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Design of Sparse Relative Sensing Networks

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***Abstract*— This paper considers the problem of designing sparse relative sensing networks (RSN) subject to a given $\mathcal{H}_\infty$-performance constraint. The topology design considers homogeneous and heterogeneous agents over weighted graphs. We develop a computationally efficient formulation of the sparse topology design via a convex ℓ_1-relaxation. This makes the proposed algorithm attractive for practical applications. We also demonstrate how this relaxation can be used to embed additional performance criteria, such as maximization of the algebraic connectivity of the RSN.**

***Index Terms*— relative sensing networks, $\mathcal{H}_\infty$-performance, ℓ_0-minimization, topology design**

I. Introduction

The analysis and control of interconnected systems is one of the great challenges of modern engineering science [21]. Most often the single systems are spatially distributed and have an inherently distributed sensing architecture. The ability of a single agent using available sensors to measure state information of the entire network can be limited by spatial constraints such as orientation, range and power requirements [9], [17], [23]. Applications for distributed sensor networks relying on relative sensing range from environmental surveillance, modeling and localization to collaborative information processing [1], [3], [22], [18].

This work focuses on single agents that rely on relative sensing to achieve their mission objectives which are called *relative sensing networks (RSN)*. Relative sensing networks, in their most general form, are a collection of autonomous systems that use sensed relative state information to achieve higher level objectives. Note that this type of model is in contrast to other multi-agent systems where the network coupling is introduced at the state level. In RSNs, a sensing topology (or graph) is induced by the spatial orientation of the agents and the capabilities of the relative sensor. In this way, the underlying sensing topology couples the agents at their outputs and an implicit 'network' is present. Such systems are relevant in formation flying applications where relative sensing is employed to measure inter-agent distances [20], [26]. The exchange of information between each agent in an RSN describes the underlying connection topology. Studying system-theoretic notions from the perspective of the underlying topology can lead to interpretations that explicitly characterize the effects of the network on the behavior of the system. Recent examples of analysis and synthesis from a graph-theoretic point of view studied closed-loop properties of multi-agent systems and the relation to the graph Laplacian [13] or graph-theoretic analysis and performance bounds for consensus systems [8], [28].

The analysis and synthesis of relative sensing networks was recently considered in [29] with respect to $\mathcal{H}_2$ and $\mathcal{H}_\infty$-performance. While there exists strong results for unweighted graphs relating network properties with the $\mathcal{H}_\infty$-performance of a graph, there are no corresponding results for *node- and edge- weighted* graphs. This leads to the main contribution of the current paper. We develop an optimization algorithm for determining the optimal topology for relative sensing networks with pre-specified properties when edge- and node-weights are present. We are especially interested in graphs with a *sparse* topology, i.e. graphs fulfilling the required properties with as few edges as possible. Different scenarios are considered and it is also possible to promote desired sub-graphs.

We combine control theoretic insight with results from compressed sensing (see e.g. [6], [10]) to systematically achieve sparse relative sensing networks with guaranteed system theoretic properties by computationally efficient algorithms. First, a combinatorial optimization problem is proposed. This optimization algorithm seeks a sparse topology guaranteeing maximal algebraic connectivity and pre-specified $\mathcal{H}_\infty$-performance of the network. This non-convex structure optimization is then relaxed using a convex weighted ℓ_1-minimization, and the resulting problem can be tackled by finding a solution of iterative convex optimization problems. Sparsity promoting optimization by ℓ_1-minimization was also recently considered for decentralized controller design in [12], [24].

The remainder of this paper is organized as follows. After introducing the mathematical preliminaries in Section II, the model of the relative sensing network is described in Section III. Section IV is devoted to the synthesis problem of sparse RSNs and an optimization algorithm for RSNs networks is presented. The paper concludes with illustrative examples for homogeneous and heterogeneous RSNs in Section V and a summary and outlook in Section VI.

II. Mathematical Preliminaries

The 0-norm of a vector $x \in \mathbb{R}^n$ is defined as

$$\|x\|_0 = \{\# \ x_i | x_i \neq 0\},$$

and corresponds to the number of non-zero entries in x. Despite not being a true norm, it is often referred to in the literature as one. A vector is called *sparse* if its 0-norm is small compared to the dimension of the vector, i.e., if most

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of its entries are zero. The 0-norm is used in the present context to achieve sparse graph topologies.

Performance is specified in this paper by the $\mathcal{H}_\infty$-norm. The $\mathcal{L}_2$-induced norm (or $\mathcal{L}_2$-gain) of a dynamical system $\mathcal{H} : \mathcal{L}_2^n \to \mathcal{L}_2^m$ is defined as

$$\|\mathcal{H}\|_{\mathcal{L}_2-\text{ind}} = \sup_{w \in \mathcal{L}_2 \setminus \{0\}} \frac{\|\mathcal{H}w\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}},$$

and corresponds for a linear system $\mathcal{H}$ to the $\mathcal{H}_\infty$-norm $\|H(s)\|_\infty = \sup_\omega\{\bar{\sigma}(H(j\omega)\}$, where $H(s) = C(sI - A)^{-1}B + D$ is a transfer function of the dynamical system $\mathcal{H}$ and $\bar{\sigma}(H(j\omega))$ denotes the largest singular value of H at a fixed frequency ω.

Graphs and the matrices associated with them will be widely used in this paper. For an in-depth treatment of graph theory, the reader is referred to [16]. An undirected (simple) graph $\mathcal{G}$ is specified by a vertex set $\mathcal{V}$ and a node set $\mathcal{E}$ whose elements characterize the incidence relation between distinct pairs of $\mathcal{V}$. Two vertices i and j are called *adjacent* (or neighbors) when $\{i, j\} \in \mathcal{E}$. An *orientation* of an undirected graph $\mathcal{G}$ is the assignment of directions to its edges, i.e. an edge e_k is an ordered pair (i, j) such that i and j are, respectively, the initial and the terminal nodes of e_k.

For this work, the $|\mathcal{V}| \times |\mathcal{E}|$ incidence matrix $E(\mathcal{G})$ for a graph with arbitrary orientation is of importance. The incidence matrix is a $\{0, \pm1\}$-matrix with rows and columns indexed by the vertices and edge of $\mathcal{G}$ such that $[E(\mathcal{G})]_{ik}$ has the value '$+1$' if node i is the initial node of edge e_k, '-1' if it is the terminal node, and '0' otherwise.

The (graph) Laplacian of $\mathcal{G}$

$$L(\mathcal{G}) := E(\mathcal{G})E(\mathcal{G})^T$$

is a rank deficient positive semi-definite matrix. The eigenvalues of the graph Laplacian are real and will be ordered and denoted as $0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_{|\mathcal{V}|}(\mathcal{G})$. Furthermore, the eigenvector corresponding to the eigenvalue at zero is $[1 \dots 1]^T$, henceforth written as $\mathbf{1}$. Similar to [25] we define the *node- and edge- weighted graph Laplacian*

$$L_w(\mathcal{G}) := Q^{-1}E(\mathcal{G})WE(\mathcal{G})^T,$$

where Q is a positive and W a non-negative diagonal matrix representing the weights associated to the nodes and edges of the graph, respectively.

Notational specifications as used in the paper are given next. A symmetric and positive definite (resp. positive semi-definite) matrix M is written as $M > 0$ (resp. $M \geq 0$); M^T and M^{-1} denote the transpose and inverse of a matrix M. A diagonal matrix with elements $m_1, \dots, m_n$ on the diagonal is abbreviated as $\text{diag}(m_1, \dots, m_n)$ or simply $\text{diag}(m_i)$. The Kronecker product between two matrices is denoted by $\otimes$.

III. Model of the Relative Sensing Network

This section introduces the model of the RSN considered in this paper and presents the theoretical background used for the optimization algorithm later on. Consider a group of g linear time-invariant dynamical systems (agents)

$$\Sigma_i = \begin{cases} \dot{x}_i(t) &= A_i x_i(t) + B_i w_i(t) \\ y_i(t) &= C_i x(t), \end{cases} \tag{1}$$

where each agent in indexed by the script i. Here, $x_i(t) \in \mathbb{R}^{n_i}$ represents the state, $w_i(t) \in \mathbb{R}^{r_i}$ the exogenous input and $y_i(t) \in \mathbb{R}^{q_i}$ the measured output. We denote the transfer-function representation of Σ_i as H_i with

$$H_i := C_i(sI - A_i)^{-1}B_i. \tag{2}$$

We assume a minimal realization for each agent and compatible output for all agents, e.g. system outputs will correspond to the same physical quantity. In the homogeneous case, it is assumed that each agent in the RSN possesses the same dynamics and is described by the same state-space realization (e.g. $\Sigma_i = \Sigma_j$ for all i, j). It should be noted, that in a heterogeneous RSN, the dimension of each agent can be different; however, using a padding argument', it can be assumed, that all agents have identical dimensions for the respective state space (e.g. $n_i = n_j = n$ for all i, j).

The parallel interconnection of all agents can be expressed by a concatenation of the corresponding system states, inputs, and outputs, and through the block diagonal aggregation of each agent's state-space matrices. We use bold-phase notation to denote the expanded state-space, e.g. $\boldsymbol{x}(t) = [x_1(t)^T, \dots, x_g(t)^T]^T$ and $\boldsymbol{A} = \text{diag}(A_1, \dots, A_g)$.

The sensed output of the RSN is the vector $\boldsymbol{y}_\mathcal{G}(t)$ containing relative state information of each agent and its neighbors and is motivated by the relative sensing problem discussed in Section I. The incidence matrix of a graph naturally captures state differences and will be the algebraic construct used to define the relative outputs of RSNs. In this paper, we will especially consider that the relative output of the RSN corresponds to a relative position' measurement between each agents as

$$\boldsymbol{y}_\mathcal{G}(t) = (WE(\mathcal{G}_c)^T \otimes I)\boldsymbol{y}(t). \tag{3}$$

Here $\mathcal{G}_c$ is the complete graph and the topology is defined by the diagonal weighting matrix $W = \text{diag}(w_1, \dots, w_{|\mathcal{E}|})$, with $w_i \in \mathbb{R}_0^+$ and the node set given as $\mathcal{V} = \{1, \dots, g\}$. Note that edge i does not exist if and only if $w_i = 0$. Furthermore, we denote the vector containing all weighs as $w = [w_1, \dots, w_{|\mathcal{E}|}]^T$. The weights w_i in this setup can be seen as the gains of the sensor used to sense the relative state. They might be used to capture the fidelity of a relative measurement. The global layer is visualized in the block diagram shown in Figure 1.

Using the above notations, we can express the heterogeneous RSN in a compact form

$$\Sigma_{het}(\mathcal{G}) \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{w}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \\ \boldsymbol{y}_\mathcal{G}(t) &= (WE(\mathcal{G}_c)^T \otimes I)\boldsymbol{y}(t). \end{cases} \tag{4}$$

The homogeneous RSN, $\Sigma_{hom}(\mathcal{G})$, can be expressed using Kronecker products, for example $\boldsymbol{A} = I_g \otimes A$. The transfer function representation of Σ_{het} is denoted as $\hat{\Sigma}_{het}$ and is

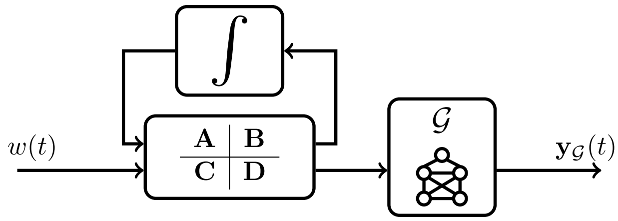


Fig. 1. Global RSN layer block diagram; the feedback connection represents an upper fractional transformation [11].

defined as in (2). As in the state space model, bold faced transfer functions denote the block diagonal aggregation of each agent's corresponding transfer function, e.g. $\boldsymbol{H}(s) = \mathrm{diag}(H_1(s), \ldots, H_g(s))$. The homogeneous system $\hat{\Sigma}_{hom}$ can also be written using Kronecker products in a similar manner as described above. We denote $T_{het}^{w\mapsto\mathcal{G}}$ and $T_{hom}^{w\mapsto\mathcal{G}}$ as the map from exogenous inputs to the RSN sensed output for homogeneous and heterogeneous systems respectively, e.g. $T_{het}^{w\mapsto\mathcal{G}} = (E(\mathcal{G})^T \otimes I_q)\boldsymbol{H}$ and $T_{hom}^{w\mapsto\mathcal{G}} = E(\mathcal{G})^T \otimes H(s)$.

We will now state some facts for homogeneous and heterogeneous RSNs used later on.

Theorem 1 ([29]): The $\mathcal{H}_\infty$ norm of a homogeneous RSN is given as

$$\|T_{hom}^{w\mapsto\mathcal{G}}\|_\infty = \|WE(\mathcal{G})\|\|H\|_\infty.$$

Theorem 1 states that the overall $\mathcal{L}_2$-gain of the system is proportional to the matrix 2-norm of the weighted incidence matrix. For heterogeneous RSNs, only upper bounds can be given.

Theorem 2 ([29]): The $\mathcal{H}_\infty$ norm of a heterogeneous RSN is bounded as

$$\|T_{het}^{w\mapsto\mathcal{G}}\|_\infty \leq \|WE(\mathcal{G})^T Q\|$$

where $Q = \mathrm{diag}(\|H_1\|_\infty, \ldots, \|H_g\|_\infty)$.
Both, Theorem 1 and Theorem 2 show that the topology of the underlying graph is of significant influence to the performance of the relative sensing network. Furthermore, for heterogeneous agents, the dynamic difference between agents is an important factor in the performance of the overall system.

IV. Synthesis of Sparse Relative Sensing Networks

This section presents the main results of the current paper. We discuss different scenarios for the synthesis of homogeneous and heterogeneous relative sensing networks. The special focus is on the design of *sparse* RSNs, i.e. networks that fulfill certain pre-specified properties with as few edges as possible. First, a problem formulation is derived that incorporates the sparsity requirements into the network design. Then, we use a weighted ℓ_1-minimization to relax the numerically exhaustive combinatorial exact solution of the original problem. Graphs with sparse topologies can then be found by the iterative solution of convex optimization problems.

A. Problem Formulation

The synthesis problem for (4), i.e. designing the topology of a RSN can be formulated as follows:

Problem 1 ($\mathcal{H}_\infty$-optimal design of RSN [29]): Given a network consisting of g agents which are coupled as given in (4), find edge weights $w_i \geq 0$, such that the $\mathcal{H}_\infty$-performance of the RSN is optimal, i.e.

$$\begin{aligned} &\text{minimize } \|T_{het/hom}^{w\mapsto\mathcal{G}}\|_\infty \\ &\text{subject to } \mathcal{G} \text{ is connected.} \end{aligned}$$

As shown in [29], this problem can be formulated as a convex optimization problem. However, the solution to this optimization problem is in general not sparse, i.e. all weights w_i are non-zero. This is not a desired solution, since it requires the implementation of a complete graph. Instead, one searches for a graph topology, where most of the edge weights are zero and only very few are non-zero and have to be implemented. Therefore, in the following, we state the problem of sparse relative sensing network design.

Problem 2 (Sparse Topology Design for RSN): Given a network consisting of g agents which are coupled as given in (4) and a predefined $\mathcal{H}_\infty$-performance γ. Find a sparse distribution of weights $w_i \geq 0$, such that the network is connected and the $\mathcal{H}_\infty$-performance is less than γ, i.e.

$$\begin{aligned} &\text{minimize } \|w\|_0 \\ &\text{subject to } \|T_{het/hom}^{w\mapsto\mathcal{G}}\|_\infty < \gamma \\ &\qquad \mathcal{G} \text{ is connected.} \end{aligned} \tag{5}$$

The problem formulation implies that the $\mathcal{H}_\infty$ performance of the relative sensing network does not exceed γ. Recall, that the 0-norm of a vector is a measure of its sparsity. In this way, minimizing $\|w\|_0$ attempts to maximize the number of zero elements in the edge weight vector w and therefore minimizes the number of actually used edge weights.

However, the minimization of the 0-norm is a non-convex problem and requires a combinatorial search (see [5]). Therefore, Problem 2 cannot be formulated as a convex optimization problem as easily as Problem 1. Inspired by the field of compressed sensing [2], [6], [10], we show how Problem 2 can be approximated by a numerically tractable convex optimization problem.

B. Optimization Algorithm

As stated before, to design sparse relative sensing networks, we impose sparsity requirements on the edge weights w_i with $\|w\|_0$. This is a common sense approach which simply seeks the sparsest w satisfying the constraints. However, such an approach is of little practical use, since the optimization problem is non-convex and NP-hard as its solution requires a combinatorial search which grows faster than polynomial as $|\mathcal{E}|$ grows [5]. Similar to the convex relaxation for rank minimization in [15], we will use the convex envelope of $\|w\|_0$ defined next.

Let $f : \mathbb{X} \to \mathbb{R}$, where $\mathbb{X} \subseteq \mathbb{R}^n$. The convex envelope of f (on $\mathbb{X}$) is defined as the point wise largest convex function g such that $g(x) \leq f(x)$ for all $x \in \mathbb{X}$.

Lemma 1 ([14]): The convex envelope of the function $f = \|x\|_0 = \sum_{i=1}^{n} |\text{sign}(x_i)|$ on $\mathbb{X} = \{x \in \mathbb{R}^n | \|x\|_\infty \leq 1\}$ is $f_{\text{env}}(x) = \|x\|_1 = \sum_{i=1}^{n} |x_i|$.

With this, we can relax the non-convex ℓ_0-minimization in (5) by the convex ℓ_1-minimization $\|w\|_1$; note that this can be solved using linear programming. Additionally, this is the best possible convex relaxation since the ℓ_1-norm is the convex envelope of the ℓ_0-norm.

As described in [7], *re-weighted* ℓ_1-minimization can be used to improve the results of the minimization. In this direction, ℓ_1-weights $m_i > 0$ can be assigned to each edge w_i as

$$\sum_{i=1}^{n} m_i w_i.$$

where $m_1, m_2, \ldots, m_n$ are non-negative weights. For the described design problem, the ℓ_1-weights are free parameters. They counteract the influence of the signal magnitude on the ℓ_1-penalty function. If $m_i = 1$ for all i, the weighted ℓ_1-norm reduces to the regular ℓ_1-norm. If the ℓ_1-weights m_i are chosen to be inversely proportional to the magnitude of w_i

$$\begin{cases} m_i = 1/|w_i|, & w_i \neq 0 \\ m_i = \infty, & w_i = 0, \end{cases} \tag{6}$$

then the weighted ℓ_1-norm and the ℓ_0-norm coincide.

Additionally, in the context of Problem 2, a certain *a-priori* choice of ℓ_1-weights can be used to force the solution towards certain network topologies. This is especially important if we want to promote certain sub-graphs (e.g. path graphs or star graphs). Assigning a large initial ℓ_1-weight to specific edges has the interpretation that those edges are not desirable, while small ℓ_1-weights make it more likely that those edges appear in the graph.

As show in [29] Theorem 2 can be formulated into an LMI

$$\begin{bmatrix} \gamma^2 I & QE(\mathcal{G})W \\ WE(\mathcal{G})^T Q & I \end{bmatrix} \geq 0. \tag{7}$$

The algebraic connectivity of the graph can be expressed as the following LMI [4]

$$P^T EWE^T P > 0, \tag{8}$$

with $P = \mathbf{Im}(\mathbf{1}^\perp)$. Combining equation (7) and (8) with the relaxations of the 0-norm derived in (6) leads to the following convex optimization problem

$$\text{minimize} \quad \sum_{i=1}^{n} m_i w_i \tag{9a}$$

$$\text{subject to} \quad \begin{bmatrix} \gamma^2 I & QE(\mathcal{G})W \\ WE(\mathcal{G})^T Q & I \end{bmatrix} \geq 0 \tag{9b}$$

$$P^T EWE^T P > 0 \tag{9c}$$

$$w_i \geq 0. \tag{9d}$$

If there is an additional constraint on the maximum weight on each edge, equation (9d) can be replaced by

$$0 \leq w_i \leq w_{i,\max}. \tag{9e}$$

Additionally, one is often not only interested in connectivity of a graph, but in the *maximization* of the connectivity of the graph. Since the $\mathcal{H}_\infty$-norm of the single agents $q_i = \|H_i^{yw}\|_\infty$ can be interpreted as node weights, maximization of the weighted algebraic connectivity of a graph can be formulated as (see [25]).

$$\begin{aligned} &\text{maximize } \mu \\ &\text{subject to } P^T(EWE^T - \mu Q)P > 0. \end{aligned} \tag{10}$$

Note that this definition slightly differs from the classical definition of the connectivity and is associated with the node- and edge- weighted graph Laplacian. To achieve a sparse topology while *simultaneously* maximizing the weighted connectivity of the graph, we combine the two objective functions (9a) and (10) to a convex sum

$$\text{minimize } (1-\alpha)\sum_{i=1}^{n} m_i w_i - \alpha\mu, \quad \alpha \in (0,1) \tag{11a}$$

$$\text{subject to } \begin{bmatrix} \gamma^2 I & QE(\mathcal{G})W \\ WE(\mathcal{G})^T Q & I \end{bmatrix} \geq 0 \tag{11b}$$

$$P^T(EWE^T - \mu Q)P > 0 \tag{11c}$$

$$w_i \geq 0. \tag{11d}$$

The weighting factor $\alpha \in [0, 1]$ is a tuning parameter for the relative emphasis on each term in the objective function. The weighting scheme described in (6) cannot be implemented directly. Therefore, we use the algorithm according to [7].

Algorithm 1 Sparse Topology Design Algorithm

1) Set $h = 0$ and choose $m_i^{(0)}$ for $i = 1, \ldots, |\mathcal{E}|$ and $\nu > 0$.
2) Solve the minimization problem (9) (or (11)) to find the optimal solution $w_i^{(h)}$.
3) Update the weights
$$m_i^{(h+1)} = (w_i^{(h)} + \nu)^{-1}$$
4) Terminate on convergence, otherwise set $h = h + 1$ and go to Step 2.

Remark 1: We introduce a small positive number ν to ensure that all weights are well defined when $m_i^{(h)} = 0$.

Remark 2: As discussed earlier, the first step of the algorithm, e.g. the initial choice of ℓ_1 weights, greatly influences the solution. From a design standpoint, this is preferable, since the initial weights themselves can implicitly include additional constraints such as the economic cost of adding a certain edge. The authors further noticed that the influence of the initial ℓ_1-weights is larger when homogeneous RSNs are considered, while for heterogeneous RSNs the node weights seem to be more important.

Algorithm 1 provides a computationally tractable solution to the original problem of sparse relative sensing network design proposed in Problem 2. The exhaustive combinatorial search of the 0-norm to achieve a sparse structure of the controller was relaxed by the computationally attractive weighted ℓ_1-minimization. In the next section, we will apply our results to an illustrative example.

V. Example

To illustrate the previous results, we design the topology of homogeneous and heterogeneous agents. Algorithm 1 was solved using SeDuMi [27] and YALMIP [19] in Matlab. First, we consider a network of $g = 10$ homogeneous agents. In this example, we show how an initial choice of ℓ_1-weights influences the topology of the graph and therefore, how a sub-graph can be promoted. Here, we wanted to promote a path graph as a sub-graph. Therefore, the initial ℓ_1-weights of the edges associated with a path were chosen to $m_i = 1e^{-4}$, while other initial ℓ_1-weights were set to $m_j = 1$. The edges corresponding to the path graph are the edges 1, 3, 6, 10, 15, 21, 28, 36 and 45. The predefined $\mathcal{H}_\infty$-performance was set to $\gamma = 10$. Figure 2 shows the number of non-zero weights and their corresponding weighted connectivity. As can be seen, for decreasing sparsity, the weighted connectivity is increasing. In Figure 3 each column corresponds to a bar in Figure 2 with the corresponding weighted connectivity. As can be seen, the path as a subgraph is present for all weighted connectivity levels, while for increasing weighted connectivity, additional edges are added.

As a second example, we consider an RSN with $g = 10$ heterogeneous SISO systems (generated randomly in MATLAB) with $\mathcal{H}_\infty$-performance $\|H_i\|_\infty \in [0.17, 7.48]$. The $\mathcal{H}_\infty$-performance of the RSN was specified as $\gamma = 10$ and for varying α a tradeoff between sparsity and weighted connectivity was computed. As can be seen in Figure 4, for increasing sparsity of the RSN, the weighted connectivity decreases. Furthermore, compared to the tree (nine edges), the weighed connectivity increases by more than 100%, when allowing 16 edges instead, while there is almost no improvement of weighted connectivity when allowing 37 edges instead of only 23. Figure 5(a), 5(b), 5(c) show the graphs for weighted connectivity level of 0.21, 0.48 and 0.84, respectively. The numbers beside the nodes correspond to the $\mathcal{H}_\infty$-norm of the single agent and represent the node weight. The darker the edge, the higher the edge weight. Note that the color of the edges only relate within one figure and are not comparable between figures. As can be seen in Figure 5(a), the tree is actually a star graph where the node with the highest node weight is the center node. For increasing weighted connectivity (see Figure 5(b) and Figure 5(c)), the edges connecting the next larger nodes are added.

VI. Summary and Outlook

This paper considers the design of sparse relative sensing networks subject to an $\mathcal{H}_\infty$-bound on the performance. This problem is closely related to the problem of edge weight design for node and edge weighted graphs. While there exist theoretical results for $\mathcal{H}_\infty$-performance and connectivity for certain topologies of unweighted graphs, such results do not exist for weighted graphs. To overcome this, the synthesis problem was formulated in terms of an optimization

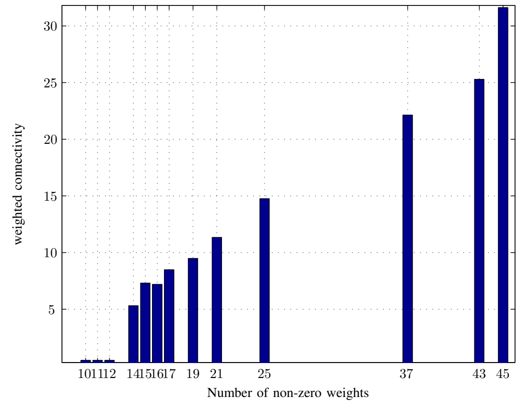


Fig. 2. Number of non-zero edges for homogeneous RSN for increasing connectivity with $\gamma = 10$.

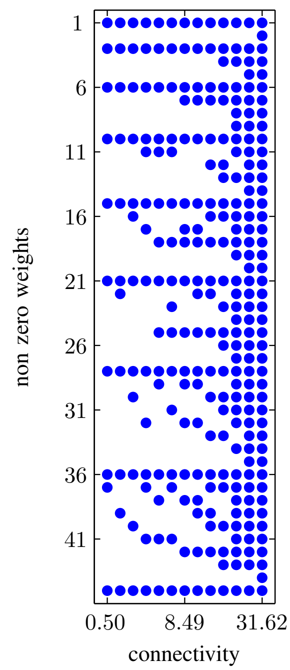


Fig. 3. Non-zero edges for homogeneous RSN for increasing connectivity with $\gamma = 10$.

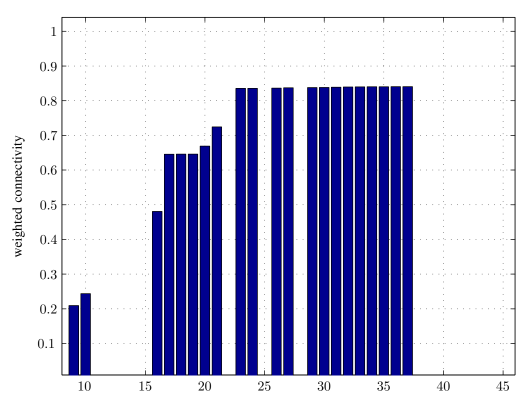
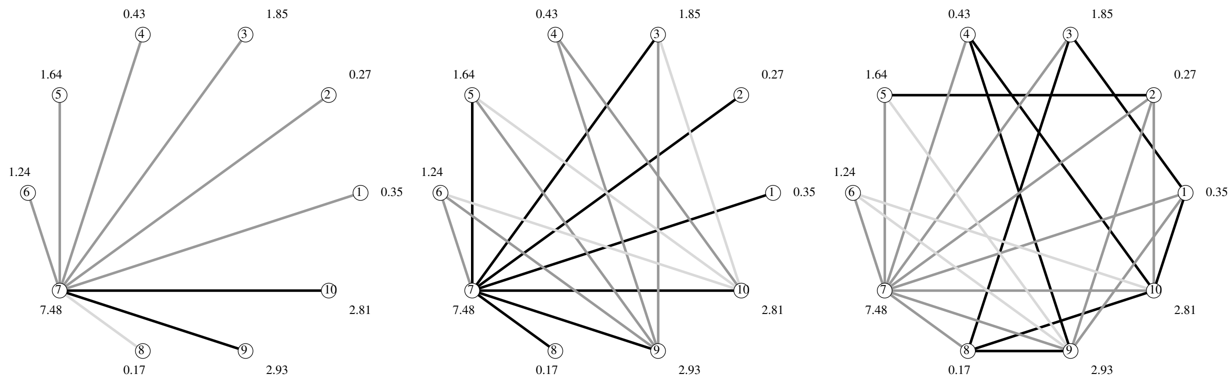


Fig. 4. Number of non-zero edges for heterogeneous RSN for increasing connectivity with $\gamma = 10$..



(a) 9 edges, $\mu = 0.21$, $0.19 \le w_i \le, 0.78$. (b) 16 edges, $\mu = 0.48$, $0.13 \le w_i \le 0.65$. (c) 23 edges , $\mu = 0.84$, $0.29 \le w_i \le 1.53$.

Fig. 5. Graphs with increasing number of edges and increasing connectivity wiht $\gamma = 10$.

problem. Special emphasis was put on the sparsity of the delivered graphs, i.e. graphs with as few edges as possible that fulfill certain pre-specified properties. Sparsity of the graph was achieved by 0-minimization of the edge weight vector. For the resulting combinatorial optimization problem, computationally tractable convex relaxations have been provided. The provided relaxation can also be used to embed additional performance criteria, such as the maximization of the algebraic connectivity of the relative sensing network. With the resulting convex optimization problem a tradeoff between sparsity and algebraic connectivity can be achieved. Future work considers the design of robust relative sensing networks, with possible uncertainties in the edge weights as well as in the node weights.

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