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Robust Design of Sparse Relative Sensing Networks

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***Abstract*— This paper considers the robust design of sparse relative sensing networks subject to a given $\mathcal{H}_\infty$-performance constraint. The topology design considers heterogenous agents over weighted graphs. We develop a robust counterpart to the uncertain optimization problem and formulate the sparsity constraint via a convex ℓ_1-relaxation. We also demonstrate how this relaxation can be used to embed additional performance criteria, such as the maximization of the algebraic connectivity of the relative sensing network.**

***Index Terms*— relative sensing networks, robust $\mathcal{H}_\infty$-performance, ℓ_0-minimization, topology design**

I. INTRODUCTION

The ability of a single agent using available sensors to measure state information of an entire network can be limited by spatial constraints such as orientation, range and power requirements [8], [19]. Applications for distributed sensor networks relying on relative sensing range from environmental surveillance, modeling and localization to collaborative information processing [1], [16]. Additionally, such systems are relevant in formation flying applications where distributed sensing is employed to measure inter-agent distances [18], [23].

In this work, we focus on single agents that rely on relative sensing to achieve a common mission. Such systems are called *relative sensing networks* (RSN). While in other multi-agent systems, the network is often coupled at state level, this is not the case for relative sensing networks. Here, the underlying sensing topology couples the agents at their outputs and therefore introduces an implicit 'network'. In recent years, increasing interest was given on how the underlying graph topology affects system theoretic notions and the behavior of the system. Closed loop properties of multi-agent systems and relation to the graph Laplacian are studied in e.g. [12] and performance bounds of consensus systems are given in [25].

The analysis and synthesis of relative sensing networks was recently considered in [26] with respect to $\mathcal{H}_2$- and $\mathcal{H}_\infty$-performance and with respect to sparsity constraints in [21]. Strong results exist for unweighted homogenous graphs relating network properties with the $\mathcal{H}_\infty$-performance of the network. However, the dynamics of a single agent influences the performance of the network and the links between the agents differ in importance or fidelity. The agent dynamic can be interpreted as a node weight on the graph, whereas the link dynamic (or static gain) is an edge weight. Unfortunately, there not many analytical results considering *node and edge weighted* graphs. Another important topic when dealing with networks of dynamic systems is robustness to uncertainties. The network should retain certain properties such as the connectivity of the graph or $\mathcal{H}_\infty$-performance even though parts of the network topology is unknown. Robustness of network topologies is also considered in the area of survivable network design [11], [15], but there, control theoretic aspects are often not taken into account. This leads to the main contribution of the current paper. We present a first step towards the design of robust relative sensing networks. We develop an optimization algorithm for determining the optimal robust topology for relative sensing networks with pre-specified properties when edge and node weights are present. We are especially interested in graphs with a sparse topology, i.e. graphs fulfilling the required properties with as few edges as possible, in the presence of uncertainties in the edge weights.

By combining control theoretic insight with results from compressed sensing (see e.g. [6], [9]) we systematically achieve sparse relative sensing networks with guaranteed system theoretic properties, such as algebraic connectivity of the graph and robustness against network uncertainties. This paper extends our previous work [21] on sparse design of relative sensing network to uncertainties in the topology of the network. First, a combinatorial optimization problem is proposed. This optimization algorithm seeks a sparse topology guaranteeing (maximal) algebraic connectivity and pre-specified $\mathcal{H}_\infty$-performance of the network in the face of uncertain edge weights. This non-convex formulation is then relaxed by using a robust counterpart and a convex weighted ℓ_1-minimization, and the resulting problem can be tackled by finding a solution of iterative convex optimization problems. Sparsity promoting optimization by ℓ_1-minimization was also recently considered for decentralized controller design in [20].

The remainder of the paper is organized as follows: Section II introduces the mathematical preliminaries and notation of the paper. In Section III the model of the relative sensing network is described and Section IV deals with the synthesis problem of sparse RSN that are robust against edge weight uncertainties and an optimization algorithm for topology design is presented. The paper concludes with an example in Section V and a summary and outlook in Section VI.

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II. Mathematical Preliminaries

The 0-norm of a vector $x \in \mathbb{R}^n$ is defined as

$$\|x\|_0 = \{\# \; x_i | x_i \neq 0\},$$

and corresponds to the number of non-zero entries in x. Despite not being a true norm, it is often referred to in the literature as one. A vector is called *sparse* if its 0-norm is small compared to the dimension of the vector, i.e., if most of its entries are zero.

We use the $\mathcal{H}_\infty$-norm to analyze the performance of an RSN in this paper. The $\mathcal{L}_2$-induced norm (or $\mathcal{L}_2$-gain) of a dynamical system $\mathcal{H} : \mathcal{L}_2^n \to \mathcal{L}_2^m$ is defined as

$$\|\mathcal{H}\|_{\mathcal{L}_2-\text{ind}} = \sup_{w \in \mathcal{L}_2 \setminus \{0\}} \frac{\|\mathcal{H}w\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}},$$

and corresponds for a linear system $\mathcal{H}$ to the $\mathcal{H}_\infty$-norm $\|H(s)\|_\infty = \sup_\omega \{\bar{\sigma}(H(j\omega))\}$, where $H(s) = C(sI - A)^{-1}B + D$ is a transfer function of the dynamical system $\mathcal{H}$ and $\bar{\sigma}(H(j\omega))$ denotes the largest singular value of H at a fixed frequency ω.

Graphs and the matrices associated with them will be widely used in this paper. An in-depth treatment of graph theory is given in e.g. [14]. An undirected (simple) graph $\mathcal{G}$ is specified by a vertex set $\mathcal{V}$ and a node set $\mathcal{E}$ whose elements characterize the incidence relation between distinct pairs of $\mathcal{V}$. Two vertices i and j are called *adjacent* (or neighbors) when $\{i, j\} \in \mathcal{E}$. For this work, the $|\mathcal{V}| \times |\mathcal{E}|$ incidence matrix $E(\mathcal{G})$ for a graph with arbitrary orientation is of importance. The incidence matrix is a $\{0, \pm 1\}$-matrix with rows and columns indexed by the vertices and edge of $\mathcal{G}$ such that $[E(\mathcal{G})]_{ik}$ has the value '+1' if node i is the initial node of edge e_k, '-1' if it is the terminal node, and '0' otherwise.

The (graph) Laplacian of $\mathcal{G}$,

$$L(\mathcal{G}) := E(\mathcal{G})E(\mathcal{G})^T,$$

is a positive semi-definite matrix. The eigenvalues of the graph Laplacian are real and will be ordered and denoted as $0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_{|\mathcal{V}|}(\mathcal{G})$. Furthermore, the eigenvector corresponding to λ_1 is $[1 \ldots 1]^T$, henceforth written as $\mathbf{1}$. Similar to [22] we define the *node- and edge-weighted graph Laplacian*

$$L_w(\mathcal{G}) := Q^{-1}E(\mathcal{G})WE(\mathcal{G})^T,$$

where Q is a positive-definite and W a positive semi-definite diagonal matrix representing the weights associated to the nodes and edges of the graph, respectively.

III. Model of the Relative Sensing Network

In this section we introduce the model of the RSN and present the theoretical background for the optimization algorithm presented in Section IV. Consider a group of g linear time-invariant dynamical systems (agents)

$$\Sigma_i = \begin{cases} \dot{x}_i(t) &= A_i x_i(t) + B_i w_i(t) \\ y_i(t) &= C_i x(t), \end{cases} \tag{1}$$

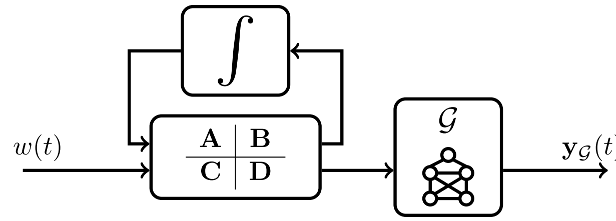


Fig. 1. Global RSN layer block diagram; the feedback connection represents an upper fractional transformation [10].

where each agent is indexed by the script i. Here, $x_i(t) \in \mathbb{R}^{n_i}$ represents the state, $w_i(t) \in \mathbb{R}^{r_i}$ the exogenous input and $y_i(t) \in \mathbb{R}^{q_i}$ the measured output. We denote the transfer-function representation of Σ_i as H_i with

$$H_i := C_i(sI - A_i)^{-1}B_i. \tag{2}$$

We assume compatible output for all agents, e.g. system outputs will correspond to the same physical quantity. It should be noted, that in a heterogeneous RSN, the dimension of each agent can be different.

The parallel interconnection of all agents can be expressed by a concatenation of the corresponding system states, inputs, and outputs, and through the block diagonal aggregation of each agent's state-space matrices. We use bold-phase notation to denote the expanded state-space, e.g. $\boldsymbol{x}(t) = [x_1(t)^T, \ldots, x_g(t)^T]^T$ and $\boldsymbol{A} = \text{diag}(A_1, \ldots, A_g)$.

The sensed output of the RSN is the vector $\boldsymbol{y}_\mathcal{G}(t)$ containing relative state information of each agent and its neighbors and is motivated by the relative sensing problem discussed in Section I. The incidence matrix of a graph naturally captures state differences and will be the algebraic construct used to define the relative outputs of RSNs, i.e.

$$\boldsymbol{y}_\mathcal{G}(t) = (WE(\mathcal{G}_c)^T \otimes I)\boldsymbol{y}(t). \tag{3}$$

Here $\mathcal{G}_c$ is the complete graph and the topology is defined by the diagonal weighting matrix $W = \text{diag}(w_1, \ldots, w_{|\mathcal{E}|})$, with $w_i \in \mathbb{R}_0^+$ and the node set given as $\mathcal{V} = \{1, \ldots, g\}$. Note that edge i does not exist if and only if $w_i = 0$. Furthermore, we denote the vector containing all weighs as $w = [w_1, \ldots, w_{|\mathcal{E}|}]^T$. The weights w_i in this setup can be seen as the gains of the sensor used to sense the relative state. They might be used to capture the fidelity of a relative measurement. The global layer is visualized in the block diagram shown in Figure 1.

Using the above notations, we can express the heterogeneous RSN in a compact form

$$\Sigma_{het}(\mathcal{G}) \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{w}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \\ \boldsymbol{y}_\mathcal{G}(t) &= (WE(\mathcal{G}_c)^T \otimes I)\boldsymbol{y}(t). \end{cases} \tag{4}$$

The transfer function representation of $\Sigma_{het}(\mathcal{G})$ is denoted as $\hat{\Sigma}_{het}(\mathcal{G})$ and is defined as in (2). As in the state space model, bold faced transfer functions denote the block diagonal aggregation of each agent's corresponding transfer function, e.g. $\boldsymbol{H}(s) = \text{diag}(H_1(s), \ldots, H_g(s))$. We denote $T_{het}^{w \mapsto \mathcal{G}}$ as

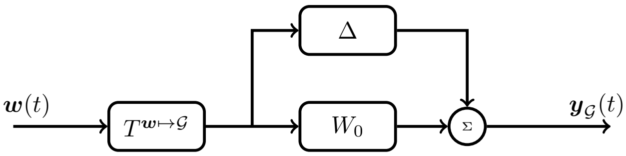


Fig. 2. Multiplicative uncertainty for the RSN.

the map from exogenous inputs to the RSN sensed output $T_{het}^{w \mapsto \mathcal{G}} = (WE(\mathcal{G}_c)^T \otimes I_q)\boldsymbol{H}$. We will now state some facts for RSNs used later on.

Theorem 1 ([26]): The $\mathcal{H}_\infty$ norm of a heterogeneous RSN is bounded from above by

$$\|T_{het}^{w \mapsto \mathcal{G}}\|_\infty \leq \|WE(\mathcal{G}_c)^T Q\|_2$$

where $Q = \mathrm{diag}(\|H_1\|_\infty, \ldots, \|H_g\|_\infty)$.
Theorem 1 shows that the topology of the underlying graph is of significant influence to the performance of the relative sensing network. Furthermore, the dynamic difference between agents is an important factor in the performance of the overall system, since the matrix Q captures the dynamics of the single agents and acts as a weighting on the matrix norm.

We are especially interested in the *robustness* of certain topologies in face of uncertainties in the network. In the following, we assume that the agent dynamics are known exactly, whereas for the edge weight, only a nominal weight is known. We assume that the edge weights in (4) are given as $W = W_0 + \Delta$, where $\Delta \in \boldsymbol{\Delta}_w$ is a structured uncertainty on each edge weight. This can be considered as an output-multiplicative uncertainty. The uncertainty set is defined as

$$\boldsymbol{\Delta}_w = \{\mathrm{diag}(\delta_1, \ldots, \delta_{|\mathcal{E}|}) : \delta \in \mathbb{R}^{|\mathcal{E}|}, \|\delta\|_2 \leq 1\} \quad (5)$$

For the synthesis of robust RSN, we will use the term of robust connectivity defined next.

Definition 1 (Robust Connectivity): A graph is called *robustly connected* under the uncertainty set $\boldsymbol{\Delta}_w$, if and only if the graph remains connected for all $\Delta \in \boldsymbol{\Delta}_w$.
In the next section, we use Theorem 1 to synthesize robust RSNs.

IV. Robust Synthesis of Sparse Relative Sensing Networks

In this section, we present the main result of the current paper. We focus on the robust design of sparse relative sensing networks that fulfill certain pre-specified network properties such as algebraic connectivity and $\mathcal{H}_\infty$-performance with as few edges as possible. At the same time, we consider uncertainties in the network topology. First, a problem formulation is derived that incorporates the sparsity requirements into the network design. This leads to an uncertain optimization problem. A robust counterpart is formulated and the numerically exhaustive combinatorial exact solution caused by the sparsity constraint is relaxed by a weighted ℓ_1-minimization. This results in an iterative solution of convex optimization problems to design graphs with sparse topology.

A. Problem Formulation

We consider the network as given in (4) with edge weight uncertainties as in (5). Designing the topology for this network can be formulated as follows:

Problem 1 ($\mathcal{H}_\infty$-optimal design of RSN [26]): Given a network consisting of g agents which are coupled as given in (4) with edge weight uncertainties as given in (5), find nominal edge weights $w_{0_i} \geq 0$, such that the

$$\min_{w_{0_i} \geq 0} \max_{\|\delta\|_2 \leq 1} \quad \|QE(\mathcal{G}_c)W\|_2$$
$$\text{subject to } \mathcal{G} \text{ is robustly connected.}$$

As shown in [26], this problem can be formulated as a convex optimization problem using robust optimization techniques. However, the solution to this optimization problem is in general not sparse, i.e. all weights w_{0_i} are non-zero. This is not a desired solution, since it requires the implementation of a complete graph. Instead, it is more desirable to search for a graph topology that is sparse and only requires the implementation of a few non-zero edges. To take this into account, we now state the *sparse* relative sensing network design problem with edge weight uncertainties:

Problem 2 (Sparse Topology Design for RSN): Given a network consisting of g agents which are coupled as given in (4), edge weight uncertainty as given in (5) and a predefined $\mathcal{H}_\infty$-performance γ. Find a sparse distribution of the nominal weights $w_{0_i} \geq 0$, such that the network is connected and the $\mathcal{H}_\infty$-performance is less than γ, i.e.

$$\begin{aligned} \min_{w_{0_i} \geq 0} \max_{\|\delta\|_2 \leq 1} & \|w_0\|_0 \qquad (6) \\ \text{subject to } & \|T_{het}^{w \mapsto \mathcal{G}}\|_\infty < \gamma \\ & \mathcal{G} \text{ is robustly connected.} \end{aligned}$$

The problem formulation implies that the network remains connected and that the $\mathcal{H}_\infty$ performance of the relative sensing network does not exceed γ in the presence of edge weight uncertainties. Recall, that the 0-norm of a vector is a measure of its sparsity. In this way, minimizing $\|w\|_0$ attempts to maximize the number of zero elements in the edge weight vector w and therefore minimizes the number of actually used edge weights.

As show in [26] Theorem 1 can be formulated as a uncertain LMI

$$\begin{bmatrix} \gamma^2 I & QE^1(W_0 + \Delta) \\ (W_0 + \Delta)E^T Q & I \end{bmatrix} \geq 0. \qquad (7)$$

The algebraic connectivity of the graph can be expressed as the following uncertain LMI [4]

$$P^T E(W_0 + \Delta)E^T P > 0, \qquad (8)$$

[1]To simplify notation, we use E instead of $E(\mathcal{G}_c)$ from now on.

with $P = \mathbf{Im}(\mathbf{1}^{\perp})$. Combining equation (7) and (8) leads to the following *robust optimization problem*

$$\min_{w_0} \max_{\|\delta\|_2 \leq 1} \|w_0\|_0 \tag{9a}$$

$$\text{subject to } \begin{bmatrix} \gamma^2 I & QE(W_0+\Delta) \\ (W_0+\Delta)E^T Q & I \end{bmatrix} \geq 0 \tag{9b}$$

$$P^T E(W_0+\Delta)E^T P > 0 \tag{9c}$$

$$w_i \geq 0. \tag{9d}$$

If there is an additional constraint on the maximum weight on each edge, equation (9d) can be replaced by

$$0 \leq w_{0_i} \leq w_{0_i,\max}. \tag{9e}$$

Additionally, one is often not only interested in connectivity of a graph, but in the *maximization* of the connectivity of the graph. Since the $\mathcal{H}_\infty$-norm of the single agents $q_i = \|H_i^{yw}\|_\infty$ can be interpreted as node weights, maximization of the weighted algebraic connectivity of a graph can be formulated as (see [22])

$$\max_{w_0,\mu} \max_{\|\delta\|_2 \leq 1} \mu \tag{10}$$
$$\text{subject to } P^T(E(W_0+\Delta)E^T - \mu Q)P > 0.$$

To achieve a sparse topology while *simultaneously* maximizing the connectivity of the graph, we combine the two objective functions (9a) and (10) to a convex sum

$$\min_{w_0,\mu} \max_{\|\delta\|_2 \leq 1} (1-\alpha)\|w_0\|_0 - \alpha\mu, \quad \alpha \in (0,1) \tag{11a}$$

$$\text{subject to (9)} \tag{11b}$$

The weighting factor $\alpha \in (0,1)$ is a tuning paramter for the relative emphasis on each term in the objective function.

The optimization problem in (11) cannot directly be solved as a semidefinite program, due to two reasons. The 0-norm is a non-convex objective function and the constraints lead to an infinite-dimensional problem. In the following, we will show, how both, the objective function and the constraints can relaxed. For the constraints, a robust counterpart [3] can be formulated. To apply these results, the constraints in (11) must be rewritten in the following form

$$F^j(w,\delta) = F_0^j + \sum_i^{|\mathcal{E}|} \delta_i F_i^j(w), \quad j = 1,2$$

where

$$F_0^1 = \begin{bmatrix} \gamma^2 I & QEW_0 \\ W_0 E^T Q & I \end{bmatrix}$$
$$F_i^1 = \begin{bmatrix} 0 & QEM_i \\ M_i E^T Q & 0 \end{bmatrix}$$
$$[M_{kl}^i] = \begin{cases} 1, & k = l = i \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F_0^2 = P^T(EW_0E^T - \mu Q)P$$
$$F_i^2 = P^T e_i e_i^T P.$$

The above expressions can now be applied to the results in [3] to obtain the following robust counterpart

$$\min_{w,\mu,T^j,S^j} (1-\alpha)\|w_0\|_0 - \alpha\mu, \quad \alpha \in (0,1) \tag{12a}$$

$$\text{subject to } \begin{bmatrix} S^j & F_1^j & \dots & F_{|\mathcal{E}|}^j \\ F_1^j & T^j & & \\ \vdots & & \ddots & \\ F_{|\mathcal{E}|}^j & & & T^j \end{bmatrix} \geq 0, \quad j = 1,2 \tag{12b}$$

$$S^j + T^j \leq 2F_0^j, \quad j = 1,2 \tag{12c}$$

$$w_i \geq 0. \tag{12d}$$

Remark 1: It is also possible to consider $\|\delta\|_\infty \leq 1$, but this would increase the number of decision variables in the semidefinite program (12) even more. Therefore, we follow the slightly stricter assumption of $\|\delta\|_2 \leq 1$.

B. Optimization Algorithm

The minimization of the 0-norm is a non-convex optimization problem and requires a combinatorial search. Therefore, Problem 2 cannot be formulated as a convex optimization problem as easily as Problem 1. Inspired by the field of compressed sensing [2], [6], [9], we show how Problem 2 can be approximated by a numerically tractable convex optimization problem.

With the introduction of the cost function $\|w_0\|_0$, we impose a sparsity requirement on w_{0_i} to design sparse RSNs. While this is a common sense approach, it is of little practical use. The optimization problem is non-convex and NP-hard as its solution requires a combinatorial search which grows faster than polynomial as $|\mathcal{E}|$ grows [5]. It is well known, that ℓ_1-minimization leads to sparse results. This is also motivated by the fact that the 1-norm is the convex envelope of the 0-norm, and therefore its best *convex* relaxation [13]. As described in [7], *re-weighted* ℓ_1-minimization can be used to improve the results of the minimization. In this direction, ℓ_1-weights $m_i > 0$ can be assigned to each edge w_{0_i} as

$$\sum_{i=1}^{n} m_i w_{0_i}.$$

where $m_1, m_2, \dots, m_n$ are non-negative weights. For the described design problem, the ℓ_1-weights are free parameters. They counteract the influence of the signal magnitude on the ℓ_1-penalty function. If $m_i = 1$ for all i, the weighted ℓ_1-norm reduces to the regular ℓ_1-norm. If the ℓ_1-weights m_i are chosen to be inversely proportional to the magnitude of w_i

$$\begin{cases} m_i = 1/|w_i|, & w_i \neq 0 \\ m_i = \infty, & w_i = 0, \end{cases} \tag{13}$$

then the weighted ℓ_1-norm and the ℓ_0-norm coincide.

Additionally, in the context of Problem 2, a certain *a-priori* choice of ℓ_1-weights can be used to force the solution towards certain network topologies. This is especially important if we want to promote certain sub-graphs (e.g. path graphs or star graphs). Assigning a large initial ℓ_1-weight to specific edges has the interpretation that those edges are not desirable, while small ℓ_1-weights make it more likely that those edges appear in the graph.

With this, we can now state the convex optimization problem

$$\max_{\|\delta\|_2 \leq 1} \min_{w_0, \mu} \; (1-\alpha) \sum_{i=1}^{n} m_i w_{0_i} - \alpha \mu, \quad \alpha \in (0,1) \quad (14a)$$

$$\text{subject to constraints in (12)} \quad (14b)$$

The weighting as in (13) cannot be applied directly, since it requires the solution of the optimization problem in (14). Therefore, we propose an optimization algorithm to seek the optimal weights m_i:

Algorithm 1 Sparse Robust Topology Design Algorithm

1) Set $h = 0$ and choose $m_i^{(0)}$ for $i = 1, \ldots, |\mathcal{E}|$ and $\nu > 0$.
2) Solve the minimization problem (14) to find the optimal solution $w_i^{(h)}$.
3) Update the weights
$$m_i^{(h+1)} = (w_i^{(h)} + \nu)^{-1}$$
4) Terminate on convergence, solve the optimization problem (14) with the fixed structure obtained in Step 3. Otherwise set $h = h + 1$ and go to Step 2.

Remark 2: Due to the auxiliary variables introduced by the robust counterpart in (12), the size of the problems grows very fast with the number of nodes. Even though interior-point methods offer polynomial-time algorithms, solving the optimization problem (12) for very large problems might lead to numerical issues.

V. Example

To illustrate the previous results, we design the topology of relative sensing networks with heterogenous agents. The presented algorithm was implemented in Matlab using SeDuMi [24] and Yalmip [17]. First, we consider an RSN with g=10 heterogeneous SISO systems, randomly generated in Matlab with $\|H_i\|_\infty \in [0.17, 7.48]$. Using the optimization algorithm presented in [26], the minimum $\mathcal{H}_\infty-$performance of the RSN is $\gamma_{nom} = 19.2$. The graph with the optimal performance is a complete graph with 45 non-zero edge weights. Next, we allow a slightly larger $\mathcal{H}_\infty$-performance and apply Algorithm 1, with $\mu = 0$ and $w_{\max} = 2$ and four iteration steps. For $\gamma = 19.34$, we can reduce the graph to 34 non-zero edges, and for $\gamma = 19.45$ to 29 edges. The results are also depicted in Fig. 3. Darker lines correspond to larger edge weights (note that the lines are only comparable within on graph, not between different graphs). The number of non-zero edges can be reduced by 35% by only allowing a performance degradation of 1.3%.

The second example shows the tradeoff between sparsity and weighted connectivity. Seven heterogenous SISO systems were randomly generated in Matlab with $\|H_i\|_\infty \in [0.44, 3.88]$. The $\mathcal{H}_\infty$-performance of the RSN was specified as $\gamma = 10$ and for varying α a tradeoff between sparsity and weighted connectivity was computed. As can be seen in Figure 4, for increasing sparsity of the RSN, the weighted connectivity decreases. In Figure 4 each column corresponds to a bar in Figure 5 with the corresponding weighted connectivity. Note, that if we would solve Problem 2 as a combinatorial problem, we would have to check 1.86 Mio possibilities of connected graph topologies. Here, we can clearly see the advantages of the weighted ℓ_1-minimization.

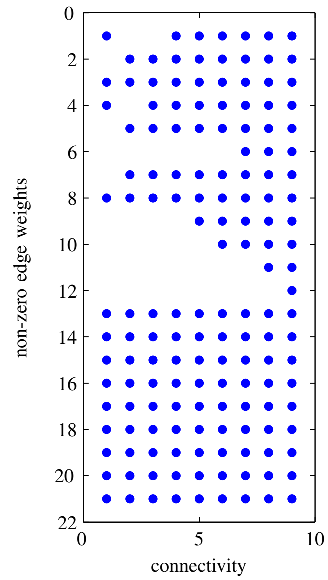


Fig. 4. Non-zero edge weights for increasing connectivity level. Each column depicts the non-zero edge weights for the corresponding connectivity levels in Figure 5.

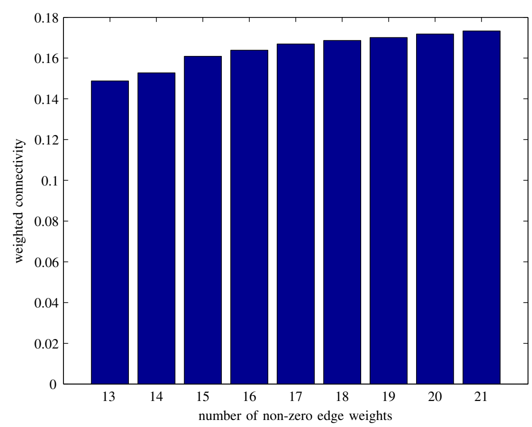
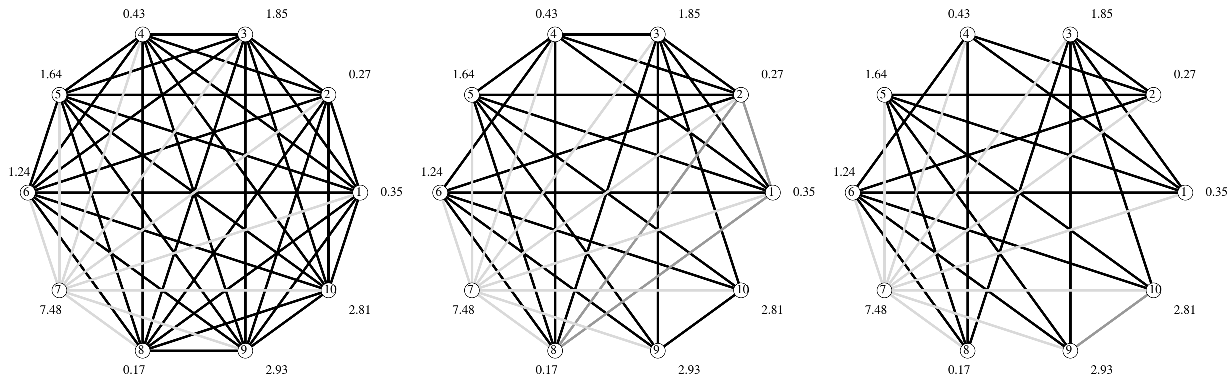


Fig. 5. Number of non-zero edge weights for increasing weighted connectivity for $\gamma = 10$.

VI. Summary and Outlook

This paper is a first step towards the design of sparse robust relative sensing networks subject to an $\mathcal{H}_\infty$-bound on the performance. This problem is closely related to the problem of edge weight design for node and edge weighted



(a) $\gamma = 19.2$, 45 edges (no sparsity constraints).

(b) $\gamma = 19.39$, 34 edges.

(c) $\gamma = 19.45$, 29 edges.

Fig. 3. Increasing sparsity for increasing performance level γ ($w_{\max} = 2$).

graphs. The problem was formulated as an optimization problem with special emphasis on the sparsity of the delivered graphs and robustness against edge weight uncertainties. A robust counterpart was formulated to deal with the uncertain optimization problem. Sparsity of the graph was achieved by 0-minimization of the edge weight vector. For the resulting combinatorial optimization problem, computationally tractable convex relaxations have been provided by means of ℓ_1-minimization. Additional performance criteria such as the maximization of the algebraic connectivity can be embedded into the resulting optimization problem. With the resulting convex optimization problem a tradeoff between sparsity and algebraic connectivity can be achieved while at the same time robustness against edge weight uncertainties is guaranteed.

Future work will try to reduce the additional variables introduced by the robust counterpart to allow for larger problem sizes. Additionally, uncertainties in the agent dynamics will be considered.

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