertible for $\overrightarrow{\mathcal{G}}_{\mathbf{x}}$.

We first show that $\det A(\mathbf{t})$ is multi-affine. This follows from Lemma IV.6 once we observe that each variable t_{ij} appears in at most one column of $Z(\mathbf{t})$ and that P is a fixed constant. Because $A(\frac{1}{2}\mathbf{1}) = P^\top RR^\top P$ is positive definite, $\det A(\frac{1}{2}\mathbf{1}) > 0$. Lemma IV.5 then implies that there is some $\mathbf{x} \in \{0,1\}^{\mathcal{E}}$ such that $\det A(\mathbf{x}) \neq 0$. This shows that the directed orientation $\overrightarrow{\mathcal{G}}_{\mathbf{x}}$ is admissible at the target p^*. That $\overrightarrow{G}_{\mathbf{x}}$ is generically admissible now follows from Theorem III.3. $\square$

C. Acyclic orientations and the vertex flag

Acyclic orientations play an important role in the literature on directed formation control. The edge-based perspective in this paper allows us to give a simple explanation of why this is so and unify much of what is known.

Theorem IV.7 (Stability of acyclic controllers)**.** *If $\overrightarrow{\mathcal{G}}$ is acyclic, then the following are equivalent:*

(a) $\overrightarrow{\mathcal{G}}$ is weakly admissible;
(b) $\overrightarrow{\mathcal{G}}$ is admissible;
(c) $\overrightarrow{\mathcal{G}}$ is strongly admissible;
(d) the map $-\eta^*|_{T_{m^*}\mathcal{Q}}$ is Hurwitz at every generic target (p^*, m^*).

$\triangle$

The proof is an application of Theorem III.8 to a flag naturally associated with the orientation that we now describe. Suppose that $\overrightarrow{\mathcal{G}}$ is an acyclic directed graph, and that the vertices are topologically ordered so that the edge ij has orientation $\overrightarrow{ij}$ if and only if $i < j$; such an ordering exists if and only if $\overrightarrow{G}$ is acyclic. We define the following subspaces of $\mathbb{R}^{|\mathcal{E}|}$: $V_0 = \{0\}$, and, for $1 \leq j \leq |\mathcal{V}|$, V_j is the coordinate subspace spanned by coordinates corresponding to edges $\overrightarrow{ij}$ with their tail in the initial segment $1 \leq i \leq j$. This implies that $V_{|\mathcal{V}|} = \mathbb{R}^{|\mathcal{E}|}$. Plainly the V_i form a flag of subspaces which we call the *vertex flag* of $\overrightarrow{\mathcal{G}}$.

Lemma IV.8. *Let $\overrightarrow{\mathcal{G}}$ be an acyclic directed graph and let V_j be the vertex flag. Then, for all targets $p^* \in \mathcal{C}$ of the associated directed controller with $Z = 2R(p^*)\overrightarrow{R}(p^*)^\top$,*

(a) each V_j is Z-invariant;
(b) the induced maps $Z_j : W_j \to W_j$ along the quotients $W_j = V_j/V_{j-1}$ are PSD and satisfy (16)*;*
(c) the vertex flag satisfies the decoupling condition.

Proof. We require some additional definitions. Define X_j to be the coordinate subspace of $\mathbb{R}^{|\mathcal{E}|}$ corresponding to the edges out of vertex j. Then $V_j = V_{j-1} \oplus X_j$. Let $\pi_j : V_j \to W_j := V_j/V_{j-1}$ be the quotient map. Since $\pi_j(X_j) = W_j$ and $X_j \cap \ker \pi_j = X_j \cap V_{j-1} = \{0\}$, the restriction $\pi_j|_{X_j} : X_j \to W_j$ is injective. Since $\dim X_j = \dim V_j - \dim V_{j-1} = \dim W_j$, $\pi_j|_{X_j}$ is a linear isomorphism. We also define U_j to be the coordinate subspace of $\mathbb{R}^{dn}$ corresponding to the vertices i such that $1 \leq i \leq j$. Finally, we define Q_j to be the matrix whose rows are the edge vectors emanating from vertex j.

We first prove (b). Note that on X_j, $\overrightarrow{R}(p)^\top$ acts like $Q_j^\top$. On U_j, $R(p)$ acts like Q_j. From the previous observation, $Z_j = 2Q_jQ_j^\top$, which is PSD. In particular, Z_j satisfies (16).

Now we prove (a). Since V_j is supported on edges whose tails lie in $\{1, \dots, j\}$, and each row of $\overrightarrow{R}(p)$ is supported only on the coordinates of its tail vertex, we have $\overrightarrow{R}(p)^\top V_j \subseteq U_j$. Conversely, if $u \in U_j$, then $(R(p)u)_{kl}$ can be nonzero only if at least one of k or l is at most j. Since the vertices are topologically ordered, the tail of such an edge is the endpoint with smaller index, and hence is also at most j. Thus $R(p)U_j \subseteq V_j$. It follows that $ZV_j = 2R(p)\overrightarrow{R}(p)^\top V_j \subseteq V_j$, so V_j is Z-invariant.

We finish with the decoupling condition (c) using criterion (b) from Lemma III.7. Let $v \in V_j$ and suppose that $Zv \in V_{j-1}$. Using the splitting of V_j, we write $v = w + x$, with $w \in V_{j-1}$ and $x \in X_j$. We have $0 = \pi_j(Zv) = \pi_j(Zw) + \pi_j(Zx)$. By invariance, $Zw \in V_{j-1}$, so $\pi_j(Zw)$ vanishes. This leaves $0 = \pi_j(Zx) = Z_j\pi_j(x)$. Since the kernel of the PSD Z_j is equal to $\ker Q_j^\top$, $\pi_j(x) \in \ker Q_j^\top$. On X_j, $\overrightarrow{R}(p)^\top$ acts like $Q_j^\top$, and so $\overrightarrow{R}(p)^\top x = 0$. Hence $Zx = 0$, and so $Zv = Zw$. This verifies the decoupling condition, completing the proof. □

Proof of Theorem IV.7. Let p^* be a generic target, and assume that the directed controller from $\overrightarrow{\mathcal{G}}$ is weakly admissible. Because it is a constant, the vertex flag depends rationally on the node state. The remaining hypotheses of Theorem III.8 are supplied by Lemma IV.8, since they hold pointwise, and so at p^*, in particular. Because the spectrum of η^* is the union of the spectra of the PSD Z_j by Lemma III.9, the non-zero eigenvalues of η^* have positive real parts. Hence, $-\eta^*|_{T_{m^*}\mathcal{Q}}$ is Hurwitz by Theorem III.8. Since p^* was arbitrary, (a) implies (d) for every generic p^*. The implications from bottom to top are immediate for any generic p^*, so the proof is complete. □

We can apply the same machinery to other invariant flags associated with a directed orientation. The following construction generalizes a related approach from [11].

Definition IV.9 (Root extension)**.** *Let $\overrightarrow{\mathcal{G}}$ be a directed graph and let $\overrightarrow{\mathcal{H}} \subseteq \overrightarrow{\mathcal{G}}$ be a vertex-induced subgraph. We say that $\overrightarrow{\mathcal{G}}$ is a* root extension *of $\overrightarrow{\mathcal{H}}$ if*

(a) $\overrightarrow{\mathcal{G}} \setminus \overrightarrow{\mathcal{H}}$ is acyclic; and

(b) there are no directed edges from a vertex of $\overrightarrow{\mathcal{H}}$ to a vertex of $\overrightarrow{\mathcal{G}} \setminus \overrightarrow{\mathcal{H}}$.

If, in addition, $\overrightarrow{\mathcal{H}}$ is admissible and every vertex of $\overrightarrow{\mathcal{G}} \setminus \overrightarrow{\mathcal{H}}$ has out-degree at least d, then $\overrightarrow{\mathcal{G}}$ is a stable root extension *of $\overrightarrow{\mathcal{H}}$.*

The construction is illustrated in Fig. 5.

The definition of a root extension implies that there is an ordering $v_1, \dots, v_n, w_1, \dots, w_\ell$ of the vertices of $\overrightarrow{\mathcal{G}}$, where the v_i correspond to vertices of $\overrightarrow{\mathcal{G}} \setminus \overrightarrow{\mathcal{H}}$ and the w_j correspond to vertices of $\overrightarrow{\mathcal{H}}$, such that for each edge $\overrightarrow{v_iv_j}$, $i < j$ and there are no directed edges $\overrightarrow{w_iv_j}$. Using such an order, which extends a topological order on the vertices of $\overrightarrow{\mathcal{G}'}$, we define a flag V_j of subspaces of $\mathbb{R}^{|\mathcal{E}|}$ by defining V_j to be the coordinate subspace corresponding to edges with their tail in $\{v_1, \dots, v_j\}$

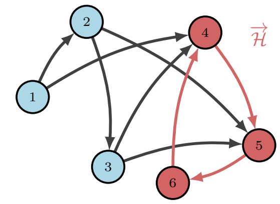


Fig. 5: A stable root extension by the admissible directed subgraph $\overrightarrow{\mathcal{H}}$ (red). The subgraph $\overrightarrow{\mathcal{G}} \setminus \overrightarrow{\mathcal{H}}$ is acyclic, all edges between the two subgraphs are directed toward $\overrightarrow{\mathcal{H}}$, and the vertices outside of $\overrightarrow{\mathcal{G}} \setminus \overrightarrow{\mathcal{H}}$ have out-degree at least $d = 2$.

for $0 \le j \le n$ and $V_{n+1} = \mathbb{R}^{|\mathcal{E}|}$. We call this flag V_j the *stable extension flag*.

Theorem IV.10 (Stability of root extensions)**.** *Let $\overrightarrow{\mathcal{G}}$ be a k vertex stable root extension of an ℓ vertex admissible directed graph $\overrightarrow{\mathcal{H}}$. Then $\overrightarrow{\mathcal{G}}$ is admissible. If for some generic target p^*, the non-zero eigenvalues of $Z_{\overrightarrow{\mathcal{H}}}$ have positive real parts, where $Z_{\overrightarrow{\mathcal{H}}}$ is the matrix of the edge operator at the restriction of p^* to $\overrightarrow{\mathcal{H}}$, then $\overrightarrow{\mathcal{G}}$ is strongly admissible.* △

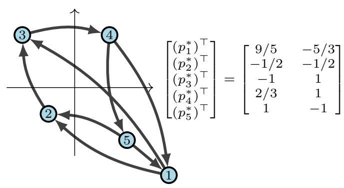
Proof. Let $D = \binom{d+1}{2}$. Let $n = k - \ell$, and order the vertices of $\overrightarrow{\mathcal{G}}$ as $v_1, \dots, v_n, w_1, \dots, w_\ell$ with the conditions as discussed above. With an eye towards applying Theorem III.8, we first observe that, as a constant flag, the stable extension flag depends rationally on the state. It is also a flag of Z invariant subspaces, because it agrees with the vertex flag up to V_n, so the proof of invariance applies verbatim, except for V_{n+1}, where invariance is trivial.

Next we look at the induced maps Z_j. For $1 \le j \le n$, $Z_j = 2Q_jQ_j^\top$, as in the proof of Lemma IV.8, with the same argument. Since every vertex outside $\overrightarrow{\mathcal{H}}$ has out-degree at least d, generically, $\operatorname{rk} Z_j = d$ for $1 \le j \le n$. We also get $Z_{n+1} = Z_{\overrightarrow{\mathcal{H}}}$, since, by construction, $\overrightarrow{\mathcal{H}}$ has no out edges. By admissibility and Theorem III.6, $Z_{\overrightarrow{\mathcal{H}}}$ satisfies (16).

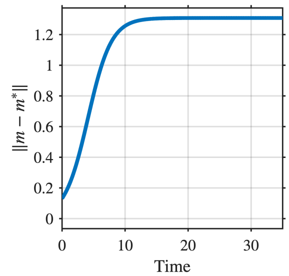
From the general rank inequality $\operatorname{rk} Z \ge \operatorname{rk} Z_1 + \cdots + \operatorname{rk} Z_{n+1} = dn + \operatorname{rk} Z_{\overrightarrow{\mathcal{H}}}$. Since $\overrightarrow{\mathcal{H}}$ has $k - n$ vertices and $\overrightarrow{\mathcal{H}}$ is admissible, generically, $\operatorname{rk} Z \ge dk - D$. Since $\operatorname{im} Z \subseteq T_{m^*}\mathcal{Q}$, which has dimension at most $dk - D$, we conclude both that $\overrightarrow{\mathcal{G}}$ is d-rigid and that equality holds. This shows that $\overrightarrow{\mathcal{G}}$ is weakly admissible, and, by Lemma III.7 (d) that the decoupling condition holds.

At this point, we have established all the hypotheses of Theorem III.8, so we conclude that $\overrightarrow{\mathcal{G}}$ is admissible. If, at some generic target the non-zero eigenvalues of $Z_{\overrightarrow{\mathcal{H}}}$ have positive real parts, the same is true for Z because the other Z_j are PSD, using Lemma III.9. The strong admissibility conclusion then follows. □

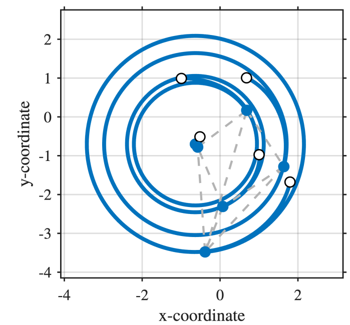
As a last application of vertex flags, we prove a combinatorial theorem about weakly admissible acyclic orientations. We define the *defect* of vertex i as $\delta_i := \max\{0, d - \overrightarrow{\deg}(i)\}$, and the *defect of an acyclic graph* to be the sum of its vertex defects.



(a) Persistent directed target.



(b) Edge-error norm.



(c) Node trajectories (open circles indicate the initial positions).

Fig. 6: A persistent directed formation for which the directed controller is locally unstable. The orientation in (a) is persistent. The edge error (b) converges to a nonzero value, while the node trajectories (c) approach a limit cycle.

Theorem IV.11 (Admissible acyclic orientations characterization)**.** *Let $\overrightarrow{\mathcal{G}}$ be an acyclic orientation of an $n \geq d$ vertex d-rigid graph $\mathcal{G}$. Then the following are equivalent:*

(a) $\overrightarrow{\mathcal{G}}$ is admissible;
(b) the defect of $\overrightarrow{\mathcal{G}}$ is exactly $\binom{d+1}{2}$;
(c) $\overrightarrow{\mathcal{G}}$ is formed from a directed complete d vertex subgraph $\overrightarrow{\mathcal{H}}$ by a sequence of stable root extensions.

$\triangle$

Proof. For convenience, set $D = \binom{d+1}{2}$. Fix any topological ordering $v_1, \ldots, v_n$ of the vertices of $\overrightarrow{\mathcal{G}}$, and let $V_1, \ldots, V_n$ be the associated vertex flag. In a topological ordering, the out-neighbors of any vertex appear after it. In particular, each vertex v_{n-k} for $0 \leq k \leq d-1$ has at most k out neighbors. Summing over k, the total vertex defect in any acyclic orientation is at least D. Moreover, defect D is attained if and only if each of these v_{n-k} for $0 \leq k \leq d-1$ have out-degree exactly k and all other defects are zero. From this it follows that conditions (b) and (c) are equivalent.

Meanwhile, by Lemma IV.8 the vertex flag satisfies the decoupling condition. Hence, from Lemma III.7 (d), generically $\operatorname{rk} Z = \operatorname{rk} Z_1 + \cdots + \operatorname{rk} Z_n$. In the notation of the proof of Lemma IV.8, $Z_j = 2Q_jQ_j^\top$, which generically has rank $d - \delta_i$. Since each Z_j is positive semidefinite, condition (16) holds for each induced map. Hence, $\overrightarrow{\mathcal{G}}$ is weakly admissible, and hence, from Theorem III.8 admissible, if and only if the total defect is D. $\square$

Theorems IV.7 and IV.11 together provide an explanation for the "leader first-follower" directed control architecture appearing in works such as [8] and [10]. Theorem IV.11 shows that the defect is exactly D if and only if the directed graph has a (generalized) leader first-follower structure. We note that, in dimensions $d \geq 2$ there are generically rigid bipartite graphs. By Theorem IV.11, these cannot have acyclic admissible orientations, so all the orientations supplied by Theorem IV.4 must contain directed cycles.

D. Comparison to persistence

We first compare admissibility with the classical notion of persistence [6]. For acyclic orientations, the two notions coincide.

Corollary IV.12 (Acyclic persistent orientations)**.** *An acyclic directed graph is admissible if and only if it is persistent.*

Proof. Suppose that $\mathcal{G}$ is admissible and acyclic. By Theorem IV.11, $\mathcal{G}$ contains a K_d subgraph carrying the total defect D and every other vertex has degree at least d. Removing edges with their tail at a vertex of degree at least $d+1$ preserves this property, so removing edges until all degrees are at most d preserves admissibility (using Theorem IV.11) and, hence, rigidity of the underlying graph. This shows that $\mathcal{G}$ is persistent.

Now suppose that $\mathcal{G}$ is persistent. Remove edges until every vertex has degree at most d. The graph that is left is minimally rigid, so, by counts, it has defect exactly D. In an acyclic orientation, this is the minimal defect, so the defect of $\mathcal{G}$ is exactly D. By Theorem IV.11, $\mathcal{G}$ is admissible. $\square$

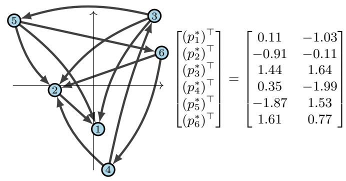
For $d = 2$, Corollary IV.12, together with Theorem IV.11, recovers the classical characterization of persistent acyclic graphs: there is a leader of out-degree zero, a first follower of out-degree one, and every remaining vertex has out-degree at least two [6, Thm. 5].

The numerical characterization developed thus far provides a target-dependent test for local convergence that is independent of the notion of persistence [6]. Persistence is a structural property of a directed formation, as it characterizes whether the directed distance constraints are compatible with maintaining a rigid shape. It does not, however, characterize the stability of a particular feedback law. Indeed, previous stability results for persistent formations have required either controller design or additional restrictions on the directed architecture [10], [11].

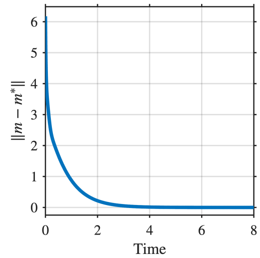
Theorem IV.13. *Persistence is neither necessary nor sufficient for local exponential stability of the directed controller* (19). $\triangle$

The proof is given through the construction of two counter examples given below.

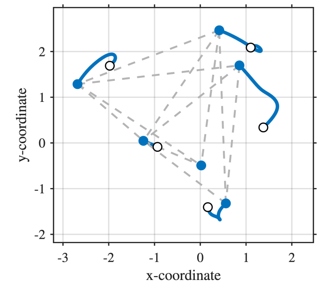
Example IV.14 (Persistence is not sufficient)**.** *Consider the persistent directed wheel and target framework shown in Figure 6a. The underlying undirected wheel graph is generically rigid in dimension 2, and the out-degrees are* $(2, 1, 1, 2, 2)$,



(a) Directed target.



(b) Edge-error norm.



(c) Node trajectories (open circles indicate the initial positions).

Fig. 7: Non-persistent orientation of Example IV.15. The edge error converges to zero and the node trajectories converge to a congruent realization.

so the orientation is persistent. For this target, computation gives $\min \operatorname{Re} \lambda(A) \approx -0.522 < 0$. *Hence* $-A$ *is not Hurwitz, and by Theorem II.7 the target is not locally exponentially stable under the directed controller. Figures 6b-6c illustrate the resulting local instability using an initial perturbation along an unstable mode of the linearization.* $\Diamond$

We note that the orientation in Figure 6a is strongly admissible (certified by the given configuration), and that, for an open set of targets (not shown), $-A$ is Hurwitz. The second example shows that local exponential stability at a generic target does not imply persistence.

Example IV.15 (Persistence is not necessary)**.** *Consider the non-persistent directed graph and target framework shown in Figure 7a. The underlying undirected graph is rigid, but the orientation is not persistent. Indeed, vertices* 3 *and* 5 *both have out-degree three, while the graph has* $10 = 2n - 2$ *edges. Any minimally persistent spanning subgraph would therefore be obtained by removing a single edge and, in the plane, would require every vertex to have out-degree at most two [6]. Since removing one edge can reduce the out-degree of at most one of vertices* 3 *and* 5*, no minimally persistent spanning subgraph exists. Thus the directed graph is not constraint consistent, and hence is not persistent [6].*

Although the orientation fails the persistence criterion, the directed controller converges locally to the desired edge measurements, as shown in Figure 7b; one can also check computationally that $-A$ *is Hurwitz. Consequently, the controller is locally exponentially stable at the target (Figure 7c), in accordance with Theorem II.5. Thus, persistence is not necessary for local exponential stability of the directed distance-based controller.* $\Diamond$

V. Edge Weight Design via SDP

We have seen up to now that for directed sensing architectures, local stability of a compatible controller can depend on both the target geometry and the chosen orientation of the graph. The stability certificates of Theorem II.7, on the other hand, suggest that for a fixed target and graph orientation, it may still be possible to shape the dynamics to recover desirable properties. A natural approach to accomplish this is to introduce a scaled feedback law where each directed measurement is assigned an independent gain.

In this direction, we begin with the compatible directed formation controller presented in (18). We now associate a scalar gain $w_{\overrightarrow{ij}}$ with each directed edge $\overrightarrow{ij}$. The corresponding weighted controller is

$$u_i = \sum_{\overrightarrow{ij} \in \mathcal{E}} w_{\overrightarrow{ij}} (\|p_i - p_j\|^2 - m_{ij}^*)(p_j - p_i).$$

Thus, the weighting changes only the magnitude of each existing feedback term while the sensing architecture and the information available to each agent remain unchanged.

Let $W = \operatorname{diag}(w) \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ collect the edge weights in the ordering used for the rows of $\overrightarrow{R}(p)$. The weighted controller can then be written in aggregate form as

$$\dot{p} = -\overrightarrow{R}(p)^\top W (R(p)p - m^*).$$

This defines a compatible formation controller in the sense of Definition II.3. Indeed, setting

$$\nu(p, m) = \overrightarrow{R}(p)^\top W, \text{ and } \eta(p, m) = 2R(p)\overrightarrow{R}(p)^\top W,$$

gives $\eta(p, m) = 2R(p)\nu(p, m)$. The corresponding edge dynamics are therefore

$$\dot{m} = 2R(p)\overrightarrow{R}(p)^\top W(m^* - m).$$

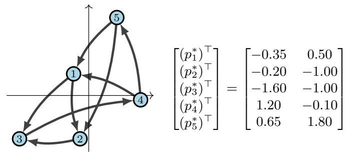
For a fixed target (p^*, m^*), let $Z := 2R(p^*)\overrightarrow{R}(p^*)^\top$, as in Section IV. If the columns of P form an orthonormal basis for $T_{m^*}\mathcal{Q}$, then the linearization of the weighted edge dynamics is

$$\dot{\delta m} = -A_W \delta m, \text{ with } A_W := P^\top Z W P.$$

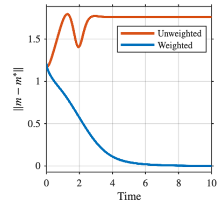
By Theorem II.7, the weighted controller is locally exponentially stable if $-A_W$ is Hurwitz. A sufficient condition is $\frac{1}{2}(A_W + A_W^\top) \succ 0$. Since $\frac{1}{2}(A_W + A_W^\top) = \frac{1}{2}P^\top \left(ZW + WZ^\top\right) P$, this condition is affine in the edge weights.

Since the matrix inequality is affine in the optimization variables w, the design problem can be formulated as the semidefinite program

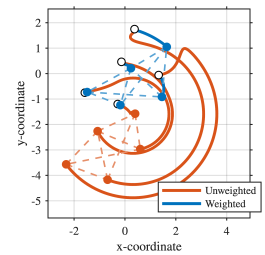
$$\begin{aligned} \max_{w,\gamma} \quad & \gamma \\ \text{subject to} \quad & P^\top \left(ZW + WZ^\top\right) P \succeq \gamma I, \\ & \mathbf{1}^\top w = |\mathcal{E}|, \ W = \operatorname{diag}(w) \succeq 0. \end{aligned} \tag{20}$$



(a) Directed target.



(b) Edge-error norm.



(c) Node trajectories .

Fig. 8: Edge-weight synthesis (Example V.1). The unweighted controller (red) is locally unstable; the SDP-designed controller (blue) satisfies the quadratic certificate and converges.

We restrict attention here to nonnegative edge gains and impose $\mathbf{1}^\top w = |\mathcal{E}|$ to remove the arbitrary scaling of the controller gains. The nonnegativity constraint is not required by the stability condition itself. If signed edge weights are allowed, a sign-independent normalization, such as $\|w\|_1 \le |\mathcal{E}|$, may be used instead.

The optimal value $\gamma^\star$ gives the largest quadratic stability margin under this gain normalization. In particular, if $\gamma^\star > 0$, then the synthesized edge weights satisfy the sufficient condition of Theorem II.7 (b), and the corresponding directed controller locally exponentially stabilizes the target edge state.

We emphasize that infeasibility of (20) does not preclude local exponential stability. Rather, it indicates only that the stronger Lyapunov certificate of Theorem II.7 (b) cannot be established through any choice of *diagonal* edge weights. The controller may nevertheless satisfy the Hurwitz condition of Theorem II.7 (a). In this sense, (20) should be viewed as a constructive test for the stronger quadratic Lyapunov certificate, complementing the existence results of [10] for stabilizing diagonal weight matrices. We conclude this section with an example.

Example V.1 (Stabilization by edge-weight design)**.** *Consider the directed target framework shown in Figure 8a. With unit edge weights, the reduced operator* $A = P^\top Z P$ *satisfies* $\min \operatorname{Re} \lambda(A) = -0.1156 < 0$ *and* $\lambda_{\min}\big(\tfrac{1}{2}(A + A^\top)\big) = -0.9434$*, so the unweighted directed controller is locally unstable and does not satisfy the quadratic certificate.*

We therefore solve the semidefinite program (20)*. The optimal value is* $\gamma^\star = 0.4926 > 0$*, with edge weights*

$$w^\star \approx \begin{bmatrix}0.244 & 0.110 & 1.058 & 2.665 & 0.388 & 0.245 & 0.195 & 3.096\end{bmatrix}^\top .$$

For the resulting weighted operator $A_{W^\star} = P^\top Z W^\star P$*, we obtain* $\min \operatorname{Re} \lambda(A_{W^\star}) = 0.6993 > 0$ *and* $\lambda_{\min}\big(\tfrac{1}{2}(A_{W^\star} + A_{W^\star}^\top)\big) = 0.4926 > 0$*. Hence the optimized weights satisfy the quadratic certificate of Theorem II.7 (b) and locally exponentially stabilize the target.*

Figure 8b compares the edge-error trajectories from the same initial condition, while Figure 8c shows the corresponding node trajectories. The unweighted controller moves away from the target and enters a limit cycle, whereas the weighted controller converges to a realization congruent to it. $\diamond$

VI. Further directions

Several questions remain open. The most immediate is to characterize when diagonal gains W exist for which $-P^\top Z W P$ is Hurwitz, since our examples show Hurwitz stability can hold even when the quadratic certificate $P^\top (ZW + WZ^\top)P \succ 0$ fails. It is also natural to ask whether such gains admit a graph-theoretic characterization, and for which graph classes the Hurwitz and quadratic notions of stabilizability coincide. The other question of dynamical interest is whether, for every strongly admissible compatible controller, there is a non-empty open set of target configurations p^* on which $-\eta^*|_{T_{m^*}\mathcal{Q}}$ is Hurwitz.

There are also several open questions concerning admissibility of directed controllers and directed orientations. The most immediate is a combinatorial characterization of admissible and strongly admissible orientations of d-rigid graphs. It would also be interesting to know for which directed graph classes all the admissibility conditions coincide.

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APPENDIX A. Proof of Lemma II.6

Let $x^* := (p^*, m^*)$. Since V is an open neighborhood of x^* in the smooth manifold $\mathcal{S}$, there is an $r > 0$ such that the closed ball $\overline{W} := \{(p,m) \in \mathcal{S} : \|(p,m) - x^*\| \leq 2r\}$ is compact and contained in V. Let W denote the open ball that is the interior of $\overline{W}$. Since ν is smooth, there exists $N > 0$ such that $\|\nu_{(p,m)}\| \leq N$, for $(p,m) \in \overline{W}$.

Now, choose $T_0 > 0$ large enough that $C\left(2 + \frac{N}{c}\right) e^{-cT_0} < 1$. We will use the continuous dependence of the trajectories on the finite interval $[0\ T_0]$, and then use the exponential estimate (6) to bound their total displacement for $t \geq T_0$. Since x^* is an equilibrium of (5), there exists $0 < \rho < r$ such that every solution initialized at $x^0 := (p^0, m^0) \in U := \{x \in \mathcal{S} : \|x - x^*\| < \rho\}$ satisfies $\|x(t) - x^*\| < r$, $0 \leq t \leq T_0$. In particular, every such trajectory remains in W on $[0\ T_0]$.

Fix an initial condition $x^0 \in U$, and define the escape time $T := \sup\{\tau \geq T_0 : x(t) \in W \text{ for all } t \in [T_0, \tau]\}$. The supremum is over a set containing T_0, so T is well-defined. Fix any τ with $T_0 \leq \tau < T$. Since the trajectory remains in $W \subseteq V$ for all $0 \leq t \leq \tau$, the estimate (6) holds throughout this interval.

For $T_0 \leq t \leq \tau$, the triangle inequality and (6) give

$$\begin{aligned}\|m(t) - m(T_0)\| &\leq \|m(t) - m^*\| + \|m(T_0) - m^*\| \\ &\leq C\left(e^{-ct} + e^{-cT_0}\right)\|m^0 - m^*\| \leq 2Ce^{-cT_0}\rho,\end{aligned}$$

where we used $t \geq T_0$ and $\|m^0 - m^*\| \leq \|x^0 - x^*\| < \rho$.

Next, compatibility and (6) give

$$\begin{aligned}\|\dot{p}(t)\| &= \|\nu_{(p(t),m(t))}(m^* - m(t))\| \leq N\|m^* - m(t)\| \\ &\leq NCe^{-ct}\|m^0 - m^*\| < NCe^{-ct}\rho,\end{aligned}$$

for $T_0 \leq t \leq \tau$. Integrating, we get

$$\|p(t) - p(T_0)\| \leq \int_{T_0}^{\infty} \|\dot{p}(s)\| \mathrm{d}s \leq (CN/c)e^{-cT_0}\rho.$$

Combining these bounds, we obtain

$$\begin{aligned}\|x(t) - x(T_0)\| &= \|(p(t), m(t)) - (p(T_0), m(T_0))\| \\ &\leq \|p(t) - p(T_0)\| + \|m(t) - m(T_0)\| \\ &\leq C\left(2 + \frac{N}{c}\right) e^{-cT_0}\rho < \rho,\end{aligned}$$

for $T_0 \leq t \leq \tau$, where the last inequality follows from the earlier choice of T_0. Since $\|x(T_0) - x^*\| < r$, the triangle inequality gives

$$\|x(t) - x^*\| \leq \|x(t) - x(T_0)\| + \|x(T_0) - x^*\| < \rho + r < 2r$$

for all $T_0 \leq t \leq \tau$. Thus the trajectory remains strictly inside W on $[T_0\ \ \tau]$.

Suppose, for contradiction, that $T < \infty$. Since the estimate above holds for every $\tau < T$, letting $\tau \to T$ from below and using continuity of the trajectory gives $\|x(T) - x^*\| \leq r + \rho < 2r$. Hence $x(T) \in W$. Because W is open and the vector field is smooth, the solution can be continued on some interval $[T\ \ T + \varepsilon)$ while remaining in W. This contradicts the definition of T as the escape time, and therefore $T = \infty$. Combined with the continuous-dependence bound on $[0\ \ T_0]$, every solution initialized in U remains in W for all $t \geq 0$. Thus $U \subseteq W \subseteq V$ satisfy the conclusion of the lemma. $\square$

APPENDIX B. Proof of Lemma III.4

The characteristic polynomial $c_{\alpha(\mathbf{t})}(x)$ has coefficients in $\mathbb{Q}(\mathbf{t})$. By Cayley–Hamilton, the transformation is invertible if and only if the constant term $\det \alpha(\mathbf{t})$ does not vanish. If it vanishes identically, then $\alpha(\mathbf{t})$ is generically non-invertible. Otherwise, $\det \alpha(\mathbf{t}) \neq 0$, so has the form $p(\mathbf{t})/q(\mathbf{t})$ with $p, q \in \mathbb{Q}[\mathbf{t}]$ both non-zero. Away from the set of $\mathbf{t}$ such that $p(\mathbf{t})$ or $q(\mathbf{t})$ vanish, $\alpha(\mathbf{t})$ is invertible, and, hence, $\alpha(\mathbf{t})$ is generically invertible. This is statement (a).

Statement (b) is similar, except we add to the exceptional set the numerator and product of the denominators in the resultant $\operatorname{Res}(c_{\alpha(\mathbf{t})}(x), c_{\alpha(\mathbf{t})}(-x); x)$, which vanishes exactly when $\alpha(\mathbf{t})$ has a pair of eigenvalues that sum to zero. When $\alpha(\mathbf{t}_0)$ is Hurwitz for some specialization $\mathbf{t}_0$, then the resultant is generically non-vanishing, since the sum of any pair of eigenvalues of $\alpha(\mathbf{t}_0)$ has negative real part. $\square$

APPENDIX C. Proof of Lemma III.5

Let $A \in \mathbb{R}(\mathbf{t})^{m\times m}$ be a matrix for the family $\alpha(\mathbf{t})$ in the standard basis, and suppose that the rank of A is r. By Gaussian elimination, there is a basis $\{v_1, \ldots, v_r, w_1, \ldots, w_{m-r}\}$ of $\mathbb{R}(\mathbf{t})^m$ so that the v_i span $\operatorname{im}\alpha(\mathbf{t})$. Hence, the projection $\pi(\mathbf{t})$ to $\operatorname{im}\alpha(\mathbf{t})$ is also rational in $\mathbf{t}$, as is the inclusion $\pi^*(\mathbf{t}) : \operatorname{im}\alpha(\mathbf{t}) \hookrightarrow \mathbb{R}(\mathbf{t})^m$. Since the image is an invariant subspace, the restriction is given by $\pi(\mathbf{t})\circ\alpha(\mathbf{t})\circ\pi^*(\mathbf{t})$, which, as a composition of rational maps is rational. $\square$.

APPENDIX D. Proof of Lemma III.7

That (a) implies (b) is immediate from the definitions. For (b) implies (a), let $v \in V_j$ be given and suppose that $v = \alpha(w)$. Among the pre-images w, choose one such that $w \in V_{j'}$ with j' minimal. If $j' > j$, (b) implies that there is a $w' \in V_{j'-1}$ such that $\alpha(w) = \alpha(w')$. This contradicts minimality of j', so (b) implies (a).

Now we address (c). From the intertwining relation $\pi_j \circ \alpha = \alpha_j \circ \pi_j$ and surjectivityy of $\pi_j : V_j \to W_j$, we have $\pi_j(\alpha(V_j)) = \operatorname{im}\alpha_j$. Since the kernel of $\pi_j \circ \alpha|_{V_j} = \alpha(V_j) \cap V_{j-1}$, rank-nullity gives $\operatorname{rk}\alpha|_{V_j} = \operatorname{rk}\alpha_j + \dim(\alpha(V_j)\cap V_{j-1})$. By invariance of V_{j-1}, $\alpha(V_{j-1}) \cap V_{j-1} \subseteq \alpha(V_j) \cap V_{j-1}$, and so we get the inequality $\operatorname{rk}\alpha|_{V_j} \geq \operatorname{rk}\alpha_j + \operatorname{rk}\alpha|_{V_{j-1}}$. The inequality is an equality if and only if $\alpha(V_{j-1}) \cap V_{j-1} = \alpha(V_j) \cap V_{j-1}$. This latter condition is (b).

Finally (d) is equivalent to (c) by telescoping: (d) can hold if and only if each of the inequalities from above $\operatorname{rk}\alpha|_{V_j} \geq \operatorname{rk}\alpha_j + \operatorname{rk}\alpha|_{V_{j-1}}$ is an equality. $\square$.

APPENDIX E. Proof of Lemma III.9

The spectral statement is standard and follows from the fact that the characteristic polynomial $c_\alpha(x) = c_{\alpha_1}(x)\cdots c_{\alpha_k}(x)$.

Statement (a) follows from the fact that Lemma III.7 (c) expresses the decoupling condition in terms of ranks of maps that are rational in $\mathbf{t}$ and their restrictions to rationally defined spaces. Hence, the decoupling condition is a generic property by Lemmas III.4 and III.5. The proof of (b) is similar, noting that (16) is equivalent to the statement that $\operatorname{rk}\alpha^2 = \operatorname{rk}\alpha$. $\square$

APPENDIX F. Proof of Lemma III.10

We first suppose that (16) does not hold for α; i.e., that there is a non-zero $v \in \operatorname{im}\alpha \cap \ker\alpha$. Let j be minimal such that $v \in V_j$; in particular $v \notin V_{j-1}$, so $\pi_j(v) \neq 0$. By the decoupling condition and Lemma III.7 (b), there is a $w \in V_j$ such that $\alpha(w) = v$. We now obtain, from the intertwining relation $\pi_j \circ \alpha = \alpha_j \circ \pi_j$ on V_j, that $\pi_j(v) = \pi_j(\alpha(w)) = \alpha_j(\pi_j(w))$, which shows that $\pi_j(v) \in \operatorname{im}\alpha_j$. We also have, because $v \in \ker\alpha$, $0 = \pi_j(\alpha(v)) = \alpha_j(\pi_j(v))$. Hence, $\pi_j(v) \in \operatorname{im}\alpha_j \cap \ker\alpha_j$ and is non-zero, so (16) fails for α_j.

Now suppose that for some j, (16) does hold for α. For convenience, set $M = \operatorname{im}\alpha$ and $M_j = M \cap V_j$. The decoupling condition says that $M_j = \alpha(V_j)$. Injectivity of α on M descends to M_j, and so $\alpha(M_j) \subseteq M_j$ is promoted to the equality $\alpha(M_j) = M_j$. This yields a slight strengthening of the decoupling condition: if $v \in M_j$, then there is a $u \in M_j$ (not just V_j) such that $\alpha(u) = v$.

Fix a j and let $w \in \operatorname{im}\alpha_j \cap \ker\alpha_j$ be given. By the intertwining relation $\operatorname{im}\alpha_j = \pi_j(\alpha(V_j)) = \pi_j(M_j)$, so there is a $v \in M_j$ such that $\pi_j(v) = w$. Since $w \in \ker\alpha_j$, intertwining gives $0 = \alpha_j(w) = \pi_j(\alpha(v))$. Hence, $\alpha(v) \in M_{j-1}$. From above, there is a $u \in M_{j-1}$ such that $\alpha(u) = \alpha(v)$. Injectivity of α on M_j implies that $u = v$, and so $0 = \pi_j(u) = \pi_j(v) = w$. This shows that (16) holds for α_j. $\square$

APPENDIX G. Proof of Lemma IV.5

If A vanishes identically, it does so on every $\mathbf{x} \in \{0,1\}^m$. For the other direction, the key observation is that, because A is multiaffine, the variable supports of its monomials are distinct. Suppose that $A \neq 0$, and let M be a monomial with support of minimum cardinality. Setting every variable in the support of M to be one and the rest of the variables to be zero produces an $\mathbf{x} \in \{0,1\}^m$ such that $A(\mathbf{x}) = M(\mathbf{x}) \neq 0$, as required. $\square$

APPENDIX H. Proof of Lemma IV.6

Using Cauchy–Binet twice, we compute

$$\begin{aligned}\det B^\top A(\mathbf{t})B &= \sum_S \det B^\top[-,S]\det(A(\mathbf{t})B)[S,-]\\ &= \sum_S\sum_T \det B[S,-]\det B[T,-]\det A(\mathbf{t})[S,T],\end{aligned}$$

where S and T are subsets of $[m]$ with cardinality p, and $A[S,T]$ denotes the submatrix with rows indexed by S and columns indexed by T, and $B[S,-]$ denotes the submatrix formed from the rows indexed by S. This exhibits the determinant on the l.h.s. as a linear (the minors of B are scalars) combination of $p\times p$ minors of $A(\mathbf{t})$, which are multiaffine by hypothesis. Multiaffine polynomials are closed under linear combinations, giving the desired statement. $\square$