A Characterization of All Linear Passivizing Input-Output Transformations of a Passive-Short System: The SISO Case

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***Abstract*—Passivity theory is one of the cornerstones of control theory providing a systematic way to study the stability of interconnected systems. It is well known that many systems are not passive, and must be passivized in order to be included in the framework of passivity theory. Input-output (loop) transformations are the most general tool for passivizing systems. In this letter, we propose a characterization of all possible input-output transformations that map a system with given shortage of passivity to a system with prescribed excess of passivity. We do so by using the connection between passivity theory and cones for SISO systems.**

***Index Terms*—Feedback passivation, passivity theory, nonlinear systems.**

I. Introduction

OVER the last few decades, many engineering systems have become much more complex, as networked systems and large-scale systems turned common, and "system-of-systems" evolved into a leading design methodology. To address the ever-growing complexity of systems, many researchers suggested various component-level tools that guarantee system-level properties. One important example of such a notion is passivity, which can be informally stated as "energy-based control" [1]. Passivity theory has proven to be a powerful tool across many application domains, including networked systems [2], [3] and cyber-physical systems [4].

In practice, however, many systems are not passive. Examples include systems with input/output delays (such as chemical processes), human operators, generators, and power networks [5], [6], [7]. The lack of passivity is often quantified using passivity indices. In order to use passivity-based design techniques, one needs to passivize the system under consideration. A behavioral model approach to the synthesis of passivizing transformations using quadratic forms was considered in [8]. Other common methods for passivation include gains, output-feedback, input-feedthrough, or a combination thereof [9], [10], [11].

More generally, a passivation method relying on an input-output (I/O) transformation was suggested in [12]. An I/O transformation is a concise formulation aggregating output-feedback, input-feedthrough, and gains. Namely, [12] generalized the well known Cayley Transform [1] to show that any system with finite $\mathcal{L}_2$-gain can be passivized using an I/O transformation, found algebraically. More recently, [13] used a geometric approach to prescribe a passivizing I/O transformation for SISO systems. In that letter, one constructs an I/O transformation mapping a system with known passivity indices to a system with prescribed passivity indices. This was achieved using a connection between passivity and cones through the notion of projective quadratic inequalities (PQIs), which can be seen as a specific case of sector bounds [14].

In this letter, we use the geometric approach of [13] to give a parameterization of all passivizing I/O transformations of a given SISO system. More precisely, we give a concise description of all I/O transformations that map a system with known passivity indices to a system with prescribed excess of passivity. This is done by understanding the action of the group of (invertible) I/O transformations on the collection of cones in the plane, which is a byproduct of the geometric approach of [13], allowing us to use standard group theory methods. We show that any transformation mapping a system with known passivity indices to a system with prescribed excess of passivity can be written (up to a scalar) as a product of three matrices - one depending on the original passivity indices, one depending on the desired excess of passivity, and a matrix whose entries are all non-negative.

Our results can be seen as an analogue of the Youla parameterization [15], dealing with passivizing I/O transformations instead of stabilizing controllers. We also consider multiple application domains of our results. First, we consider transformations simultaneously passivizing multiple systems, and explore applications in fault mitigation and plug-and-play control. Second, we consider the problem of passivizing

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a plant with respect to multiple equilibria, which relates to equilibrium-independent passivity [16].

The rest of this letter is organized as follows. Section II presents the geometric approach of [13] and formulates the problem. Section III characterizes all passivizing transformations of a given passive-short SISO system. Section IV provides possible applications of the achieved results.

Notations: We denote the group of all invertible matrices $T \in \mathbb{R}^{d\times d}$ as $GL_d(\mathbb{R})$. Given a linear transformation $S : \mathbb{R}^d \to \mathbb{R}^d$ and a basis $\mathcal{B}$ for $\mathbb{R}^d$, we denote the representing matrix of S in the basis $\mathcal{B}$ as $[S]_{\mathcal{B}}$. Furthermore, given two bases $\mathcal{B}_1, \mathcal{B}_2$ of $\mathbb{R}^d$, we denote the change-of-base matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ by $I_{\mathcal{B}_1\to\mathcal{B}_2} \in GL_d(\mathbb{R})$. We note that $I^{-1}_{\mathcal{B}_1\to\mathcal{B}_2} = I_{\mathcal{B}_2\to\mathcal{B}_1}$. Moreover, for any linear transformation $S : \mathbb{R}^d \to \mathbb{R}^d$, we have that $[S]_{\mathcal{B}_2} = I_{\mathcal{B}_1\to\mathcal{B}_2}[S]_{\mathcal{B}_1} I_{\mathcal{B}_2\to\mathcal{B}_1}$. Lastly, we denote the unit circle inside $\mathbb{R}^2$ by $\mathbb{S}^1$.

II. Background and Problem Formulation

We consider SISO dynamical systems given by the state-space representation $\dot{x} = f(x, u)$, $y = h(x, u)$, where $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, and $x \in \mathbb{R}^{n_x}$ is the state of the system. We recall the definition of passivity below.

Definition 1 [17]: Let Σ be a SISO dynamical system with $u = 0, y = 0$ as an equilibrium. We say Σ is *passive* if there exists a positive semidefinite C^1-smooth function (i.e., a storage function) $S(x)$ such that:

$$\frac{dS(x(t))}{dt} = \nabla S(x(t))\dot{x}(t) \le u(t)y(t), \tag{1}$$

holds for any trajectory $(u(t), x(t), y(t))$ of the system.

The notion of passivity stems from energy-based control, as $S(x)$ can be thought of as the potential energy stored inside the system, so (1) implies that the change in the energy stored in the system cannot be greater than the supplied power. We can expand the notion of passivity to consider both the case of total energy dissipation, and the case of (bounded) total energy gain, by adding either a negative or a positive term to the right-hand side of (1).

Definition 2: Let Σ be a SISO dynamical system with $u = 0, y = 0$ as an equilibrium, and $\rho, \nu \in \mathbb{R}$. The system is

i) *output ρ-passive* if there exists a positive semidefinite C^1-smooth function storage function S such that the inequality,
$$\frac{dS(x(t))}{dt} \le u(t)y(t) - \rho y(t)^2, \tag{2}$$
holds for any trajectory $(u(t), x(t), y(t))$ of the system;

ii) *input ν-passive* if there exists a positive semidefinite C^1-smooth function storage function S such that the inequality,
$$\frac{dS(x(t))}{dt} \le u(t)y(t) - \nu u(t)^2, \tag{3}$$
holds for any trajectory $(u(t), x(t), y(t))$ of the system;

iii) *input-output (ρ, ν)-passive* if $\rho\nu < 1/4$ and there's a positive semidefinite C^1-smooth function storage function S such that,
$$\frac{dS(x(t))}{dt} \le u(t)y(t) - \rho y(t)^2 - \nu u(t)^2, \tag{4}$$
holds for any trajectory $(u(t), x(t), y(t))$ of the system.

Remark 1: The demand $\rho\nu < 1/4$ is made to assure that the right-hand side of (4) is not always positive, nor always negative, as it would either imply that all static nonlinearities are I/O (ρ, ν)-passive, or that no system is I/O (ρ, ν)-passive, both of which are absurd. See [18, Appendix Lemma 3] for a detailed discussion.

The case in which $\rho, \nu > 0$ is usually referred to as *strict passivity* (or "excess of passivity"), and the case in which $\rho, \nu < 0$ is usually called *passive short* (or "shortage of passivity"). The definition above allows us to consider both cases in a unified framework. In order to incorporate them into passivity-based control schemes, one usually passivizes them using a loop transformation [9], [13]. We consider a transformed plant $\tilde{\Sigma}$ with new input $\tilde{u}$ and output $\tilde{y}$, which are connected to u, y via

$$\begin{bmatrix} u(t) \\ \tilde{y}(t) \end{bmatrix} = \hat{T}\begin{bmatrix} y(t) \\ \tilde{u}(t) \end{bmatrix}, \tag{5}$$

for some invertible matrix $\hat{T}$. This transformation is an aggregation of a constant gain input-feedthrough, constant gain output-feedback, and cascade with a constant gain (see [13] for further details). In this letter, we prefer to work with a different representation of $\hat{T}$, mapping signals $(u(t), y(t))$ to $(\tilde{u}(t), \tilde{y}(t))$, as

$$\begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) \end{bmatrix} = T\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}. \tag{6}$$

This is always possible for invertible $\hat{T}$ by noting that

$$\hat{T}\begin{bmatrix} y(t) \\ \tilde{u}(t) \end{bmatrix} - \begin{bmatrix} u(t) \\ \tilde{y}(t) \end{bmatrix} = \begin{bmatrix} -1 & t_{11} \\ 0 & t_{21} \end{bmatrix}\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} t_{12} & 0 \\ t_{22} & -1 \end{bmatrix}\begin{bmatrix} \tilde{u}(t) \\ y(t) \end{bmatrix} = 0$$

where $t_{ij} = [\hat{T}]_{ij}$.

We wish to understand the effect of these I/O transformations (6) on the passivity of the transformed system. Formally, the problem we consider is given below.

Problem 1: Let Σ be a dynamical system with equal input and output dimensions, which is I/O (ρ, ν)-passive, and let $\rho_\star, \nu_\star$ be numbers such that $\rho_\star\nu_\star < 1/4$. Characterize all I/O transformations of the form (6) such that the transformed system $\tilde{\Sigma}$ is I/O $(\rho_\star, \nu_\star)$-passive.

In order to address this problem, we consider the geometric approach to passivity of [13]. It considers an abstraction of the inequalities appearing in Definition 2, called projective quadratic inequalities:

Definition 3 (PQI): A *projective quadratic inequality* (PQI) is an inequality in the variables $\xi, \chi \in \mathbb{R}$ of the form

$$0 \le a\xi^2 + b\xi\chi + c\chi^2 := \mathbf{f}_{(a,b,c)}(\xi, \chi), \tag{7}$$

for some numbers a, b, c, not all zero. The inequality is called *non-trivial* if $b^2 - 4ac > 0$. The associated solution set of PQI is the set of all points $(\xi, \chi) \in \mathbb{R}^2$ satisfying the inequality.

PQIs are a special case of the sector bound formulation for passivity, see, e.g., [14]. We note that a PQI resembles (4), in which ξ, χ, a, b, c are replaced by $u, y, -\nu, 1, -\rho$ respectively. For this reason, we denote $\mathbf{f}_{(a,b,c)}(\xi, \chi)$ as $\varphi_{\rho,\nu}(\xi, \chi)$, and the

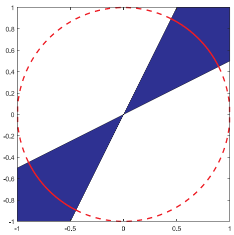

Fig. 1. A double cone (in blue), and the associated symmetric section (in solid red). The parts of $\mathbb{S}^1$ outside the symmetric section are presented by the dashed red line.

corresponding solution set as $C_{\rho,\nu}$. More explicitly, we define $\varphi_{\rho,\nu}$ as

$$\varphi_{\rho,\nu}(\xi,\chi) = -\nu\xi^2 + \xi\chi - \rho\chi^2,$$

and $C_{\rho,\nu}$ as $\{(\xi,\chi) \in \mathbb{R}\times\mathbb{R} : \varphi_{\rho,\nu}(\xi,\chi) \geq 0\}$.

Remark 2: The definition of PQIs allows an abstraction of the inequality defining passivity. It also encapsulates more sophisticated variants of passivity, such as shifted passivity, incremental passivity [19], equilibrium-independent passivity [16] and maximal equilibrium-independent passivity [20]. Hence, the results of this letter also apply to these variants.

As noted in [13], I/O transformations give rise to an action of the group of 2×2 invertible matrices, $GL_2(\mathbb{R})$, on the collection of solution sets of PQIs. This allows us to use standard group theory methods, which will be manifested in Proposition 1 in the next section. In particular, let A be the solution set of a PQI, $A = \{(\xi,\chi) : 0 \leq \mathbf{f}_{(a,b,c)}(\xi,\chi)\}$. For any invertible matrix $T \in GL_2(\mathbb{R})$, the solution set of the transformed PQI is given by $T(A)$, the image of A under T. In fact, one can show that an I/O transformation maps an I/O (ρ,ν)-passive system to an I/O $(\rho_\star,\nu_\star)$-passive system if and only if it maps the PQI $0 \leq \varphi_{\rho,\nu}(\xi,\chi)$ to the PQI $0 \leq \varphi_{\rho_\star,\nu_\star}(\xi,\chi)$ (or to a stricter inequality).

Definition 4: A *symmetric section* S on the unit circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is defined as the union of two closed disjoint sections that are opposite to each other, i.e., $S = B \cup (-B)$, where B is a closed section of angle $< \pi$. A *symmetric double cone* is defined as $A = \{\lambda s : \lambda > 0, s \in S\}$ for some symmetric section S.

The connection between cones and passivity theory is intricate, stemming from the notion of sector-bounded nonlinearities [21], [22]. An example of a symmetric section and the associated symmetric double-cone can be seen in Fig. 1. These are of interest due to their relationship with PQIs.

Theorem 1 [13]: The solution set of any non-trivial PQI is a symmetric double cone. Moreover, any symmetric double-cone is the solution set of some non-trivial PQI, which is unique up to a multiplicative positive constant.

As a corollary, we conclude that a map transforms an I/O (ρ,ν)-passive system to an I/O $(\rho_\star,\nu_\star)$-passive system if and only if it sends $C_{\rho,\nu}$ into $C_{\rho_\star,\nu_\star}$, which we denote by $C_{\rho,\nu} \hookrightarrow C_{\rho_\star,\nu_\star}$ Thus, we wish to characterize maps $C_{\rho,\nu} \hookrightarrow C_{\rho_\star,\nu_\star}$. One possible option is given in the following theorem:

Theorem 2 [13]: Let ρ,ν,$\rho_\star$, $\nu_\star$ be any numbers such that $\rho\nu$, $\rho_\star\nu_\star < 1/4$. Let (ξ_1,χ_1) and (ξ_2,χ_2) be two non-colinear solutions to $\varphi_{\rho,\nu}(\xi,\chi) = 0$. Moreover, let (ξ_3,χ_3) and (ξ_4,χ_4) be two non-colinear solutions to $\varphi_{\rho_\star,\nu_\star}(\xi,\chi) = 0$. Define

$$T_1 = \begin{bmatrix}\xi_3 & \xi_4\\ \chi_3 & \chi_4\end{bmatrix}\begin{bmatrix}\xi_1 & \xi_2\\ \chi_1 & \chi_2\end{bmatrix}^{-1}, T_2 = \begin{bmatrix}\xi_3 & -\xi_4\\ \chi_3 & -\chi_4\end{bmatrix}\begin{bmatrix}\xi_1 & \xi_2\\ \chi_1 & \chi_2\end{bmatrix}^{-1}.$$

Let α_1 be equal to 1 if $\varphi_{\rho,\nu}(\xi_1+\xi_2, \chi_1+\chi_2) \geq 0$ and zero otherwise. Moreover, let α_2 be equal to 1 if $\varphi_{\rho_\star,\nu_\star}(\xi_3+\xi_4, \chi_3+\chi_4) \geq 0$ and zero otherwise.

i) If $\alpha_1 = \alpha_2$, then T_1 is $C_{\rho,\nu} \hookrightarrow C_{\rho_\star,\nu_\star}$.

ii) If $\alpha_1 \neq \alpha_2$, then T_2 is $C_{\rho,\nu} \hookrightarrow C_{\rho_\star,\nu_\star}$.

Theorem 2 provides the foundation for the main result of this letter. In the next section we leverage this result to parameterize all passivizing transformations of the form (6) from an I/O (ρ,ν)-passive system to an arbitrary I/O $(\rho_\star,\nu_\star)$-passive system.

III. A Characterization of All Passivizing Transformations for SISO Systems

We wish to characterize all I/O transformations mapping an arbitrary dynamical system Σ to an I/O $(\rho_\star,\nu_\star)$-passive system. Namely, we assume that the given system is I/O (ρ,ν)-passive (for some known ρ,ν), and seek all transformations that force the transformed system to be I/O $(\rho_\star,\nu_\star)$-passive. We do so by finding all transformations that map a given double cone $C_{\rho,\nu}$ into $C_{\rho_\star,\nu_\star}$. Theorem 2 provides one way to build a map from an arbitrary cone into another arbitrary cone, but does not prescribe a general method to find all such maps. However, we can use Theorem 2 to show that all maps from an arbitrary cone into another arbitrary cone can be built using maps from $C_{0,0}$ into itself.

Proposition 1: Let ρ, ν, $\rho_\star$, $\nu_\star$ be any four numbers such that $\rho\nu, \rho_\star\nu_\star < 1/4$, and let T be any matrix $C_{\rho,\nu} \hookrightarrow C_{\rho_\star,\nu_\star}$. Let $S_{\rho,\nu}, S_{\rho_\star,\nu_\star}$ be the invertible matrices $C_{0,0} \hookrightarrow C_{\rho,\nu}, C_{0,0} \hookrightarrow C_{\rho_\star,\nu_\star}$ respectively, as built using Theorem 2. Then there exists a matrix Q, which is $C_{0,0} \hookrightarrow C_{0,0}$, such that $T = S_{\rho_\star,\nu_\star} Q S_{\rho,\nu}^{-1}$ holds.

Proof: Theorem 2 shows that $S_{\rho,\nu}^{-1}, S_{\rho_\star,\nu_\star}^{-1}$ map $C_{\rho,\nu}$ and $C_{\rho_\star,\nu_\star}$ into $C_{0,0}$, respectively. Define $Q = S_{\rho_\star,\nu_\star}^{-1} T S_{\rho,\nu}$. Then Q is invertible as a product of invertible matrices. Moreover, it maps $C_{0,0}$ into itself as

$$C_{0,0} \overset{S_{\rho,\nu}}{\hookrightarrow} C_{\rho,\nu} \overset{T}{\hookrightarrow} C_{\rho_\star,\nu_\star} \overset{S_{\rho_\star,\nu_\star}^{-1}}{\hookrightarrow} C_{0,0}.$$

∎

Proposition 1 gives a prescription for finding all matrices mapping $C_{\rho,\nu}$ into $C_{\rho_\star,\nu_\star}$. It contains two main ingredients, namely the matrices $S_{\mu,\tau}$, and matrices mapping $C_{0,0}$ into itself. We start by finding all matrices in $GL_2(\mathbb{R})$ mapping $C_{0,0}$ into itself.

Proposition 2: A matrix $T \in GL_2(\mathbb{R})$ sends $C_{0,0}$ into itself if and only if all of the entries of T have the same sign, i.e., $T_{ij}T_{kl} \geq 0$ for every $i,j,k,l \in \{1,2\}$.

Proof: We first show that if T sends $C_{0,0}$ into itself, then all of the entries of $T = (T_{ij})_{i,j}$ have the same sign. We recall that $C_{0,0}$ contains all points (ξ,χ) such that $\xi\chi \geq 0$, i.e., $C_{0,0}$ is a union of $\{0\}$, the first quadrant, and the third quadrant. We note that $e_1 = (1,0)^\top$ and $e_2 = (0,1)^\top$ are in $C_{0,0}$, hence

$Te_1 = (T_{11}, T_{21})^\top$ and $Te_2 = (T_{12}, T_{22})^\top$ are also in $C_{0,0}$. This implies that T_{11}, T_{21} have the same sign, and that T_{12}, T_{22} have the same sign, and in each pair not both elements are zero (as T is invertible). We note that by switching between T and $-T$, we may assume without loss of generality that T_{11}, T_{21} are both non-negative. We want to show that T_{12}, T_{22} are also both non-negative.

Assume the contrary, that is, that T_{12}, T_{22} are both non-positive. Moreover, as $Te_1, Te_2 \neq 0$, we conclude that Te_1 lies in the first quadrant of $\mathbb{R}^2$, and that Te_2 lies in the third quadrant. We note that the line between e_1, e_2 lies inside $C_{0,0}$, so the same is true for the line between Te_1, Te_2, as T is linear and maps $C_{0,0}$ into itself. However, as Te_1 is in the first quadrant and Te_2 is in the third, the straight line between them passes either through zero, the second quadrant or the fourth quadrant. The latter two cases are impossible, as $C_{0,0}$ contains no points from these quadrants, and the former case is impossible as it would imply that the invertible transformation T maps a non-zero point to zero. We thus conclude all entries of T have the same sign.

Conversely, assume that all of the entries of T have the same sign. By replacing $-T$ with T, we assume without loss of generality that $T_{ij} \geq 0$ for all $i, j \in \{1, 2\}$, so that Te_1, Te_2 are both in the first quadrant. Take any point $x \in C_{0,0}$. If $x = 0$ then $Tx = 0 \in C_{0,0}$. If x is in the first quadrant, then it is a linear combination of e_1, e_2 with non-negative coefficients, not both zero. Thus Tx is a linear combination of Te_1, Te_2 with non-negative coefficients (not both zero), hence Tx is in the first quadrant. If x is in the third quadrant, then $-x$ is in the first quadrant, so $T(-x) = -Tx$ is in the first quadrant, hence Tx is in the third quadrant. As we showed that $Tx \in C_{0,0}$ for all $x \in C_{0,0}$, this concludes the proof. ■

Remark 3: More generally, given some ρ, ν, one could ask for a characterization of all matrices $T \in GL_2(\mathbb{R})$ that are $C_{\rho,\nu} \hookrightarrow C_{\rho,\nu}$. Mimicking the proof above, one can show that a map $T \in GL_2(\mathbb{R})$ is $C_{\rho,\nu} \hookrightarrow C_{\rho,\nu}$ if and only if all of the elements of the matrix $I_{e\to\mathcal{B}} T I_{e\to\mathcal{B}}^{-1}$ possess the same sign, where $\mathcal{B}$ is composed of the non-colinear solutions to the equation $-\nu\xi^2 + \xi\chi - \rho\chi^2 = 0$ and e is the standard basis. A more explicit form for the basis $\mathcal{B}$ can be achieved by taking the columns of the matrix $S_{\rho,\nu}$, as seen in Proposition 3 below.

We now clarify the second component appearing in Proposition 1, namely the matrices $S_{\mu,\tau}$:

Proposition 3: Let μ, τ be any two numbers such that $\mu\tau < 1/4$. Recall that $S_{\mu,\tau}$ is a map $C_{0,0} \hookrightarrow C_{\mu,\tau}$, as constructed in Theorem 2. Define $R = \sqrt{1 - 4\tau\mu}$.

i) If $\tau < 0$, we can choose $S_{\mu,\tau} = \frac{1}{2\tau}\begin{bmatrix} -1-R & 1-R \\ -2\tau & 2\tau \end{bmatrix}$.

ii) If $\tau > 0$, we can choose $S_{\mu,\tau} = \frac{1}{2\tau}\begin{bmatrix} 1+R & 1-R \\ 2\tau & 2\tau \end{bmatrix}$.

iii) If $\tau = 0$, we can choose $S_{\mu,\tau} = \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$.

Proof: We use Theorem 2 to build $S_{\mu,\tau}$. As we consider a map $C_{0,0} \hookrightarrow C_{\mu,\tau}$, we take $(\xi_1, \chi_1) = (1, 0)$ and $(\xi_2, \chi_2) = (0, 1)$. As $(\xi_1 + \xi_2, \chi_1 + \chi_2) = (1, 1)$ satisfies the PQI $\xi\chi \geq 0$, we choose:

$$S_{\mu,\tau} = \begin{cases} \begin{bmatrix} \xi_3 & \xi_4 \\ \chi_3 & \chi_4 \end{bmatrix}, & \alpha_2 = 1 \\ \begin{bmatrix} \xi_3 & -\xi_4 \\ \chi_3 & -\chi_4 \end{bmatrix}, & \alpha_2 \neq 1 \end{cases},$$

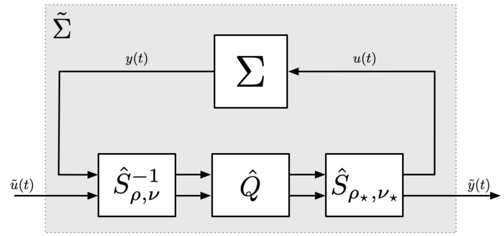

Fig. 2. A general transformation mapping a SISO I/O (ρ, ν)-passive system to a SISO I/O $(\rho_\star, \nu_\star)$-passive system. The entries of the matrix $Q \in \mathbb{R}^{2\times 2}$ all have the same sign.

where we recall that $(\xi_3, \chi_3), (\xi_4, \chi_4)$ are two non-colinear solutions to $-\tau\xi^2 + \xi\chi - \mu\chi^2 = 0$, and $\alpha_2 = 1$ if and only if $(\xi_3 + \xi_4, \chi_3 + \chi_4)$ satisfies the PQI $-\tau\xi^2 + \xi\chi - \mu\chi^2 \geq 0$. We first assume that $\tau \neq 0$, so we write $-\tau\xi^2 + \xi\chi - \mu\chi^2 = 0$ as $-\tau(\xi - a_1\chi)(\xi - a_2\chi) = 0$, where a_1, a_2 are given by

$$a_1 = \frac{-1 + \sqrt{1 - 4\tau\mu}}{-2\tau} = \frac{1 - R}{2\tau}$$
$$a_2 = \frac{-1 - \sqrt{1 - 4\tau\mu}}{-2\tau} = \frac{1 + R}{2\tau},$$

where we note that $a_1 \neq a_2$ as $\mu\tau < 1/4$. Choose $(\xi_3, \chi_3) = (-a_2, -1), (\xi_4, \chi_4) = (a_1, 1)$. We have that the point $(\xi_3 + \chi_3, \xi_4 + \chi_4) = (a_1 - a_2, 0)$ satisfies the PQI $-\tau\xi^2 + \xi\chi - \mu\chi^2 \geq 0$ if and only if $\tau < 0$, as we assumed $\tau \neq 0$. We therefore conclude the desired result for $\tau \neq 0$ from Theorem 2.

Suppose now that $\tau = 0$. We note that $(\xi_3, \chi_3) = (1, 0)$ and $(\xi_4, \chi_4) = (\mu, 1)$ are two non-colinear solutions to $\xi\chi - \mu\chi^2 = 0$, and that $(\xi_3 + \xi_4, \chi_3 + \chi_4) = (1 + \mu, 1)$ satisfies the PQI $-\tau\xi^2 + \xi\chi - \mu\chi^2 \geq 0$, as $\xi\chi - \mu\chi^2 = \chi(\xi - \mu\chi)$. This completes the proof. ■

We now conclude with the main result.

Theorem 3: Let Σ be a SISO I/O (ρ, ν)-passive system, and let $T \in GL_2(\mathbb{R})$ be an invertible matrix inducing an I/O transformation of the form (6). The transformed system $\tilde{\Sigma}$ is I/O $(\rho_\star, \nu_\star)$-passive if and only if there exists a matrix $M \in GL_2(\mathbb{R})$ such that $M_{ij} \geq 0$ for all $i, j \in \{1, 2\}$ and some $\theta \in \{\pm 1\}$ such that $T = \theta S_{\rho_\star,\nu_\star} M S_{\rho,\nu}^{-1}$, where $S_{\rho,\nu}, S_{\rho_\star,\nu_\star}$ are given in Proposition 3. In other words, the transformed system $\tilde{\Sigma}$ is I/O $(\rho_\star, \nu_\star)$-passive if and only if all of the entries of the matrix $S_{\rho_\star,\nu_\star}^{-1} T S_{\rho,\nu}$ have the same sign.

Proof: Proposition 1 implies that for an invertible matrix $T \in GL_2(\mathbb{R})$, the transformed system $\tilde{\Sigma}$ is I/O $(\rho_\star, \nu_\star)$-passive if and only if there exists an invertible matrix $Q \in GL_2(\mathbb{R})$ which is $C_{0,0} \hookrightarrow C_{0,0}$ such that $T = S_{\rho_\star,\nu_\star} Q S_{\rho,\nu}^{-1}$. By Proposition 2, a matrix Q is $C_{0,0} \hookrightarrow C_{0,0}$ if and only if all of its entries possess the same sign. By letting $\theta \in \{\pm 1\}$ be that sign, we can write any matrix Q sending $C_{0,0}$ into itself as $Q = \theta M$, where $M \in GL_2(\mathbb{R})$ and $M_{ij} \geq 0$ for all $i, j \in \{1, 2\}$. Thus, the transformed system $\tilde{\Sigma}$ is I/O $(\rho_\star, \nu_\star)$-passive if and only if there exists some $\theta \in \{\pm 1\}$ and $M \in GL_2(\mathbb{R})$ with non-negative entries such that $T = \theta S_{\rho_\star,\nu_\star} M S_{\rho,\nu}^{-1}$. ■

A block diagram visualizing Theorem 3 can be seen in Fig. 2. We note that the block diagram uses the operator $\hat{S}_{\rho_\star,\nu_\star} \hat{Q} \hat{S}_{\rho,\nu}^{-1}$, which is akin to the operator $\hat{T}$ in (5). The operator $T = S_{\rho_\star,\nu_\star} Q S^{-1}_{\rho,\nu}$ can be recovered from $\hat{T}$ in the manner described earlier.

Remark 4: These results are effective even if a "poor" storage function is chosen as a certificate for passivity of the original system. Indeed, the proposed transformations aims to passify the system to a prescribed passivity index, the utility of which is illustrated in Section IV.

IV. Applications

In this section, we consider possible applications of the achieved characterization for synthesis. We explore two possible applications, including multi-purpose transformations, and passivation with respect to multiple equilibria.

A. Multiple Purpose Transformations

Suppose we are provided with different systems $\{\Sigma_i\}_{i\in\mathcal{I}}$ which are I/O (ρ_i, ν_i)-passive. Our goal is to design a transformation T mapping each system Σ_i to an I/O $(\rho_i^\star, \nu_i^\star)$-passive system (for $i \in \mathcal{I}$). One scenario this may arise is when we wish to control a system which can fault. Our goal is to find a transformation which makes the faultless system as strictly passive as possible, but also passivizes any faulty version of the system. When connecting the system with a strictly passive feedback controller, the first part improves the convergence rate, and the second part ensures stability of the closed-loop system even when the system malfunctions.

Proposition 4: Consider the SISO systems $\{\Sigma\}_{i\in\mathcal{I}}$, which are I/O (ρ_i, ν_i)-passive. Let $(\rho_i^\star, \nu_i^\star)$ be real numbers such that for each i, $\rho_i^\star \nu_i^\star < 1/4$. Consider a general I/O transformation T of the form (6). The transformed systems $\{\tilde{\Sigma}\}_{i\in I}$ are I/O $(\rho_i^\star, \nu_i^\star)$-passive for all i, if and only if there exists matrices $M_i \in GL_2(\mathbb{R})$ with $[M]_{ij} \geq 0$ for all $i, j \in \{1, 2\}$, and numbers $\theta_i \in \{\pm 1\}$ such that $T = \theta_i S_{\rho_i^\star,\nu_i^\star} M_i S^{-1}_{\rho_i,\nu_i}$.

The proposition immediately follows from Theorem 3. In particular, if we only wish to passivize the systems $\{\Sigma_i\}$ (i.e., $\rho_i^\star = \nu_i^\star = 0$), we obtain the following set of equations:

$$T = \theta_i M_i S^{-1}_{\rho_i,\nu_i},\ i \in \mathcal{I}. \tag{8}$$

Example 1: Consider the SISO system Σ which is the parallel interconnection of two linear and time-invariant SISO systems Σ_1, Σ_2 given by the transfer functions $G_1(s) = \frac{s-1}{s+1}$ and $G_2(s) = \frac{-s^3+6s+5}{s^3+4s^2+5s+2}$. The system Σ is linear and time-invariant, and its transfer function can be computed to be $G(s) = \frac{2s+3}{s^2+3s+2} = \frac{1}{s+2} + \frac{1}{s+1}$. It is easy to verify that Σ is passive, and actually output-strictly passive with a parameter $\rho = \frac{2}{3}$. However, the component corresponding to Σ_2 is unreliable, and may fault. When it does, the transfer function changes to $G_1(s)$, which is not passive as it is non-minimum phase. However, it does have a finite $\mathcal{L}_2$-gain equal to $\max_{\mathrm{Re}(s)>0} \|G_1(s)\| = 1$. It is shown in [13] that a system with a finite $\mathcal{L}_2$-gain equal to β is input ν-passive for $\nu = -\beta^2 - 0.25$. Thus, the faultless system is output $\frac{2}{3}$-passive, and the faulty system is input (-1.25)-passive.

Suppose we want to find a transformation T that maps the faultless system to an output 2-passive system, and the faulty system to a passive system. If we define $T_1 = S^{-1}_{2,0} T S_{\frac{2}{3},0}$ and $T_2 = S^{-1}_{0,0} T S_{0,-\frac{5}{4}}$, then we want the entries of both T_1 and T_2 to have the same sign. We choose

$$T = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}.$$

A simple computation shows that:

$$T_1 = S^{-1}_{2,0} T S_{\frac{2}{3},0} = \frac{1}{15}\begin{bmatrix} 3 & 2 \\ 6 & 7 \end{bmatrix}$$
$$T_2 = S^{-1}_{0,0} T S_{0,-\frac{5}{4}} = \frac{1}{25}\begin{bmatrix} 10 & 10 \\ 3 & 5 \end{bmatrix},$$

meaning both T_1, T_2 have entries which have the same sign. Thus, T satisfies the requirements we established, which is a fact we now verify independently of the computation above.

First, we note that the map T can be written as the product

$$T = \begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.04 \end{bmatrix}\begin{bmatrix} 1 & 0.4 \\ 0 & 1 \end{bmatrix},$$

meaning it operates in the following way: It first implements a constant output feedback with gain equal to 0.4, it then implements a gain on the output of size 0.04, and at the end it implements a constant input feed-through with gain 0.4. Thus, it transforms the faultless transfer function G to

$$\tilde{G}(s) = \frac{0.04}{\frac{1}{G} + 0.4} + 0.4 = \frac{0.4s^2 + 1.6s + 1.4}{s^2 + 3.8s + 3.2},$$

and simultaneously transforms the faulty transfer function $G_1(s)$ to

$$\tilde{G}_1(s) = \frac{0.04}{\frac{1}{G_1} + 0.4} + 0.4 = \frac{0.6s + 0.2}{1.4s + 0.6}.$$

One could easily check (e.g., using the MATLAB command "getPassiveIndex") that $\tilde{G}(s)$ is output-strictly passive with index $\rho \approx 2.2857 > 2$, and that $\tilde{G}_1(s)$ is passive, and actually output-strictly passive with index $\rho \approx 2.333$. Thus, the transformation T maps the faultless system to an output 2-passive system, and the faulty system to a passive system, as required.

B. Passivation With Respect to Multiple Equilibria and Equilibrium-Independent Passivity

In several occasions, one wishes to study the behavior of a system around more than one equilibrium, such as in multi-agent networks. In this direction, one can consider passivity (or shortage thereof) with respect to an arbitrary steady-state I/O pair (u, y). Indeed, the notions of output ρ-passivity, input ν-passivity and I/O (ρ, ν)-passivity can be extended to other steady-states by replacing $y(t)$ by $y(t) - \mathrm{y}$ and $u(t)$ by $u(t) - \mathrm{u}$ in (2), (3), and (4) respectively. When designing controllers for systems which can operate around more than one equilibrium, we need to consider passivity (or I/O (ρ, ν)-passivity) with respect to each equilibrium. The same system can behave differently around different equilibria. To remedy this problem, various extensions to passivity theory were introduced; see Remark 2. Under these assumptions, it is possible to prove that certain multi-agent networks converge without specifying a limit ahead of time. In some cases, we might know that there are some ρ, ν such that the system is I/O (ρ, ν)-passive with respect to all equilibria, and in that case we can use the method of [13] to passivize the system with respect to all equilibria. However, we can consider a more general case, where different equilibria are associated with different corresponding dissipation inequalities.

Before moving forward, we note that the inequalities defining I/O (ρ, ν)-passivity with respect to any steady-state can also be written as PQIs in exactly the same way used for I/O (ρ, ν)-passivity with respect to the origin. Thus, our results characterize all transformations T that passivize a plant with respect to any fixed equilibrium. In that direction, we consider a system Σ and a collection of steady-state I/O pairs $\{(\mathrm{u}_i, \mathrm{y}_i)\}$. Our goal is to find a transformation T that passivizes the system Σ with respect to all (transformed) steady-state pairs simultaneously.

Unsurprisingly, this problem is very similar to the multiple objective transformation considered in Section IV-A. We can prove the following proposition:

Proposition 5: Consider a SISO system Σ, let $\{(\mathrm{u}_i, \mathrm{y}_i)\}$ be a collection of steady-state I/O pairs of Σ, and let $(\rho_i, \nu_i), (\rho_i^\star, \nu_i^\star)$ be real numbers such that for each i, $\rho_i\nu_i, \rho_i^\star\nu_i^\star < 1/4$. Suppose that for each i, the system Σ is I/O (ρ_i, ν_i)-passive with respect to the steady-state I/O pair $(\mathrm{u}_i, \mathrm{y}_i)$. Consider a general I/O transformation T of the form (6), and consider the new system $\tilde{\Sigma}$ and the new steady-state pairs $\{T(\mathrm{u}_i, \mathrm{y}_i)\}$. Then $\tilde{\Sigma}$ is I/O $(\rho_i^\star, \nu_i^\star)$-passive with respect to $T(\mathrm{u}_i, \mathrm{y}_i)$, for all i, if and only if there exists matrices $M_i \in GL_2(\mathbb{R})$ with $[M]_{ij} \geq 0$, and numbers $\theta_i \in \{\pm 1\}$ such that $T = \theta_i S_{\rho_i^\star, \nu_i^\star} M_i S_{\rho_i, \nu_i}^{-1}$.

The proof follows immediately from Theorem 3.

V. Conclusion

This letter considers the notion of (ρ, ν)-passivity, which contains both shortage and excess of passivity. We characterized all I/O transformations mapping an I/O (ρ, ν)-passive system to an I/O $(\rho_\star, \nu_\star)$-passive system. We used the geometric approach of [13] to convert the problem into characterizing all linear transformations that map a given symmetric double-cone to a desired symmetric double-cone and studying the action of the collection of invertible 2×2 matrices, $GL_2(\mathbb{R})$, on the collection of symmetric double-cones. This culminated in a result showing that any I/O transformation mapping an I/O (ρ, ν)-passive system to an I/O $(\rho_\star, \nu_\star)$-passive system can be written (up to a sign) as the product of three matrices, $S_{\rho,\nu}$, $S_{\rho_\star,\nu_\star}^{-1}$ and a non-negative matrix. Future work will extend these results of the MIMO case and further explore potential applications including optimal passiving transformations with respect to specified costs.

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