Robustness of Consensus over Weighted Digraphs

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Abstract—This paper investigates the robustness of consensus protocols over weighted directed graphs using the Nyquist criterion and small gain theorem for agents with single and double integrator dynamics. For single integrators, the linear consensus protocol, described by the weighted Laplacian, is considered, while for double integrators a new consensus protocol is presented which also uses the weighted Laplacian. For both single and double integrators, the allowable bound on a single edge weight perturbation, while consensus among the agents can be achieved, is derived. Specific results are obtained for a directed acyclic graph and the directed cycle graph along with their graph theoretic interpretations. For double integrators, a dual problem is formulated and solved, whereby it is shown that, subject to certain conditions, perturbing a single edge weight may stabilize the consensus protocol. Simulations support the theoretical results.

Index Terms—Consensus protocol, algebraic graph theory, uncertainty, stability analysis, robustness

1 Introduction

In recent times, a number of researchers have looked into networked systems whose underlying graphs contain negative edge weights [1], [2], [3], [4], [5], [6]. These could be in the context of finding an optimal solution to obtain the fastest converging linear iteration in distributed averaging, in studying the phenomenon of clustering, or to model antagonistic interactions in a social network. In this work the presence of negative weights is investigated in the context of the consensus problem over weighted directed graphs. Although negative couplings were considered in [3], the focus was on achieving clustering behavior by building networks from subnetworks that were balanced and strongly connected. However, this paper focuses on the effect of negative edge weights on a consensus-seeking system and clustering occurs as a transitional behavior between consensus and lack thereof, in the system, at a critical value of the edge weight.

Consensus protocol occupies an important position in the domain of multi-agent systems and has been investigated from various perspectives [7]. One direction of investigation has been the robustness of consensus over weighted undirected graphs [8], [9] where concepts from graph theory and robust control have been merged. These analyses involve the application of small gain theorem to the networked dynamic system described by the graph Laplacian and the edge Laplacian matrices. Particularly, [9] considered the possibility of negative edge weights. Further, an interpretation of the allowable perturbation on an edge weight was provided in terms of *equivalent graph resistances*. Reference [8] showed how the edge Laplacian assisted in studying the roles of certain subgraphs such as cycles and spanning trees, in the agreement problem, and laid the foundations for robustness studies of the consensus problem over undirected graphs. For undirected graphs, whose Laplacians are symmetric and therefore have special properties, such an analysis is feasible. But for a directed graph, the Laplacians are not symmetric in general.

In works such as [10] and [11] the effects of delays and dynamic uncertainties in the communication channel have been investigated, but as in most of the related literature, the focus has been on undirected topologies. Although the present work also relies on a Nyquist based approach for robustness studies over directed networks, [10] uses Nyquist conditions to analyze the effects of time delays in an undirected network. While the Integral Quadratic Constraint (IQC) does provide a broad framework for the study of uncertainties, [10] does not provide any graph theoretic interpretation. In [11], the authors conclude that their methods of analysis, being reliant on the symmetry of the undirected network, cannot be extended to directed networks. The present work, in contrast, deals with consensus over directed graphs. Some researchers have looked into the robustness of linear consensus over uncertain networks [11] for discrete time systems while others have extended the notion of effective graph resistances to directed graphs [12], [13]. However, it has been shown that the notion of graph resistance cannot be exploited in interpreting perturbation bounds on edge weights of general digraphs [14].

Although most of the literature related to consensus considers agents modeled as single integrators, some researchers have also considered agents modeled as double integrators. Some relevant stability results on consensus of double integrators can be found in [15], [16], [17], [18], [19], [20], [21], but they mostly consider undirected topology and

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do not consider perturbations on edge weights. As remarked in [19], in many practical cases the agents such as aircrafts or robots are described by double integrator dynamics. Hence, the study of consensus in a system of double integrators is of significant interest from a practical perspective. While the consensus over single integrators depends directly on the spectrum of the graph Laplacian matrix, for double integrators too, several consensus protocols may be considered which require explicit knowledge of the graph Laplacian spectrum. But merely having the eigenvalues of the Laplacian in the right half of the complex plane (rhp) may not necessarily guarantee consensus of double integrators [16].

The contributions of this paper can be summarized as follows. The networked system is transformed to edge variables, which denote the difference between the node states, leading to a directed *edge agreement protocol* over weighted digraphs. The directed edge Laplacian matrix [22], [23], [24] is introduced along with its algebraic properties. Second, the robust stability of directed and weighted edge agreement protocol, where uncertainty is introduced in the form of perturbations to edge weights, is considered. Third, this work reveals that even in the absence of symmetric Laplacians, robust stability analysis is possible for single or double integrators over a weighted directed graph. Fourthly, the robust stability result for single integrators over a weighted digraph is derived using the Nyquist criteria. Further analysis, along with graph-theoretic interpretations, are given for two specific classes of graphs: the directed acyclic graph (DAG), and the directed cycle graph. For the directed cycle, the perturbation bound on an edge weight agrees with the one in the literature [25], [26]. Some preliminary results are available in [27], although many proofs, and the complete analysis for the cycle digraph are not presented there. This work also discusses how the analysis of an edge weight perturbation on a consensus-seeking system aids in determining whether multiple edge weight perturbations of known magnitudes disrupt consensus or not. Fifth, a new consensus protocol for double integrators is presented and analyzed. An existing protocol [16] is a special case of this new protocol. Finally, as a dual to the analysis problem, the possibility of designing edge weights of a consensus protocols for double integrators is discussed. Consensus for double integrators is shown to be achievable over a digraph by perturbing a suitably chosen edge weight sufficiently, when the nominal system does not attain consensus. Although transparent graph theoretic interpretations for edge weight perturbations are hard to obtain for double integrators, except for a DAG, this work provides a framework for analyzing robustness of consensus protocols.

Section 2 presents some mathematical background and commonly used notation and then the edge Laplacian is described for a weighted directed graph along with some of its important properties. Thereafter, the consensus protocols for single and double integrators are presented. The corresponding consensus models with uncertainty are next presented in Section 3. The robust stability of the uncertain edge protocol over a weighted digraph is analyzed in Section 4 for the presented protocols, and graph theoretic interpretations are provided for single integrators over two specific digraphs along with an algorithm to check if consensus fails for multiple edge weight perturbations. In Section 5, a method for designing edge weights for consensus of double integrators is illustrated. Section 6 presents relevant simulations and Section 7 concludes the paper.

2 Digraphs and Consensus Protocols

2.1 Preliminaries

Some notions related to digraphs are reviewed in this section, followed by some commonly used notation. Many of these graph theoretic concepts can be found in [28]. A directed graph, $\mathcal{G}$, consists of a vertex set, V, an edge set, $\mathcal{E}$, which is an ordered pair of distinct vertices of $\mathcal{G}$, and a diagonal matrix of edge weights, W. When the weights are all unity, the graph is represented by V and $\mathcal{E}$ only. Throughout this paper, it is assumed that $|V| = n$ and $|\mathcal{E}| = m$. Two edges that are outbound from the same node (parent node) are defined as *sibling edges*. A node $v \in V$ that can be reached by a directed path from every other node in a digraph $\mathcal{G}$ is termed a *globally reachable node*. For any graph containing at least one globally reachable node, one can define a spanning subgraph $\mathcal{G}_\tau \subseteq \mathcal{G}$ termed a *rooted in-branching*, such that there exists a directed path from every node to a globally reachable node (or *root*), and all nodes, except this root, have out-degree equal to unity, while the root has out-degree equal to zero in $\mathcal{G}_\tau$. For a digraph with a rooted in-branching, another subgraph, $\mathcal{G}_c$, with the same vertex set, V, as $\mathcal{G}$, is defined such that $\mathcal{G}_\tau \cup \mathcal{G}_c = \mathcal{G}$. The rooted in-branching, $\mathcal{G}_\tau$, has $n-1$ directed edges in the edge set, $\mathcal{E}_\tau$, while the remaining $m-n+1$ edges constitute the edge set $\mathcal{E}_c$ corresponding to $\mathcal{G}_c$ (with $\mathcal{E} = \mathcal{E}_\tau \cup \mathcal{E}_c$ and $\mathcal{E}_\tau \cap \mathcal{E}_c = \emptyset$). A *signed path vector* corresponding to an edge $e_i \in \mathcal{E}_c$ between nodes a and b in $\mathcal{G}$ is a signed path in $\mathcal{G}_\tau$ represented as a vector $z \in \mathbb{R}^{n-1}$ such that the jth index of z takes the value $+1$ if the edge $e_j \in \mathcal{E}_\tau$ is traversed positively, -1 if it is traversed negatively and 0 if the edge e_j is not used in the path. The graph and edge Laplacian matrices are defined in terms of the incidence matrix, $E(\mathcal{G}) \in \mathbb{R}^{n\times m}$. The incidence matrix is defined such that $[E(\mathcal{G})]_{ij} = 1$ if edge e_j is outgoing from vertex i, $[E(\mathcal{G})]_{ij} = -1$ if edge e_j is incoming at vertex i, and $[E(\mathcal{G})]_{ij} = 0$ otherwise. The graph Laplacian for a directed graph can be defined as $L_g = \mathcal{A}(\mathcal{G})E(\mathcal{G})^T$, where $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{n\times m}$ is such that $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ if the edge e_j is outgoing from vertex i and is 0 otherwise [22], [23], [24]. Similarly, $L_e = E(\mathcal{G})^T\mathcal{A}(\mathcal{G})$ is defined as the *directed edge Laplacian*. Matrices $E(\mathcal{G})$ and $\mathcal{A}(\mathcal{G})$ are also denoted as E and $\mathcal{A}$ for brevity. The weighted graph Laplacian and the edge Laplacian are given by $\bar{L}_g = \mathcal{A}(\mathcal{G})WE(\mathcal{G})^T$ and $\bar{L}_e = E(\mathcal{G})^T\mathcal{A}(\mathcal{G})W$, respectively, where, $W \in \mathbb{R}^{m\times m}$ is a diagonal matrix, whose diagonal entries are the weights of the corresponding edges, that is $W_{ii} = w_i > 0\ \forall i$. The null space and range space of a matrix A are denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. The vector of all-ones and all-zeros in $\mathbb{R}^p$ are denoted by $\mathbf{1}_p$ and $\mathbf{0}_p$, respectively. The matrix obtained by removing the jth column and ith row of any matrix A is denoted as $A_{(i,j)}$.

2.2 The Directed Edge Laplacian: Algebraic Properties

The edge Laplacian plays an important role in consensus-seeking systems over undirected graphs [8]. In [27] certain useful properties of the edge Laplacian were stated without proof and used gainfully to analyze the robustness of single integrators seeking consensus over a weighted digraph. These properties are occasionally discussed here, while their formal statements and proofs are in the Appendix, which can be found on the Computer Society Digital Library at http://doi.ieeecomputersociety.org/10.1109/TNSE.2018.2866780. It will be shown in Section 4 that the edge Laplacian for directed graphs provides the correct algebraic construction to analyze the robustness of consensus protocols over digraphs for both single and double integrators. The following observation is relevant before some important properties of $\bar{L}_e$ are discussed. For a nonsingular W, $\mathbf{dim}[\mathcal{N}(\mathcal{A})] = \mathbf{dim}[\mathcal{N}(\mathcal{A}W)]$ and $\mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{A}W)$. Some key properties of $\bar{L}_e$ and $\bar{L}_g$ are in the Appendix, in Lemmas A1 through A4, available in the online supplemental material. From Lemma A1, available in the online supplemental material, it follows that if $\mathbf{1_n} \in \mathcal{R}(\mathcal{A})$, then $\mathcal{N}(\bar{L}_e) \neq \mathcal{N}(\mathcal{A}W)$. By Lemma A4, available in the online supplemental material, a digraph with multiple globally reachable nodes has a directed cycle among the globally reachable nodes. Hence, every node has a positive out-degree. Combining Lemmas A1 and A4, available in the online supplemental material, $\mathcal{N}(\bar{L}_e) \neq \mathcal{N}(\mathcal{A}W)$ for cycle digraphs. This paper considers only digraphs with at least a single globally reachable node, because consensus is achievable only in such digraphs [29]. To understand the graph theoretic relation between the edges in $\mathcal{G}_\tau$ and $\mathcal{G}_c$ and to characterize the latter in terms of the former, the matrix $E(\mathcal{G})$, and $\mathcal{A}(\mathcal{G})$ for some special graphs, can be factorized in certain forms. These factorizations aid in the subsequent analysis in Section 4 and are therefore presented here. The following labeling helps in the subsequent factorizations.

Suppose that for a particular $\mathcal{G}_\tau$, the edges in $\mathcal{E}_\tau$ are labeled e_1 through e_{n-1} while their corresponding parent nodes are labeled 1 through $n-1$, respectively. Thus, the node with zero out-degree in $\mathcal{G}_\tau$, corresponding to a globally reachable node in $\mathcal{G}$, is labeled n. The incidence matrix is

$$E(\mathcal{G}) = [E(\mathcal{G}_\tau)\ E(\mathcal{G}_c)] = E(\mathcal{G}_\tau)[I_{n-1}\ T_\tau] = E(\mathcal{G}_\tau)R, \quad (1)$$

where $T_\tau \in \mathbb{R}^{(n-1)\times(m-n+1)}$ may be given by

$$T_\tau = (E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{-1} E(\mathcal{G}_\tau)^T E(\mathcal{G}_c). \quad (2)$$

Matrices $E(\mathcal{G}_\tau)$ and $E(\mathcal{G}_c)$ capture the incidence relations in $\mathcal{G}_\tau$ and $\mathcal{G}_c$, respectively. The matrix $E(\mathcal{G}_\tau)^T \in \mathbb{R}^{(n-1)\times n}$ has full row rank and so its right inverse, $E(\mathcal{G}_\tau)(E(\mathcal{G}_\tau)^T\ E(\mathcal{G}_\tau))^{-1}$, exists. Similarly, for a directed acyclic graph, $\mathcal{G}$, that has a unique globally reachable node so that every edge in $\mathcal{E}_c$ has a sibling in $\mathcal{E}_\tau$ (implying $\mathcal{R}(\mathcal{A}(\mathcal{G}_c)) \subseteq \mathcal{R}(\mathcal{A}(\mathcal{G}_\tau))$),

$$\mathcal{A}(\mathcal{G}) = [\mathcal{A}(\mathcal{G}_\tau)\ \mathcal{A}(\mathcal{G}_c)] = \mathcal{A}(\mathcal{G}_\tau)[I_{n-1}\ \tilde{T}_\tau] = \mathcal{A}(\mathcal{G}_\tau)\tilde{R}, \quad (3)$$

where $\tilde{T}_\tau \in \mathbb{R}^{(n-1)\times(m-n+1)}$, is given by

$$\tilde{T}_\tau = (\mathcal{A}(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}_\tau))^{-1} \mathcal{A}(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}_c). \quad (4)$$

The first $n-1$ columns of $\tilde{R}(\mathcal{G})$ and $R(\mathcal{G})$, corresponding to the edges in $\mathcal{E}_\tau$, contain the identity matrix. For R, the last $m-n+1$ columns (i.e., the columns of the matrix T_τ) show how the $m-n+1$ edges in $\mathcal{E}_c$ are represented in terms of the $n-1$ edges in $\mathcal{E}_\tau$ by a signed path vector [30], as shown in Fig. 1. Denote the ith columns of $\tilde{R}$ and R as $\tilde{r}_i$ and r_i, respectively, with $r_i(k)$ being the kth entry of the vector r_i. In Fig. 1, $e_8 \in \mathcal{E}_c$ is encoded in terms of e_2, e_3, e_6, $e_5 \in \mathcal{E}_\tau$. The corresponding entries in $r_8 \in \mathbb{R}^7$, are non-zero with the sign indicating the sense in which these edges are traversed ($r_8(2) = r_8(3) = +1$, and $r_8(6) = r_8(5) = -1$) while other entries are zero. Since every edge $e_q \in \mathcal{E}_c$ is a sibling to an edge $e_r \in \mathcal{E}_\tau$, the column in $\tilde{R}$ corresponding to e_q will be identical to the that corresponding to its sibling edge, e_r. So, in Fig. 1, $\tilde{r}_8 = \tilde{r}_2$. Hence, with this labeling, for $n \le i \le m$, there exists some j satisfying $1 \le j \le n-1$, such that $\tilde{r}_i = \tilde{r}_j$, where, edge $e_i \in \mathcal{E}_c$ and edge $e_j \in \mathcal{E}_\tau$ are sibling edges. Moreover,

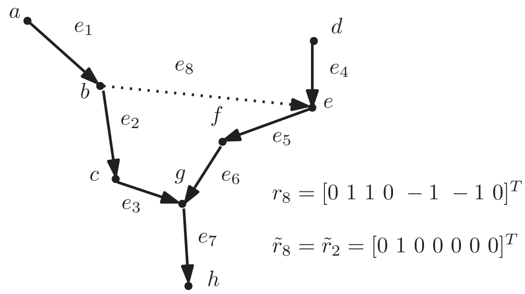


Fig. 1. Dotted edge e_8 (sibling to edge e_2, with parent node b) encoded in terms of the edges in the rooted in-branching.

$$r_i(k) = \begin{cases} +1, & \text{if } e_k \text{ is traveled in the} + \text{sense},\\ -1, & \text{if } e_k \text{ is traveled in the} - \text{sense},\\ 0, & \text{if } e_k \text{ is not traversed,} \end{cases} \quad (5)$$

is the signed path for e_i, for $n-1 < i \le m$. Using these factorizations, Lemmas A5 through A9, available in the online supplemental material, present some key relations between $\bar{L}_e$ and $\bar{L}_g$ which aid later analyses.

2.3 Consensus Protocols

Consensus protocols for single and double integrators will be considered here. Thereafter, the uncertain models will be presented for each of these cases.

2.3.1 Single Integrators

Consider the consensus dynamics over a weighted digraph driven by

$$\dot{x} = -\bar{L}_g x, \quad (6)$$

where $x \in \mathbb{R}^n$ denotes the node states while $\bar{L}_g$ is the weighted graph Laplacian. To achieve consensus, the spectrum of $\bar{L}_g$ must contain exactly one zero eigenvalue with the remaining eigenvalues in the rhp. This is a necessary and sufficient condition and leads to some conditions on the underlying digraph in graph theoretic terms.

2.3.2 Double Integrators

A Generic Protocol. The consensus protocol for double integrators is given by

$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}0 & I_n\\ -\bar{L}_{gx} & -\bar{L}_{gv}\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix} = \bar{\Psi}\begin{bmatrix}x\\ v\end{bmatrix}, \quad (7)$$

where $\bar{L}_{gx} = \mathcal{A}(\mathcal{G})W_x E(\mathcal{G})^T$, and $\bar{L}_{gv} = \mathcal{A}(\mathcal{G})W_v E(\mathcal{G})^T$ with W_x and W_v being the diagonal edge weight matrices for the position and velocity digraphs, respectively. Here, the system is required to achieve consensus in both position and velocity. In the existing consensus protocols, the weighted digraph over which the position and velocity are communicated are assumed to be identical with corresponding weights on the edges being scaled versions of each other. However, in this work a more generic consensus protocol is considered for the double integrators. Here, the edges and their directions in the position and velocity digraphs are identical, but the weights on the individual edges are different for position and velocity digraphs. Only a detailed spectral analysis of $\bar{\Psi}$ can determine the non-zero eigenvalues of this consensus protocol. Even if both the position and velocity digraphs have positive weights, consensus is not guaranteed since the relation between the spectra of $\bar{L}_{gx}$, $\bar{L}_{gv}$, and $\bar{\Psi}$ is not apparent. However, for a weighted digraph with a rooted in-branching and positive weights, similar arguments as in Lemma 4.1 of [16] dictate that algebraic and geometric multiplicities of the zero eigenvalue are 2 and 1, respectively. Thus, if the $2n-2$ non-zero eigenvalues of $\bar{\Psi}$ are in the open rhp, consensus results. This is formally stated below.

Theorem 1. *The system of double integrator agents over a weighted digraph having positive edge weights, and a rooted in branching, given by (7), achieves consensus if and only if the polynomial equation* $\det[s^2 I + s\bar{L}_{gv} + \bar{L}_{gx}] = 0$ *has all nonzero roots with negative real parts.*

This protocol offers more flexibility to the designer, since it allows one to choose $2m$ decision variables (edge weights) to assign the $2n-2$ non-zero eigenvalues of $\bar{\Psi}$.

Protocol for $W_v = \gamma W_x = \gamma W$. This is a special case of (7) and is given by:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -\bar{L}_g & -\gamma \bar{L}_g \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \Psi \begin{bmatrix} x \\ v \end{bmatrix}, \tag{8}$$

where $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ are the position and velocity vectors of the agents, $\bar{L}_g$ is the weighted graph Laplacian and $\gamma \in \mathbb{R}_+$ is a positive scalar. This model appears most commonly in the literature dealing with consensus of double integrators since it was presented in [16]. The edge weights in the position and velocity digraphs are such that the weight on each edge of the velocity digraph is a scaled version (the scaling factor being γ) of the corresponding weight on the position digraph. There are $m+1$ decision variables: m edge weights and a damping term, γ.

The consensus of the system (8) depends on the properties of $\Psi \in \mathbb{R}^{n \times n}$. If $\bar{L}_g$ has exactly one zero eigenvalue, then Ψ has exactly two eigenvalues at the origin, but the geometric multiplicity of this eigenvalue is unity, with corresponding eigenvector being $[\mathbf{1_n}^T \; \mathbf{0_n}^T]$ (Lemma 4.1, [16]). Moreover, if all the remaining eigenvalues of $-\bar{L}_g$ are in the open left half plane (lhp), then there is a lower bound on γ that is guaranteed to result in consensus. The following is a necessary and sufficient condition for consensus [31]:

$$\gamma^2 > \max_{Re(\mu_i)>0} \frac{[Im(\mu_i)]^2}{|Re(\mu_i)|([Im(\mu_i)]^2 + [Re(\mu_i)]^2)}, \tag{9}$$

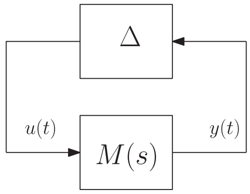


Fig. 2. Uncertain consensus protocol.

where μ_i is an eigenvalue of $\bar{L}_g$. The parameter γ may be chosen suitably with explicit knowledge of the spectrum of $\bar{L}_g$.

3 Uncertain Edge Weights

The notion of uncertainty is introduced through the edge weights in the consensus protocols. Suppose the weights on one of the m edges (or $2m$ edges for double integrators) is uncertain. The perturbations are real, and bounded about some nominal positive value. Mostly, perturbations on a single edge weight are considered in this work. The uncertainty on edge weight w_i is an additive one, $\delta_i < 0$, incorporated as $w_i + \delta_i$. The uncertainty is thus defined as

$$\Delta = \delta_i < 0, \; |\delta_i| \leq \bar{\delta} < \infty. \tag{10}$$

The perturbed model is expressed in M-Δ form as in Fig. 2.

3.1 Single Integrators

Upon pre-multiplying both sides of (6) by $E(\mathcal{G})^T$, it follows that $\dot{x}_e = -\bar{L}_e x_e$ where, $x_e = E(\mathcal{G})^T x = R^T E(\mathcal{G}_\tau)^T x \in \mathbb{R}^m$ denotes the edge states. Choosing a transformation $z = V^{-1} x_e$, where V is as described in the proof of Lemma A9, available in the online supplemental material, it turns out that $z = [((RR^T)^{-1}R)^T \; N_\tau]^T R^T E(\mathcal{G}_\tau)^T x = [x^T E(\mathcal{G}_\tau)\mathbf{0}^T_{m-n+1}]^T$. Thus, the first $n-1$ components of z represent the edge states of the rooted in-branching. Lemma A9, available in the online supplemental material, suggests that for edge agreement, it suffices to consider the edge dynamics in the rooted in-branching, say x_τ,

$$\dot{x}_\tau = -E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W R^T x_\tau. \tag{11}$$

Uncertain edge agreement protocol for single integrators is

$$\dot{x}_\tau = -E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G})(W + P\Delta P^T) R^T x_\tau, \tag{12}$$

with uncertainty as in (10) and $P \in \mathbb{R}^m$ is the ith standard basis in $\mathbb{R}^m$ if the weight on edge e_i is perturbed. Considering u as input and y as output, the system is given by

$$\dot{x}_\tau = -E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W R^T x_\tau - E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) P u \tag{13}$$

$$y = P^T R^T x_\tau, \;\; u = \Delta P^T R^T x_\tau. \tag{14}$$

This closed loop model is depicted in Fig. 2 and the transfer function, $M(s)$, between $y(s)$ and $u(s)$ is

$$M(s) = -P^T R^T [sI + E(\mathcal{G}_\tau)^T \mathcal{A} W R^T]^{-1} E(\mathcal{G}_\tau)^T \mathcal{A} P. \tag{15}$$

Remark 1. From Lemmas A5-A9 and Remark A1, available in the online supplemental material, it follows that the eigenvalues of $E(\mathcal{G}_\tau)^T \mathcal{A} W R^T$ or $E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}_\tau) \tilde{R} W R^T$ are in the open rhp for positive edge weights. Thus, $M(s)$ is Hurwitz.

3.2 Double Integrators

It is assumed that the nominal positive edge weights ensure consensus in protocols (7), (8). The general consensus protocol for double integrators in (7) is first considered, with uncertainties in the edge weights, followed by a similar treatment of the special case (8). The following results aid in deriving the uncertain model.

Lemma 1. *For a weighted digraph $\mathcal{G}$ with positive weights and a rooted in-branching, matrix $\bar{\Psi}$ is similar to*

$$\begin{bmatrix} 0 & I_n \\ -Q_x & -Q_v \end{bmatrix},$$

where

$$Q_x = \begin{bmatrix} E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W_x R^T & \mathbf{0_{n-1}} \\ \mathbf{1_n}^T \mathcal{A}(\mathcal{G}) W_x R^T & 0 \end{bmatrix},$$

and

$$Q_v = \begin{bmatrix} E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W_v R^T & \mathbf{0_{n-1}} \\ \mathbf{1_n}^T \mathcal{A}(\mathcal{G}) W_v R^T & 0 \end{bmatrix}.$$

Proof. Choosing S as in the proof of Lemma A8, available in the online supplemental material, the similarity transformation $(I_2 \otimes S)^{-1} \bar{\Psi} (I_2 \otimes S)$ proves the result. □

Lemma 2. *For a weighted digraph $\mathcal{G}$ with positive weights and a rooted in-branching, matrix $\bar{\Psi}_e$ is similar to*

$$\begin{bmatrix} 0 & I_m \\ -Q_{ex} & -Q_{ev} \end{bmatrix},$$

where

$$Q_{ex} = \begin{bmatrix} E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W_x R^T & E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W_x N_\tau \\ 0_{(m-n+1)\times(n-1)} & 0_{(m-n+1)\times(m-n+1)} \end{bmatrix},$$

and

$$Q_{ev} = \begin{bmatrix} E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W_v R^T & E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W_v N_\tau \\ 0_{(m-n+1)\times(n-1)} & 0_{(m-n+1)\times(m-n+1)} \end{bmatrix}.$$

Proof. With V as in the proof of Lemma A9, available in the online supplemental material, and similarity transformation $(I_2 \otimes V)^{-1} \bar{\Psi}_e (I_2 \otimes V)$, the result follows. □

Consider the edges states x_e and v_e given by $x_e = E(\mathcal{G})^T x \in \mathbb{R}^m$ and $v_e = E(\mathcal{G})^T v \in \mathbb{R}^m$. As in case of single integrator dynamics, the edge position states in (8), when transformed as $z = V^{-1} x_e$, where V is as described in the proof of Lemma A9, available in the online supplemental material, results in

$$z = [R^T (RR^T)^{-1} \; N_\tau]^T R^T E(\mathcal{G}_\tau)^T x = [x^T E(\mathcal{G}_\tau) \; \mathbf{0}_{m-n+1}^T]^T.$$

Thus, the first $n-1$ components of z represent the edge position states of the rooted in-branching. Same result holds for the edge velocities too. It follows from Lemma 2 that to study the consensus problem, it suffices to ensure the consensus of the $n-1$ edges in the rooted in-branching, each representing a position and velocity state. This is tantamount to $2(n-1)$ states represented by the vectors $x_{e\tau}, v_{e\tau} \in \mathbb{R}^{n-1}$, so that the dynamics of interest are

$$\begin{bmatrix} \dot{x}_{e\tau} \\ \dot{v}_{e\tau} \end{bmatrix} = \begin{bmatrix} 0 & I_{n-1} \\ -L_{ess\mathbf{x}} & -L_{ess\mathbf{v}} \end{bmatrix} \begin{bmatrix} x_{e\tau} \\ v_{e\tau} \end{bmatrix} = \bar{\Psi}_{e\tau} \begin{bmatrix} x_{e\tau} \\ v_{e\tau} \end{bmatrix}, \qquad (16)$$

where $L_{ess\mathbf{x}} = E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W_x R^T$ and $L_{ess\mathbf{v}} = E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W_v R^T$. Both of these matrices are nonsingular and thus $\bar{\Psi}_{e\tau}$ is also nonsingular. As before, the uncertainty is introduced through real, bounded perturbations in the edge weights as in (10). The edge could be in the position digraph or the velocity digraph. Thus, an additive perturbation δ_i is incorporated into an edge weight $w_{i\mathbf{x}}$ or $w_{i\mathbf{v}}$, $i = 1, 2, \ldots, m$. The uncertain edge agreement protocol is

$$\begin{bmatrix} \dot{x}_{e\tau} \\ \dot{v}_{e\tau} \end{bmatrix} = \bar{\Psi}_{e\tau} \begin{bmatrix} x_{e\tau} \\ v_{e\tau} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ -E(\mathcal{G}_\tau)^T \mathcal{A} P \end{bmatrix}}^{B_\tau} u \qquad (17)$$

$$y = C_\tau \begin{bmatrix} x_{e\tau} \\ v_{e\tau} \end{bmatrix}, u = \Delta y, \qquad (18)$$

where $C_\tau = [P^T R^T \; 0]$ or $C_\tau = [0 \; P^T R^T]$ depending on whether the perturbation is on the ith edge of the position or the velocity graph, respectively, and P is the same as in (12). This is a single input-single output system and can be analyzed using the Nyquist criterion. Hence, for a perturbation on an edge weight of the position or velocity graph, the transfer function $M(s)$ in Fig. 2 is given by

$$M(s) = -P^T R^T [s^2 I + s L_{ess\mathbf{v}} + L_{ess\mathbf{x}}]^{-1} E(\mathcal{G}_\tau)^T \mathcal{A} P \text{ or}, \qquad (19)$$

$$M(s) = -s P^T R^T [s^2 I + s L_{ess\mathbf{v}} + L_{ess\mathbf{x}}]^{-1} E(\mathcal{G}_\tau)^T \mathcal{A} P, \qquad (20)$$

respectively. Since the nominal positive weights are suitably chosen to ensure consensus, $M(s)$ is Hurwitz.

For the special case when $W_v = \gamma W_x$, (8) may be rewritten in terms of the edge variables as

$$\begin{bmatrix} \dot{x}_e \\ \dot{v}_e \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -\bar{L}_e & -\gamma \bar{L}_e \end{bmatrix} \begin{bmatrix} x_e \\ v_e \end{bmatrix} = \Psi_e \begin{bmatrix} x_e \\ v_e \end{bmatrix}, \qquad (21)$$

which eventually leads to the following representation:

$$\begin{bmatrix} \dot{x}_{e\tau} \\ \dot{v}_{e\tau} \end{bmatrix} = \begin{bmatrix} 0 & I_{n-1} \\ -L_{ess} & -\gamma L_{ess} \end{bmatrix} \begin{bmatrix} x_{e\tau} \\ v_{e\tau} \end{bmatrix} = \Psi_{e\tau} \begin{bmatrix} x_{e\tau} \\ v_{e\tau} \end{bmatrix}, \qquad (22)$$

where $L_{ess} = E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}) W R^T$ as before.

The uncertainty is introduced through real, bounded perturbations in the edge weights. Here too a single edge weight is perturbed as described in (10), but it affects both the velocity and position variables associated with the perturbed edge. The uncertain edge agreement protocol is

$$\begin{aligned} \begin{bmatrix} \dot{x}_{e\tau} \\ \dot{v}_{e\tau} \end{bmatrix} &= \Psi_{e\tau} \begin{bmatrix} x_{e\tau} \\ v_{e\tau} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 \\ -E(\mathcal{G}_\tau)^T \mathcal{A} P & -\gamma E(\mathcal{G}_\tau)^T \mathcal{A} P \end{bmatrix}}^{B_\tau} u \\ y &= \underbrace{\begin{bmatrix} P^T R^T & 0 \\ 0 & P^T R^T \end{bmatrix}}_{C_\tau} \begin{bmatrix} x_{e\tau} \\ v_{e\tau} \end{bmatrix}, u = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} y. \end{aligned} \qquad (23)$$

The system described above is thus a two input-two output system. This is consistent with Fig. 2 as the transfer matrix $\mathbf{M}(s)$ for the system (23) is given by

$$\mathbf{M}(s) = C_\tau (sI - \Psi_{e\tau})^{-1} B_\tau. \qquad (24)$$

4 Robust Stability of Uncertain Directed Consensus

4.1 Single Integrators: Nyquist Stability Analysis

The uncertain system, described by (13), (14), is represented in such a way that the uncertainty is separated from the nominal plant, as illustrated in Fig. 2. This formulation lends itself easily to a stability analysis using the Nyquist criterion and the notion of gain margin.

The single input-single output transfer function, $M(s)$, in (15) does not have any pole at the origin because the system matrix in (13) is of full rank. The uncertainty Δ is a scalar and therefore a classical Nyquist analysis of the gain margin will lead to results on robust stability.

Theorem 2. *The consensus protocol, (6), over a weighted digraph, $\mathcal{G}$, (with positive weights) having a rooted in-branching, is robustly stable to all perturbations, δ_i, on a single edge weight w_i, satisfying*

$$|\delta_i| < GM[M(s)], \tag{25}$$

where P is the ith standard basis in $\mathbb{R}^m$ and GM denotes the gain margin for a transfer function.

Proof. Consider $M(j\omega)$ in (15), as depicted in Fig. 2. Since the transfer function is of type zero, the gain margin is obtained by evaluating (15) at $s = j\omega_{pc}$ (which is the phase crossover frequency). Now, from the Nyquist criterion, stability dictates that $|\delta_i| < 1/|M(j\omega_{pc})|$. □

Remark 2. If the consensus protocol, (6), over a weighted digraph $\mathcal{G}$, as in Theorem 2, is subjected to multiple edge weight perturbations, then a sufficient condition for consensus can be derived using the small gain theorem. The sufficiency (as opposed to both necessity and sufficiency like in the case of single edge weight perturbations) stems from the general conservatism associated with the small gain theorem. The bound on the perturbation, Δ, on q edges ($q > 1$) can be given by $||\Delta||_\infty < 1/||\boldsymbol{M}(\boldsymbol{j\omega})||_\infty$, where $\boldsymbol{M}(\boldsymbol{s})$, given by (15), is now a matrix with columns of $P \in \mathbb{R}^{m\times q}$ indicating the perturbed edges.

Two special types of graphs are considered next: the DAG, having exactly one globally reachable node, and a directed cycle graph where every node is globally reachable. A graph theoretic interpretation of (25) is provided for these two special graphs.

4.1.1 *Consensus over Uncertain Directed Acyclic Graphs*

For directed acyclic graphs with a rooted in-branching, (25) has a significant graph theoretic interpretation. The factorizations of E and $\mathcal{A}$, and the subsequent interpretations of the columns of R and $\tilde{R}$ (r_i and $\tilde{r}_i$ respectively, for $i = 1, \dots, m$), presented in Section 2.2, along with the following results, aid in establishing this connection between the robust stability result and its implications in graph theoretic terms. The edges and nodes are labeled as described in Section 2.2.

Lemma 3. *For a DAG, $\mathcal{G}$, if $\tilde{r}_i = \tilde{r}_j = q_j$, $1 \le j \le n-1$, then $r_i(j) = +1$, where q_j is the jth standard basis for $\mathbb{R}^{n-1}$.*

Proof. Now, $\tilde{r}_j = q_j$, for $1 \le j \le n-1$ follows from the labeling. Now, $\tilde{r}_i = \tilde{r}_j$ implies that $i \ge n$ and $e_i \in \mathcal{E}_c$ is a sibling of $e_i \in \mathcal{E}_\tau$. Since there are no directed cycles in $\mathcal{G}$, so any edge e_i emerging from a node, say node p, cannot terminate at a node t such that there is a directed path from node t to p. Hence, the equivalent signed path, in $\mathcal{G}_\tau$, corresponding to the edge e_i, must traverse its sibling edge e_j in the positive sense. Thus, $r_i(j) = +1$. □

Lemma 4. *Consider the DAG, $\mathcal{G}$, having two edges e_s and e_t in $\mathcal{E}_c$ that are siblings to edges e_p and e_q in $\mathcal{E}_\tau$, respectively. If the signed path of e_s in $\mathcal{G}_\tau$ involves traversal of e_q, then the signed path of e_t in $\mathcal{G}_\tau$ cannot include the edge e_p.*

Proof. It suffices to prove that $r_s(q) = \pm 1$ implies $r_t(p) = 0$. From Lemma 3, $r_s(p) = +1$. Suppose $r_s(q) = +1$. Then there is a directed path through e_p to the globally reachable node with e_q appearing after edge e_p in the sequence. Hence, any edge that is a sibling of e_q (such as e_t) in $\mathcal{G}$ cannot be represented by a signed path that contains edge e_p as this will imply the existence of a directed cycle. So $r_t(p) = 0$. Next, consider $r_s(q) = -1$. This means that a sibling edge of e_p is encoded by a path that involves traversing e_q in the opposite sense. Clearly, the directed path through e_p to the globally reachable node does not include the edge e_q and vice versa. Thus, any sibling edge of e_q cannot be represented by a signed path that involves traversing e_p in the positive sense either. So, $r_t(p)$ cannot equal $+1$. Suppose $r_t(p) = -1$. But this means that there is a directed path through e_s and e_t, back to the parent node of e_p and e_s, thereby completing a directed cycle. Thus, $r_t(p) = 0$ is the only possibility. □

From (25), it is clear that an interpretation of the perturbation bound involves an investigation of the structure of $[\tilde{R}WR^T]^{-1}$. Consider the matrix $\tilde{R}WR^T = W_\tau + \tilde{T}_\tau W_c T_\tau^T$ (using (1) and (3)) where $W_\tau \in \mathbb{R}^{(n-1)\times(n-1)}$ and $W_c \in \mathbb{R}^{(m-n+1)\times(m-n+1)}$ are diagonal matrices containing the weights of the edges in $\mathcal{E}_\tau$ and $\mathcal{E}_c$, respectively. From (1), (2), (3), (4), the columns of T_τ and $\tilde{T}_\tau$ are the columns n through m of R and $\tilde{R}$, respectively. Thus, $\tilde{R}WR^T = W_\tau + \sum_{i=n}^{m} w_i \tilde{r}_i r_i^T$. Now, using the Sherman Morrison formula for inverse of rank one updates [32] iteratively, $D_{m-n+2} = (\tilde{R}WR^T)^{-1}$ can be obtained as edges in $\mathcal{E}_c$ are added one by one to the rooted in-branching, $\mathcal{G}_\tau$, with the initial value $D_1 = W_\tau^{-1}$ and the update rule given by

$$D_{i+1} = D_i - \frac{w_{n+i-1} D_i \tilde{r}_{n+i} r_{n+i-1}^T D_i}{1 + w_{n+i-1} r_{n+i-1}^T D_i \tilde{r}_{n+i-1}}. \tag{26}$$

It follows from (26) that for each additional edge $e_k \in \mathcal{E}_c$ incorporated, the jth row, corresponding to its sibling edge $e_j \in \mathcal{E}_\tau$, is updated. Moreover, only those entries of the jth row which correspond to edges in $\mathcal{G}_\tau$ that comprise the equivalent signed path of e_k are updated. For instance, in Fig. 1, when e_8 is added, only $[D_i]_{22}$, $[D_i]_{23}$, $[D_i]_{25}$ and $[D_i]_{26}$ in the second row will be updated. Besides, only rows that have already been updated at earlier iterations can be affected. This is a consequence of Lemma 4.

Further, due to Lemma 4, for a DAG the diagonal entry $[D_i]_{jj}$ is only updated if a sibling edge corresponding to e_j is added to the graph at the $i+1$th step. The following theorem provides the graph theoretic interpretation of (25) for a DAG.

Theorem 3. *The consensus protocol over a DAG, $\mathcal{G}$, with positive weights and containing a rooted in-branching, is robustly stable to all perturbations, δ_i, on an edge weight, w_i, if the sum of the out-degree weights of the parent node of edge e_i is positive.*

Proof. Consider a rooted in-branching, $\mathcal{G}_\tau$, for the DAG, $\mathcal{G}$. Such a rooted in-branching will contain several branches, b_w, each terminating in a single globally reachable node, labeled n. Suppose the labeling of the nodes on the branches follow two rules. First, any two nodes i and j along a branch b_q are labeled so that in $\mathcal{G}_\tau$, if |path length from i to n| $>$ |path length from j to n|, then $i < j$. Second, if an edge, $e_k \in \mathcal{G}$, starts from a node i in branch b_w and terminates in a node j of branch b_v, then $i < j$. These two rules do not contradict each other unless there is a directed cycle involving segments of branches b_v and b_w.

Further, consider a labeling such that the first $n-1$ edges in $\mathcal{E}$ consist of the rooted in-branching such that the parent node of edge e_i is node i, for $1 \le i \le n-1$. It follows that with this labeling, any edge $e_i \in \mathcal{E}_\tau$ terminates at node j, where $j > i$. Let the edges in $\mathcal{E}_c$ be labeled so that for any two edges $e_f, e_g \in \mathcal{E}_c$ that are siblings of $e_i, e_j \in \mathcal{E}_\tau$, respectively with $i < j$, one has $f < g$. This implies that the $k-n+1$th column of $\tilde{T}_\tau$, that is $\tilde{t}_{k-n+1}$, corresponding to edge $e_k \in \mathcal{E}_c$, will have only one non-zero entry equal to 1 at the pth position if $e_p \in \mathcal{E}_\tau$ is a sibling of e_k. Column t_{k-n+1} of T_τ, corresponding to signed path of edge $e_k \in \mathcal{E}_c$ will be such that $t_{k-n+1}(p) = +1$ (by Lemma 3) and $t_{k-n+1}(u) = 0$ for $u < p$ (by choice of labeling).

Now, $\tilde{T}_\tau W_c T_\tau^T = \sum_{i=1}^{m-n+1} w_{m-n+1+i}\tilde{t}_i t_i^T$ is a weighted sum of outer products and due to the structures of $\tilde{t}_i$ and t_i discussed above, is upper triangular. Consequently, $\tilde{R}WR^T$ is also upper triangular with the ith diagonal entry containing the sum of the out degrees of the parent node of edge $e_i \in \mathcal{E}_\tau$. Next, consider $M(s)$ given by $M(s) = -P^T R^T K(s)^{-1} E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}_\tau)\tilde{R}P$, where $K(s) = (sI + E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}_\tau)\tilde{R}WR^T)$. The matrix $E(\mathcal{G}_\tau)^T \in \mathbb{R}^{(n-1)\times n}$ is such that $[E(\mathcal{G}_\tau)^T]_{ij} = 0$ for $i > j$, $[E(\mathcal{G}_\tau)^T]_{ii} = 1$, and $\mathcal{A}(\mathcal{G}_\tau) = [I_{n-1}\ \mathbf{0}_{n-1}]^T$. Hence, $L_\tau = E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}_\tau)$, and consequently $K(s)$ are upper triangular. Moreover, $[K(s)]_{ii} = (s + \sum d_{(out)i})$ and hence, $[K(s)^{-1}]_{ii} = \frac{1}{(s+\sum d_{(out)i})}$, where, $\sum d_{(out)i} = w_i + \sum_{e_j \text{is a sibling of } e_i} w_j$. Without loss of generality, suppose e_k, the perturbed edge, is a sibling to $e_u \in \mathcal{E}_\tau$ (u may or may not be equal to k). Post-multiplication of $K(s)^{-1}$ by $\tilde{R}P$ picks out the uth column of the triangular matrix $K(s)^{-1}$ whose entries below the uth component are zero (due to triangularity of the matrix). Next, $P^T R^T$ picks out one row of R^T which corresponds to the perturbed edge e_k. Thus, $P^T R^T = r_k^T$ and $r_k(u) = +1$ (Lemma 3). Also, $r_k(s) = 0$ if $s < u$. So, $M(s)$ picks out its uth diagonal entry due to triangularity and thus $M(s) = -\frac{1}{(s+\sum d_{(out)i})}$, where $e_i \in \mathcal{E}_\tau$ is a sibling of e_k (k may or may not be equal to i). This is a first order plant and the Nyquist plot of $M(s)$ has a phase crossover at $\omega = 0$. The gain margin is thus $|\sum d_{(out)i}|$. Hence, a negative perturbation $\delta_k = -\sum d_{(out)i}$ on edge e_k precludes consensus. □

Corollary 1. *From Theorem 2, for a DAG the relations (3), (4) hold, and the perturbation bound, δ_i, on an edge weight, w_i, is:*

$$|\delta_i| < \left| \left(P^T R^T (\tilde{R}WR^T)^{-1}\tilde{R}P\right)^{-1}\right|. \tag{27}$$

Corollary 1 holds since using $\omega_{pc} = 0$ in (25), one has

$$(E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}_\tau)\tilde{R}WR^T)^{-1} = (\tilde{R}WR^T)^{-1}(E(\mathcal{G}_\tau)^T \mathcal{A}(\mathcal{G}_\tau))^{-1}.$$

Remark 3. The result of Theorem 3 may also be obtained by observing that for a DAG there exists a labeling of nodes such that $\bar{L}_g$ is upper triangular with out-degree sums on its diagonals.

4.1.2 Consensus over Uncertain Cycle Digraph

Theorem 3 deals with a graph having exactly one globally reachable node. The cycle digraph, having exactly the same number of edges as nodes ($m = n$), on the other has n globally reachable nodes. Removing any one of the edges from a cycle digraph results in a rooted in-branching. Since the cycle graph has multiple globally reachable nodes, (3), (4) do not hold. However, the condition in (25) holds for the consensus of a cycle digraph. Also, for the cycle digraph $\mathcal{A} = I_n$. The cycle digraph is also especially important as it lies at the heart of the well known cyclic pursuit algorithm [7], [25], [33], [34], [35], [36]. Some relevant results follow.

Lemma 5. *The graph Laplacian for weighted cycle digraph, $\bar{L}_g = \mathcal{A}WE(\mathcal{G})^T$ is similar to $\begin{bmatrix} E(\mathcal{G}_\tau)^T WR^T & 0 \\ 0 & 0 \end{bmatrix}$.*

Proof. Choosing matrices $S_1^{-1} = \begin{bmatrix} E(\mathcal{G}_\tau)^T \\ \mathbf{1}_n^T W^{-1} \end{bmatrix}$ and

$$S_1 = \left[WE(\mathcal{G}_\tau)(E(\mathcal{G}_\tau)^T WE(\mathcal{G}_\tau))^{-1} \quad \mathbf{1}_n \left(\sum_i \frac{1}{w_i}\right)^{-1} \right],$$

it follows that $S_1^{-1}\bar{L}_g S_1 = \begin{bmatrix} E(\mathcal{G}_\tau)^T WR^T & 0 \\ 0 & 0 \end{bmatrix}$. □

Lemma 6. *The edge Laplacian for weighted cycle digraph, $\bar{L}_e = E(\mathcal{G})^T \mathcal{A}W$ is similar to $\begin{bmatrix} E(\mathcal{G}_\tau)^T WR^T & 0 \\ 0 & 0 \end{bmatrix}$.*

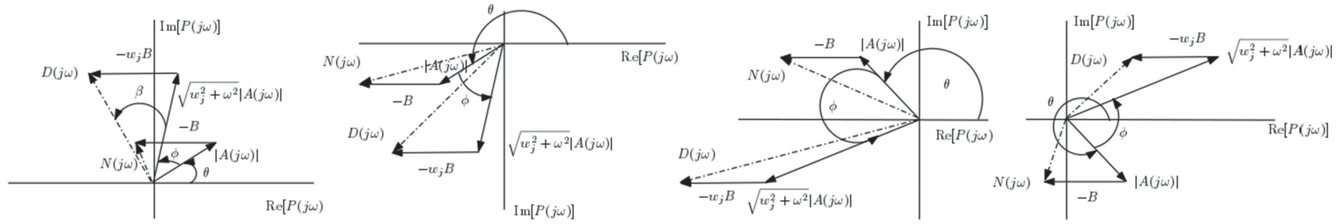
Proof. Consider the matrices $S_2 = [R^T \quad W^{-1}\mathbf{1}_n]$ and $S_2^{-1} = \begin{bmatrix} (R(\mathcal{G}_\tau)WR(\mathcal{G}_\tau)^T)^{-1}R(\mathcal{G}_\tau)W \\ \left(1/\sum_i \frac{1}{w_i}\right)\mathbf{1}_n^T \end{bmatrix}$. It follows that $S_2^{-1}\bar{L}_e S_2 = \begin{bmatrix} E(\mathcal{G}_\tau)^T WR^T & 0 \\ 0 & 0 \end{bmatrix}$. □

Lemma 7. *For the weighted cycle digraph, the edge Laplacian is similar to the graph Laplacian.*

Proof. Using the transformation $S^{-1}\bar{L}_e S$, with $S = S_2 S_1^{-1}$ and S_1, S_2 defined as above, the result follows. □

Similar to (11), (12), the reduced edge version of cycle digraph, in view of the Lemmas 5-7, can be written as

$$\dot{x}_\tau = -E(\mathcal{G}_\tau)^T WR^T x_\tau. \tag{28}$$



(a) (b) (c) (d)

Fig. 3. Four cases such that (a) $A(j\omega)$ and $(j\omega + w_1)A(j\omega)$ are both in first and/or second quadrants (b) $A(j\omega)$ and $(j\omega + w_1)A(j\omega)$ are both in third and/or fourth quadrants (c) $A(j\omega)$ is on second quadrant while $(j\omega + w_1)A(j\omega)$ is on third quadrant (d) $A(j\omega)$ is in fourth quadrant while $(j\omega + w_1)A(j\omega)$ is in first quadrant.

As in (12), considering a perturbation in edge weight w_1, the system may be described as

$$\dot{x}_\tau = -E(\mathcal{G}_\tau)^T (W + P\Delta P^T) R^T x_\tau, \tag{29}$$

with the uncertainties belonging to the set given by (10) and $P \in \mathbb{R}^n$ is a $\{0,1\}$ vector with 0-entries everywhere except at $[P]_1$. This is so chosen because in the cycle graph every edge is equivalent and without loss of generality, the perturbation may be considered in w_1.

The following result aids in computing the perturbation bound of an edge weight for a cycle digraph.

Lemma 8. *The only finite phase crossover frequency for the transfer function given by (15), for a cycle digraph with perturbation on one edge weight, is $\omega_{pc} = 0$.*

Proof. After some algebraic manipulations, it transpires that $-M(s)$, as defined in (15), can be obtained as the $(1,1)$ entry of the matrix D_{n-1}^{-1} where, D_{n-1} is given by:

$$D_{n-1} = \begin{bmatrix} (s+w_1) & -w_2 & 0 & \dots & 0 \\ 0 & (s+w_2) & -w_3 & \dots & 0 \\ 0 & 0 & (s+w_3) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_n & w_n & w_n & \dots & (s+w_{n-1}+w_n) \end{bmatrix} \tag{30}$$

for the cycle digraph. Now, $\det D_{n-1}$ is given by:

$$\det D_{n-1} = (s+w_1)\det D_{n-1(1,1)} + w_2 \det D_{n-1(1,2)}. \tag{31}$$

Consider for the cyclic pursuit system,

$$-M(s) = \frac{\det D_{n-1(1,1)}}{(s+w_1)\det D_{n-1(1,1)} + w_2 \det D_{n-1(1,2)}}.$$

Let $A(s) = \Pi_{i \neq 1}(s + w_i)$, and $B = \Pi_{i \neq 1} w_i$. Without loss of generality assume that $w_1 = 1$, while all other gains w_i, $i \neq 1$ are scaled by a factor w_1. This assumption does not affect the stability analysis of consensus, since scaling each edge weight by a constant factor only affects the rate of consensus. Next, using (30), (31), $-M(s)$ is given by:

$$-M(s) = \frac{N(s)}{D(s)} = \frac{A(s) - B}{(s+w_1)A(s) - w_1 B}. \tag{32}$$

At phase crossover frequencies the phase functions of $N(j\omega)$ and $D(j\omega)$ must differ by integral multiples of 2π. Clearly, $A(j\omega)$ and $(j\omega + w_1)A(j\omega)$ differ by an angle $\phi = \arctan(\omega/w_1) < \pi/2$ while the argument of $A(j\omega)$ is $\theta = \sum_{i \neq 1} \arctan(\frac{\omega}{w_i})$. Since $w_i > 0, \forall i, \theta$ is a monotonically increasing function of ω for $0 < \omega < \infty$. Further $B > 0$. Define the angle between $N(j\omega)$ and $A(j\omega)$ as α. Based on the different possibilities, four distinct cases are considered, as in Figs. 3a, 3b, 3c, 3d. The numerator of $-M(s)$ is the sum of $A(j\omega)$ and $-B$ while the denominator is the sum of $(j\omega + w_1)A(j\omega)$ and $-w_1 B$. Furthermore, $|A(j\omega)| < |(j\omega + w_1)A(j\omega)|$ since $w_1 = 1$.

Case 1: (Both $A(j\omega)$ and $(j\omega + w_1)A(j\omega)$ are in the first and/or second quadrants), The condition for phase crossover in Fig. 3a may be given by $\phi + \beta - \alpha = 0$. In other words, a necessary condition would be $\tan(\phi + \beta) = \tan\alpha$. Using the following expressions:

$$\tan\phi = \frac{\omega}{w_1}, \tag{33}$$

$$\tan\beta = \frac{|w_1 B| \sin(\theta + \phi)}{\sqrt{\omega^2 + w_1^2}|A(j\omega)| - |w_1 B| \cos(\theta + \phi)}, \tag{34}$$

it readily follows that

$$\tan(\phi + \beta) - \tan\alpha = \frac{\omega |A(j\omega)|/w_1}{|A(j\omega)| - |B|\cos\theta}. \tag{35}$$

Since $|A(j\omega)| > |B| > 0, \forall 0 < \omega < \infty$, and $|A(j\omega)| \to \infty$ as $\omega \to \infty$, it follows from (35) that $\tan(\phi + \beta) = \tan\alpha$ is only satisfied if $\omega = 0$ or $\omega \to \infty$ for this case.

Case 2: (Both $A(j\omega)$ and $(j\omega + w_1)A(j\omega)$ are in the third and/or fourth quadrants). Since $w_1 = 1$, $-w_1 B = -B$. Using the fact that $|A| < |A|\sqrt{\omega^2 + w_1^2}$ and the geometry of Fig. 3b, it is obvious that $N(j\omega)$ and $D(j\omega)$ cannot coincide. Hence, phase crossover is ruled out.

Case 3: ($A(j\omega)$ is in second quadrant and $(j\omega + w_1)A(j\omega)$ is in the third quadrant). Again, from the geometry in Fig. 3c, $N(j\omega)$ and $D(j\omega)$ lie in different quadrants and so no phase crossover is possible.

Case 4: ($A(j\omega)$ is in the fourth quadrant and $(j\omega + w_1)$ $A(j\omega)$ is in the first quadrant). As in Case 3, from Fig. 3d $N(j\omega)$ and $D(j\omega)$ lie in different quadrants, hence no phase crossover.

Thus, the necessary condition for a non-zero finite phase crossover frequency of $-M(s)$ is not satisfied under any circumstances. Hence, the only finite phase crossover frequency for $-M(s)$ is $\omega_{pc} = 0$. □

$M(0)$ is explicitly computed to be

$$M(0) = -\frac{\sum_{i=2}^{n} \frac{1}{w_i}}{1 + w_1 \sum_{i=2}^{n} \frac{1}{w_i}}.$$

Applying Nyquist criteria, the following expression results:

$$-w_1 - \frac{1}{\sum_{i=2}^{n} \frac{1}{w_i}} < \bar{\delta} \Rightarrow w_1 + \bar{\delta} > -\frac{1}{\sum_{i=2}^{n} \frac{1}{w_i}}. \quad (36)$$

Thus, the robust stability of cyclic pursuit may be ensured by the following theorem and it agrees with [25].

Theorem 4. *Given a perturbation on a single edge, say edge e_j (nominal weight w_j), the heterogeneous cyclic pursuit system ([25]) is stable for all perturbations greater than $\bar{\delta}$ given by:*

$$\bar{\delta} > -w_j - \frac{1}{\sum_{i=1, i\neq j}^{n} \frac{1}{w_i}}. \quad (37)$$

Corollary 2. *In any weighted digraph, $\mathcal{G}$, having a rooted in-branching, if there exists a directed cycle, $\mathcal{C}_r \subset \mathcal{G}$, comprising all r globally reachable nodes, then the bound on the perturbation on any edge of this cycle, $\mathcal{C}_r$, can be computed by applying Theorem 4 on the directed cycle, $\mathcal{C}_r$.*

Proof. It is clear that the Laplacian, $L(\mathcal{G})$, for the digraph, $\mathcal{G}$, containing a directed cycle, $\mathcal{C}_r$, comprising all the r globally reachable nodes can be written as

$$L(\mathcal{G}) = \begin{bmatrix} F & G \\ 0_{r\times(n-r)} & H \end{bmatrix},$$

where $H \in \mathbb{R}^{r\times r}$ completely captures the dynamics of the nodes in $\mathcal{C}_r$, for a suitable permutation of the nodes in $\mathcal{G}$. Thus, Theorem 4 can be applied to $\mathcal{C}_r$. □

Remark 4. The reciprocal of the edge weight is the resistance corresponding to an edge. For a cycle digraph, the bound on $\frac{1}{w_j+\delta}$ is the equivalent resistance between the nodes j and $j+1$ with the edge e_j removed. Now, for consensus over an undirected graph, a perturbed edge weight can have negative values so long as this negative value is greater than a lower bound [9] that is equal to the negative of the equivalent resistance (with the corresponding edge removed) between the nodes that the perturbed edge joins. The same interpretation is also valid for the cycle digraph.

Remark 5. So far, single edge perturbations are considered. However, graph theoretic interpretations, for a cycle digraph and a DAG, provide a broader perspective on the robustness bounds. For the DAG, if the sum of the out-degrees of each node is positive, multiple negative edge weights do not disrupt consensus. Similarly, a graph resistance based interpretation holds for the cycle digraph.

4.1.3 Multiple Edge Weight Perturbations: A Perspective

This work mostly uses Nyquist criterion which deals with single input-single output (SISO) systems. Thus, exact bounds on perturbations are obtained for a single edge weight perturbation, resulting in a SISO system in Fig. 2. But for multiple negative valued edge weight perturbations, owing to 'attacks' or disruptive elements, the method outlined here is still useful. These attacks are broadly of two types: attacks of known magnitudes, and those of unknown magnitudes. If it is required to determine whether attacks of known magnitudes on a set of q out of m edge weights will disrupt consensus or not, it suffices to employ the Nyquist criterion based results on perturbation bound q times. Assume a labeling of the edges such that edges labeled 1 through q are perturbed, with negative edge weight perturbations, $\delta_i < 0, i = 1, \ldots, q$, stacked as $\bar{\delta} = [\delta_1 \ldots \delta_q] \in \mathbb{R}^q$, and P_i is the ith standard basis in $\mathbb{R}^m$. The Algorithm 1 answers whether consensus is disrupted or not for any perturbation $\delta \succeq \bar{\delta}$, where $\succeq$ denotes element-wise comparison between two vectors.

Algorithm 1. Negative Perturbations on q Edges

1: $i = 1, j = 0$.
2: *begin loop*
3: Evaluate $M_i(s)$ with $P = P_i$, using (15).
4: Obtain $GM[M_i(s)]$.
5: If $|\delta_i| < GM[M_i(s)]$, $j = j + 1$.
6: $W = W + \delta_i P_i P_i^T$.
7: $i = i + 1$.
8: If $i < q + 1$ goto 3.
9: *end loop*
10: If $j = q$, 'Consensus'; else 'No consensus'

The following example illustrates the algorithm. Consider 5 agents connected by a cycle digraph. The nominal edge weights are $[1\ 2\ 3\ 4\ 5]$ and suppose perturbations of magnitudes -0.5 and -5.4 occur on edges 1 and 5, respectively. The sequence in which these perturbations are considered for analysis purpose does not affect the overall qualitative conclusion about whether consensus occurs or not. First consider the perturbation on edge 1. A perturbation of -0.5 does not disrupt consensus and the modified edge weights are now $[0.5\ 2\ 3\ 4\ 5]$. Now, upon calculation the tolerable perturbation on edge 5 is -5.324 which is exceeded by the given perturbation. Thus, consensus fails. Alternately, now consider first the perturbation on edge 5. The bound on the perturbation is -5.48 and so this perturbation passes the test. Next, the perturbation on edge 1 with modified edge weights $[1\ 2\ 3\ 4\ -0.4]$ is bounded by -0.294, which is exceeded by the given perturbation. It may thus be concluded again that consensus fails. In the two analyses, the overall conclusion about whether or not consensus occurs is invariant. However, the perturbation bounds computed during the two processes are different.

On the other hand, if the perturbations on the edge weights are unknown and it is required to determine how much perturbation a set of q edge weights can tolerate, one needs to apply a small gain theorem based approach, as indicated in Remark 2. But the conservatism of small gain theorem will only lead to sufficiency conditions.

4.2 Double Integrators: Robust Stability Analysis

4.2.1 General Protocol for Double Integrators

Assume that the consensus protocol (7) has suitably chosen positive edge weights so that consensus results. As with single integrators, here too the existence of a rooted in-branching, and all edges positive weights are assumed. Similar reasoning as in Theorem 2 leads to the following.

Theorem 5. *If the consensus protocol (7) achieves consensus over a weighted digraph, $\mathcal{G}$, (with different positive weights for* *position and velocity) having a rooted in-branching, then it is robustly stable to all perturbations δ_i on a single edge weight w_i or $\bar{w}_i$ corresponding to position or velocity, satisfying*

$$|\delta_i| < GM[-P^T R^T [s^2 I + sL_{ess\mathbf{v}} + L_{ess\mathbf{x}}]^{-1} E(\mathcal{G}_\tau)^T \mathcal{A} P], \quad \text{or} \quad |\delta_i| < GM[-sP^T R^T [s^2 I + sL_{ess\mathbf{v}} + L_{ess\mathbf{x}}]^{-1} E(\mathcal{G}_\tau)^T \mathcal{A} P], \tag{38}$$

where P is the ith standard basis in $\mathbb{R}^m$ and GM denotes the gain margin for a transfer function.

Remark 6. Though the nominal graph is assumed to have all weights positive throughout this work, it is not strictly necessary. If all the non-zero eigenvalues of the nominal graph Laplacian have positive real parts, the results in this paper hold even if some of the edge weights are negative. The positive nominal weights are sufficient (due to Gershgorin's theorem) for the nominal $\bar{L}_g$ to have all its non-zero eigenvalues in the open rhp.

4.2.2 *Special Case:* $W_v = \gamma W_x$

In this case, $\bar{L}_g$ has one eigenvalue at the origin and the remaining eigenvalues are in the open lhp due to the existence of a rooted in branching. Additionally, it will be assumed γ satisfies the constraint (9). Thus, the nominal system (8) will achieve consensus if the underlying graph has a rooted in-branching and all the edge weights are positive.

Using block matrix inversion, it follows that for (23),

$$(sI - \Psi_{e\tau})^{-1} = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}, \tag{39}$$

where $\psi_{11} = \frac{1}{s}I - \frac{1}{s}(s^2 I + \gamma s L_{ess} + L_{ess})^{-1} L_{ess}$, $\psi_{12} = (s^2 I + \gamma s L_{ess} + L_{ess})^{-1}$, $\psi_{21} = -(s^2 I + \gamma s L_{ess} + L_{ess})^{-1} L_{ess}$ and $\psi_{22} = s(s^2 I + \gamma s L_{ess} + L_{ess})^{-1}$. Now, using the above relations in (24), it immediately follows that

$$\mathbf{M}(s) = H(s) \begin{bmatrix} -1 & -\gamma \\ -s & -s\gamma \end{bmatrix}, \tag{40}$$

where $H(s) = P^T R^T (s^2 I + \gamma s L_{ess} + L_{ess})^{-1} E(\mathcal{G}_\tau)^T \mathcal{A} P$. The robust stability result may now be stated.

Theorem 6. *The consensus protocol over a weighted digraph having positive weights and a rooted in-branching, for double integrator agents, given by (8) is robustly stable to all perturbations δ_i on a single edge weight w_i satisfying*

$$|\delta_i| < \frac{1}{\max_\omega [\sqrt{1+\gamma^2}\sqrt{1+\omega^2}|H(j\omega)|]}, \tag{41}$$

where P is the ith standard basis in $\mathbb{R}^m$.

Proof. Using the relations (10) and (40) in conjunction with the small gain theorem, it follows that the system in Fig. 2 is robustly stable if it satisfies $||\mathbf{M}(j\omega)|||\delta_i| < 1$. Further, using some algebraic manipulations, it is at once apparent that the maximum singular value of $\mathbf{M}(j\omega)$ for a given ω is $\sqrt{1+\gamma^2}\sqrt{1+\omega^2}|H(j\omega)|$. Hence the proof. □

Note that $H(s) = -M(\frac{s^2}{\gamma s+1})/(\gamma s + 1)$, where $M(s)$ is given by (15). In other words, $H(s)$ is obtained by replacing s with $\frac{s^2}{\gamma s+1}$ in the expression for $M(s)$ given by (15) and dividing the resultant by $(\gamma s + 1)$. This also shows that $M(s)$ in (15), being strictly proper (since it is the transfer function corresponding to a state space representation that has no feedforward), implies $H(s)$ has a relative degree of at least 2 in s. Hence, $\sqrt{1+\omega^2}|H(j\omega)|$ is at least of relative degree 1 in ω and is low pass. Therefore, the maxima in (41) exists.

Remark 7. The robust stability condition in Theorem 6 is a sufficient one because of the application of the small gain theorem to arrive at (41).

Corollary 3. *For a DAG, if $W_v = W_x$, the consensus protocol (7) is robustly stable to any perturbation, δ_i, on an edge e_i of either the velocity or position graph, so long as the sum of the out-degree weights of the parent node of e_i is positive.*

Proof. The relevant transfer functions for double integrators, $-M(\frac{s}{s^2+1})/(s+1)$ and $-sM(\frac{s}{s^2+1})/(s+1)$, obtained from $M(s)$ as in (15), over a DAG (see proof of Theorem 3 for an expression of $M(s)$ corresponding to a DAG), are $-\frac{1}{s^2+\sum d_{(out)i}s+\sum d_{(out)i}}$ and $-\frac{s}{s^2+\sum d_{(out)i}s+\sum d_{(out)i}}$, respectively for perturbed position and velocity graphs. Now, the gain margins for both these transfer functions are $\sum d_{(out)i}$ and hence any negative perturbation whose magnitude is smaller than $\sum d_{(out)i}$ will not disrupt consensus. □

5 Designing Gains for Double Integrator Consensus

So far, the robustness study is presented as an analysis of the stability margin in terms of a single edge weight perturbation. However, the present framework may serve as a design tool for the consensus protocol given by (7), in case the nominal edge weights do not result in consensus.

Problem 1. *For the consensus protocol (7), given a nominal set of positive edge weights which does not lead to consensus, find a perturbation on any edge weight so as to guarantee consensus.*

Remark 8. The above problem is to be solved without explicit knowledge about the spectrum of the Laplacian.

There are $2m$ edge weights to choose, in this problem. If these $2m$ decision variables are assigned arbitrary positive values, it may result in instability of the system matrix in (7), implying no consensus. Moreover, arbitrarily choosing an edge weight for an arbitrary amount of perturbation will not ensure consensus. Therefore, a design procedure is needed to decide which edge weight to perturb and by how much, without explicit eigenvalue computations.

Remark 9. The number of zeros, z_{ol}, and the number of poles, p_{ol}, of $M(s)$, as in (19), (20), in the rhp can be checked by using Routh array for the numerator polynomial and the denominator polynomial of $M(s)$, respectively, without any pole zero cancellation. The number of zeros of $\det[s^2 I + sL_{ess\mathbf{v}} + L_{ess\mathbf{x}}]$ in rhp is also p_{ol}. Thus, evaluating p_{ol} and z_{ol} does not involve explicit knowledge of the Laplacian spectrum.

Define the discrete set Ω_p for an edge e_p as $\Omega_p = \{\omega : Im[M(j\omega)] = 0 \text{ and } Re[M(j\omega)] > 0\}$, where $M(s)$ is as defined in (19), (20) for edge e_p. Suppose $|\Omega_p| = T$. The set

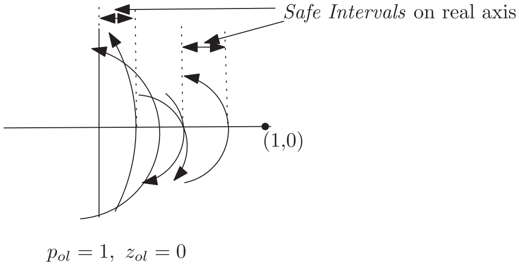


Fig. 4. Intervals where each point is encircled p_{ol} times counterclockwise for a typical $M(s)$.

Ω_p contains frequencies at which the Nyquist plot of $M(s)$ crosses the positive real axis of the $M(j\omega)$ plane either encircling the origin in a clockwise or in a counterclockwise direction. Sort the frequencies $\omega_i \in \Omega_p$ such that $i > k$ implies $Re[M(j\omega_i)] \geq Re[M(j\omega_k)]$. Clearly, there are non intersecting intervals of the form $\mathcal{I}_i = (M(j\omega_i), M(j\omega_{i+1}))$, $i = 1, \ldots, T-1$ and $\mathcal{I}_0 = (0, M(j\omega_1))$ on the positive real axis of the $M(j\omega)$-plane. For any point on interval $\mathcal{I}_i$, the number of net counterclockwise encirclements can be counted by checking the direction of encirclements corresponding to frequencies ω_{i+1} through ω_T.

Definition 1. *For a Nyquist plot of $M(s)$, as defined in (19), (20), if an interval $\mathcal{I}_i$, $i = 0, \ldots, T-1$ is such that the number of net counterclockwise encirclements about any point in $\mathcal{I}_i$ is p_{ol}, then the interval $\mathcal{I}_i$ is said to be a* safe interval.

Assumption 1. *The weighted digraph $\mathcal{G}$ over which the consensus protocol (7) runs has at least one edge for which the Nyquist plot of the transfer function $M(s)$ in (19), (20) has at least one non-empty* safe interval.

Assumption 1 would ensure that the Nyquist plot of the transfer function $M(s)$ in (19) or (20) for some edge encircles a set of points in an interval on the positive real axis of the $M(j\omega)$ plane p_{ol} times in the counterclockwise direction.

Theorem 7. *The consensus protocol (7) over a weighted digraph $\mathcal{G}$ having a rooted in branching, positive edge weights on all edges, and satisfying the Assumption 1 can be stabilized by perturbing a single edge weight.*

Proof. The polynomial $\det[s^2 I + sL_{ess\mathbf{v}} + L_{ess\mathbf{x}}]$ has p_{ol} zeros in the rhp and is the denominator of both $M(s)$ and $1 - \Delta M(s)$. Hence, if the Nyquist plot of $M(s)\Delta$ encircles the point $(1, 0)$ p_{ol} times in the counterclockwise direction, the M-Δ structure is stable. Since Assumption 1 holds for an edge, e_p, the Nyquist plot for $M(s)$ in (19) or (20), corresponding to the edge e_p, will encircle all points in some *safe interval* p_{ol} times in the counterclockwise direction. This is illustrated in Fig. 4, where any point in the two *safe intervals* is encircled $p_{ol} = 1$ times. There may be multiple such safe intervals, $\mathcal{I}_i,\ i = 0, \ldots, T-1$. This implies that there are ranges of positive gains (perturbations to edge weights are the feedback gain in the closed loop) which will stabilize the M-Δ structure. So, by perturbing one edge weight suitably within these ranges, consensus can be achieved in (7). □

Remark 10. The existence of an edge which satisfies Assumption 1 implies that the edge state corresponding to this particular edge, when chosen as an output variable, ensures that the consensus system is stabilizable and detectable. This means that all the unstable modes of the consensus system are reflected in this edge state. This may serve as further motivation for a study of observability and controllability of consensus systems in particular, and networked systems modeled by weighted directed graphs in general, in future.

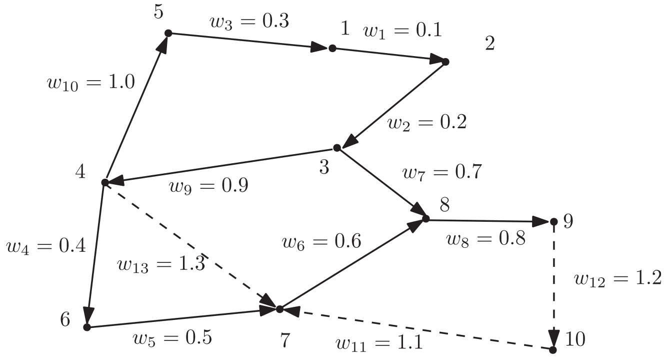


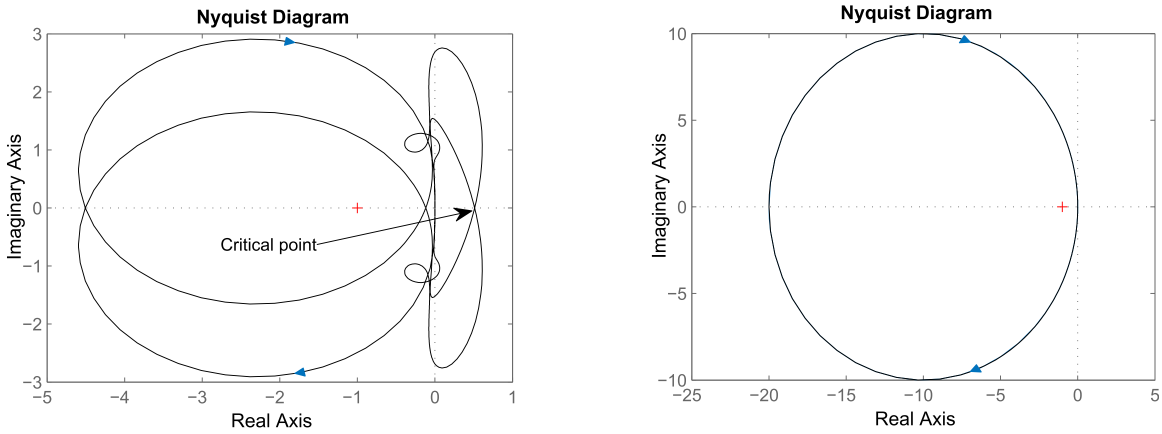
Fig. 5. Chosen weights on position digraph.

Example 1. Consider the digraph in Fig. 5 with only the bold edges, and vertices 1 through 9. The weights correspond to the position digraph. Each nominal weight is positive and there is a rooted in-branching. Hence, $\bar{L}_g$ has one zero eigenvalue while the rest are in the open rhp. Suppose each weight on the velocity graph is γ times the corresponding edge weight on the position graph. Now, the lower bound on γ can be computed from (9) using explicit knowledge about the spectrum of $\bar{L}_g$. Here $\gamma = 0.1$ is chosen arbitrarily and explicit computations, using (9), reveal that no consensus results from this choice. Suppose it is required to perturb the edge weights on the velocity graph independently, that is, instead of altering γ, which would alter all the weights on the velocity graph by the same factor, only one of the edge weights is perturbed. It may be verified, by Routh array, that for $M(s)$ corresponding to edge e_3 of the velocity graph (nominal weight $w_{v3} = 0.03$), $z_{ol} = 0$, and $p_{ol} = 2$. The Nyquist plot of this $M(s)$ is shown in Fig. 6a, from which it is clear that a non-empty *safe interval*, $\mathcal{I}_0$, exists. Hence, about any point in $\mathcal{I}_0$ there are 2 net counterclockwise encirclements. So Assumption 1 holds. A positive perturbation on the edge may be applied, akin to a positive feedback, so that the point $(1, 0)$ is encircled twice, and the closed loop system is stable. This is because both $M(s)$ and $1 - \Delta M(s)$ then have two poles and no zeros in the rhp. In this example, a minimum perturbation of 1.9531 on w_3 ensures this.

Remark 11. In Example 1, a perturbation on w_3 ensures consensus. However, this is not be true in general. Several situations may arise, mainly from rhp pole-zero cancellations of $M(s)$ leading to loss of stabilizability and/or detectability. Here, for instance, no positive perturbation on w_5 will ensure consensus, as is apparent from Fig. 6b (no *safe interval*). Same holds for w_6 and w_8.

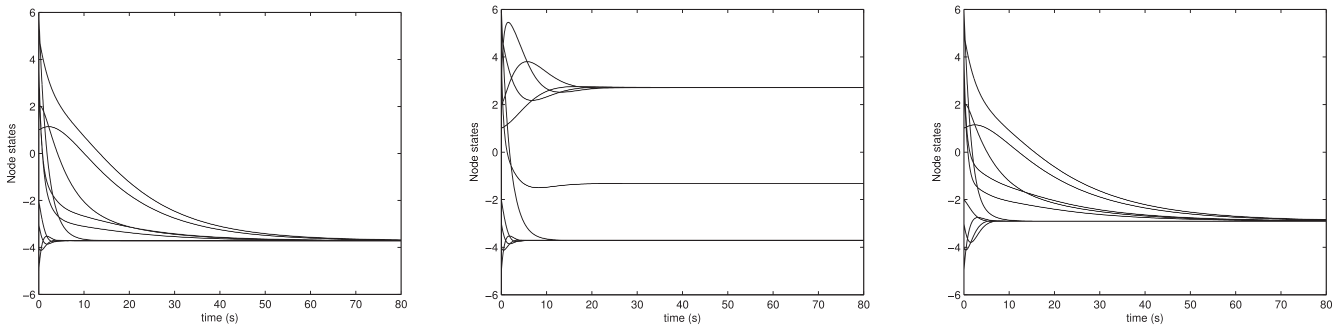
6 Simulation Results

Consider the weighted directed graph, $\mathcal{G}$, in Fig. 5 (bold and dashed portions together), with 10 nodes and 13 edges. The nodes 7, 8, 9, and 10 are globally reachable and are part of a



(a) (b)

Fig. 6. Nyquist plot of $M(s)$ for perturbation on (a) w_3 and (b) w_5 of velocity graph.



(a) (b) (c)

Fig. 7. Node states for perturbation on the weight (a) on $e_{4,5}$ within tolerable bound and (b) on edge $e_{3,8}$ at exact bound, and (c) on $e_{9,10}$ within tolerable bound.

directed cycle within the network. The nominal positive edge weights are marked on the corresponding edges. Suppose edge $e_{4,5}$ is perturbed from its nominal value. The initial node states are $[1\ 2\ 3\ 4\ 5\ 6\ -4\ -5\ -2\ -3]$. In Fig. 7a, the perturbation on the edge weight is $-.50$ so the perturbed weight is 0.50, while the tolerable bound on perturbation is -2.4995. It may be seen that consensus is achieved. In Fig. 7b, the perturbation on edge $e_{3,8}$ is exactly equal to the bound, that is -1.2667 (computed from (25)), so that the perturbed weight is -0.5667 and the nodes form clusters. Typical Nyquist plots of $M(s)\Delta$ for convergent, clustering, and divergent behaviors are shown in Fig. 8 with perturbation on edge $e_{4,5}$.

For a perturbation of -1 on the edge $e_{9,10}$ (the bound is -1.4614) Fig. 7c shows that consensus is achieved. Although the graph in Fig. 5 is not a directed cycle, yet the bound on the perturbation on edge $e_{9,10}$ can be calculated by simply considering the nominal weights on edges $e_{7,8}$, $e_{8,9}$, $e_{9,10}$, and $e_{10,7}$ and applying Theorem 4. This is not true of any arbitrary cycle in a digraph having a rooted in-branching, but holds in case the cycle comprises all globally reachable nodes as stated in Corollary 2.

Next, consider the graph in Fig. 5 with only the 10 bold edges, and nodes 1 through 9. Agents with double integrator dynamics are considered with control law (7), as in Example 1. The two simulations in Figs. 9a and 9b show the agent states without and with a positive perturbation of magnitude 2.50 on edge $e_{5,1}$ of the velocity graph (with nominal weight $w_{v_3} = 0.03$), respectively. The perturbation results in consensus, which is consistent with the results in Example 1, based on the Nyquist plot in Fig. 6a.

7 Conclusions

This paper considered consensus-seeking systems comprising single or double integrators and presented an analysis of the robustness margins for edge weights of a weighted directed graph having a rooted in-branching. Although only one weight is perturbed at a time, the framework is also suitable for analysis of multiple uncertain edge weights by employing small gain theorem, even for single integrators. Further, the proposed Nyquist based method is capable of answering whether consensus is disrupted when

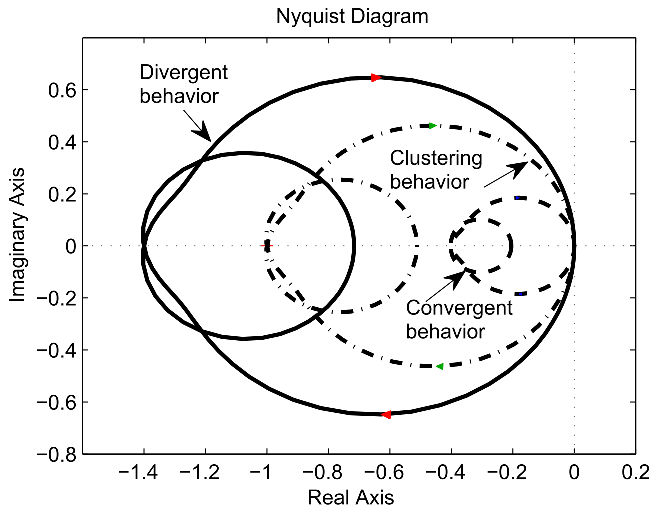


Fig. 8. Nyquist plots of $M(s)\Delta$ for the perturbed edge weight on $e_{4,5}$ exhibiting three types of behavior.

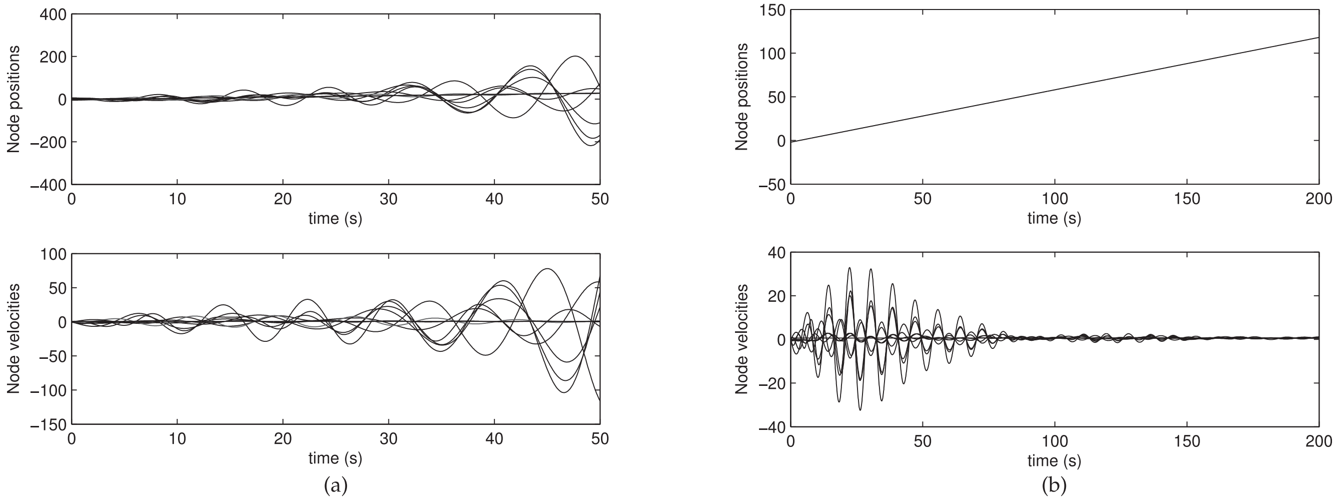


Fig. 9. Node positions and velocities (a) without any perturbations, and (b) with a perturbation on edge $e_{5,1}$ of magnitude 2.5.

multiple edge weights are perturbed by known amounts of perturbations. For double integrators, the results obtained for an existing protocol, though conservative, are the best robustness margins that can be analytically obtained. Moreover, based on the results, for any directed graph, the most vulnerable may be determined. That is, if an 'external party' wants to interrupt the consensus protocol by manipulating edge weights, the results here can help in choosing the edge weight that needs to be perturbed by the least amount. Also, the present set-up enables the graphical computation of the stability margin of the consensus protocol without explicit knowledge of the Laplacian spectrum.

For single integrators, graph theoretic interpretations of the robustness margins for a DAG and a directed cycle graph provide further insights and serve as an encouragement to interpret the result for more general graphs. A newly proposed consensus protocol for double integrators has been presented and a method for deciding the edge weights, from a designer's perspective, based on the Nyquist criterion, is outlined. It is shown that, subject to certain conditions, perturbing a single edge weight suitably can lead to consensus in a nominally unstable system.

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